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Variational Study of Ordinary and Superconducting Cosmic Strings

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Abstract. We use a variational approach to study Abelian vortices, both the ordinary and bosonic superconducting variety. We present accurate results for the energy per length. For superconducting strings we map out the parameter space of solutions, quantify the critical current, study the quench transition, and investigate the possiblity of static solutions where electromagnetic stresses balance the string tension.

In recent years there has been keen interest in cosmic strings, both the ordinary Nielsen-Olesen[1] and the superconducting[2] varieties. Even for ordinary cosmic strings there are few precise results[3-5]. In this letter we report the results of a detailed variational analysis of both ordinary cosmic strings and bosonic superconducting cosmic strings (hereafter, BSCS). The full details of our calculations will be presented elsewhere[6].

We study the strings which arise in the simple $U(1) \otimes U(1)'$ model of Witten[2], with the scalar potential

$$V(\Phi,\sigma) = -m_\Phi^2|\Phi|^2 + \lambda_\Phi|\Phi|^4/3! + 3m_\Phi^4/2\lambda_\Phi - m_\sigma^2|\sigma|^2 + \lambda_\sigma|\sigma|^4/3! + f|\Phi|^2|\sigma|^2 \qquad (1)$$

where Φ is a complex scalar field which carries no $U(1)$ charge and $U(1)'$ charge q, and σ is a complex scalar field which carries no $U(1)'$ charge and ordinary electric charge e. (Coleman-Weinberg[7] potentials are also considered in Ref. 6.)

Ordinary Cosmic Strings. First we consider ordinary $U(1)'$ vortices, both gauged and global (i.e., $q = 0$) by ignoring the σ field in Eqn(1). The potential $V(\Phi) = V(\Phi, \sigma = 0)$ has its global minimum at $|\Phi|^2 = \bar{v}^2/2$, where $\bar{v}^2 = 6m_\Phi^2/\lambda_\Phi$. The mass of the physical Higgs particle is $m_H^2 = 2m_\Phi^2$ and the mass of the vector boson is $m_V = q\bar{v}$. We define the dimensionless ratio $b = 2m_V^2/m_H^2 = 6q^2/\lambda_\Phi$. For $b > 2$ ($b < 2$) the vortices correspond to type I (type II) superconductivity in the Ginzburg-Landau theory. Note too, that the global case corresponds to the formal limit $b \to 0$.

While b is formally arbitrary, its natural range, $10^{-2} \lesssim b \lesssim 10^2$, is defined by other considerations: (1) perturbativity, which requires $\lambda_\Phi \lesssim \sqrt{4\pi}$; (2) radiative corrections[7], which, unless forbidden or cancelled due to symmetry considerations (e.g., supersymmetry[8]), require a λ_Φ term of the order of q^4 (or larger); (3) q is expected to be $O(e)$.

Assuming cylindrical symmetry about the z-axis, appropriate for long straight strings or loops with radius $\gg$ their width, the Hamiltonian per unit length is

$$\tilde{H}_\Phi = \int r dr d\theta [|\partial\Phi/\partial r|^2 + |r^{-1}\partial\Phi/\partial\theta - iqA'_\theta\Phi|^2 + V(\Phi) + B'^2/2]$$

where θ is the azimuthal angle, r the radial coordinate, and B' the magnetic flux associated with the vortex (in the global case $q = 0$ and $B' = 0$).

The general vortex solution with winding number W is of the form: $\Phi = (\bar{v}/\sqrt{2})P(r)e^{iW\theta}$. The limiting behaviour of $P(r)$ is: $P(r \to \infty) \to 1$ and $P(r \to 0) \to O(r^{|W|})$. In the gauged case: $A'_\theta(r \to \infty) \to W/qr$ and $A'_\theta(r \to 0) \to O(r)$. We restrict our analysis to the $W = 1$ vortex. In the global case and in the gauged case for $b < 2$, $|W| \geq 2$ vortices are unstable to decay into $|W|$ vortices of unit vorticity. For $b > 2$, $|W| \geq 2$ vortices are stable and may be of some cosmological interest, as one would expect them to also form in the early Universe.

As variational *ansätze* we take: $P(r) = (1 - e^{-\mu r})$ and $A'_\theta = (1 - e^{-hr})^2/qr$, where μ^{-1} and h^{-1} are the width of the vortex and magnetic flux tube respectively, and $2\pi/q$ the total flux in the flux tube. It is convenient to define the following dimensionless parameters: $a = m_\Phi^2/\mu^2$ and $s = \mu/h$. In terms of these, the size of the vortex $\mu^{-1} \sim \sqrt{a} m_\Phi^{-1}$, and the size of the magnetic flux tube $\sim s$ times that of the vortex.

Using the above *ansätze* we have solved for the variational parameters μ and h by minimizing $\tilde{H}_\Phi$. For the global vortex, $a = 1.6$ and the energy per length ε is: $\varepsilon = (0.75 + \ln \mu\Lambda)\pi\bar{v}^2$, where as expected ε is log-divergent, and Λ is the cutoff length (e.g., the distance to the nearest string or the size of a string loop). In Fig. 1 we display our results for the gauged vortex as a function of b. For $b \to 0$ the size of the vortex $a \to 1.6$, the global result. In general a is of the order of unity, and decreases with increasing b. The size of the magnetic flux tube relative to the vortex s, is about unity for $b = 2$, and as expected s decreases with increasing b.

In general, ε is rather insensitive to b, varying by less than a factor of 30 for $b = 10^{-6}$ to 10^6. Over the natural range of b, ε is well approximated (to better than 5%) by

$$\varepsilon = 1.19\,\pi\bar{v}^2\,b^{-0.195}$$

Our results for the energy per length agree with the exact result[3] for $b = 2$ ($\varepsilon = \pi\bar{v}^2$), and the very accurate numerical results[4] for $0.9 \leq b \leq 8$ to better than 2%. They agree to the accuracy that comparison is possible ($\sim 10\%$) with the semi-quantative results of Ref. 5 for the range: $0.04 \leq b \leq 400$. We have also added additional terms to our *ansätze*, and over the entire range of b our results for ε only decreased by at most 1%.

The fact that the energy per length is very insensitive to b is of some cosmological importance. Cosmic strings are a promising candidate for the origin of the primeval density inhomogeneities needed for the formation of structure in the Universe *if* the string tension ε is of the order of $10^{-6} m_{Pl}^2$, or perhaps higher[9] ($m_{Pl} \simeq 1.22 \times 10^{19}$ GeV). Cosmic strings are produced in the phase transition in which Φ acquires its vacuum expectation value (VEV); the critical temperature for this transition is: $T_c^2 \sim \bar{v}^2/(1 + b)$, which is naturally of the order of $\bar{v} \sim \sqrt{\varepsilon}$, and for the string tension of interest is about 10^{16} GeV. If the Universe underwent inflation[10], such a high value for T_c is problematic.

To avoid being diluted by inflation, strings must be produced after inflation (or near the very end of inflation). Based upon the production of long wavelength gravitational waves during inflation, there is an upper limit of $\sim 3 \times 10^{16}$ GeV to the highest temperature achieved after inflation. Of more practical concern are the highest temperatures which are typically achieved in inflationary models: owing to the scalar density perturbation constraint the maximum temperatures achieved after inflation are typically very low[10]

($\ll 10^{16}$ GeV). Thus, on the face of it is difficult to have both inflation and cosmic strings. However, our results indicate that one can have a high string tension and low value for $T_c \sim b^{-1/2}\varepsilon^{1/2}$, providing one is willing to tolerate an unnaturally large value for b (unnatural, unless the radiative corrections are suppressed for some reason).

Bosonic Superconducting Strings. We will now consider the full theory, by studying the σ condensate in the unperturbed, background vortex solution of the Φ field. We refer to this as the 'concrete vortex approximation', and will return to discuss its realm of validity. For the moment we will also take the current to be zero.

The potential $V(\Phi, \sigma)$ is specified by the 5 parameters: m_Φ, m_σ, λ_Φ, λ_σ, and f. However, we find it much more convenient to instead use only 3 of the above parameters and the two new parameters:

$$\alpha \equiv m_\sigma^2/\mu^2 = a m_\sigma^2/m_\Phi^2 \qquad \beta \equiv f\bar{v}^2/2\mu^2 = 3af/\lambda_\Phi$$

By employing α and β, the parameter space where BSCS solutions exist can be described by two algebraic constraints, and an allowed region in the α-β plane.

First the algebraic constraints: the stability of the global, unbroken $U(1)$ vacuum requires that

$$\textbf{constraint 1}: \quad m_\sigma^4/\lambda_\sigma < m_\Phi^4/\lambda_\Phi \qquad \textbf{constraint 2}: \quad \beta \geq \alpha \ \ (\text{or } f\bar{v}^2/2 \geq m_\sigma^2)$$

These 2 constraints preclude the existence of a global σ condensate, but still permit the possibility of one in the vortex where $\Phi = 0$ and the f-term no longer stabilizes the σ part of the potential. Of course, having such a condensate costs both kinetic energy (as σ must go from $\sigma \neq 0$ in the core, to $\sigma = 0$ far from the vortex) and potential energy (due to the $f|\Phi|^2|\sigma|^2$ term). The solution region of the α-β plane is determined by energetics—solutions will exist wherever a σ condensate lowers the vortex energy.

Due to the σ field there are additional terms in the Hamiltonian per length

$$\tilde{H}_{I=0} = \tilde{H}_\Phi + \tilde{H}_\sigma \, (\equiv \int r dr d\theta [|\partial\sigma/\partial r|^2 + r^{-2}|\partial\sigma/\partial\theta|^2 + V(\Phi = 0, \sigma) + f|\Phi|^2|\sigma|^2]$$

In the concrete vortex approximation we can consider the global and gauged cases together, the only difference being the value of a. For the σ field *ansatz* we use: $\sigma(t, z, r, \theta) = (\sigma_0/\sqrt{2})e^{-\kappa r}(1 + \kappa r + \kappa' r^2 + \kappa'' r^3)e^{i\phi(t,z)}$, which has four variational parameters (σ_0, κ, κ', and κ''); the full *ansatz* is only used to monitor the convergence of the truncated *ansatz*, $\sigma = (\sigma_0/\sqrt{2})e^{-\kappa r}e^{i\phi(t,z)}$. We also introduce one additional dimensionless ratio, the size of the σ condensate relative to the size of the vortex, $x \equiv \mu/\kappa$ (remember μ has already been fixed). The VEV of the charged σ field within the vortex signals that the string is superconducting; the phase $\phi(t,z)$ gives rise to the current and is the massless mode which provides the longitudinal degree of freedom for the photon on the string.

Using this *ansatz* we have mapped out the regions of α-β space where it is energetically favorable for a σ condensate to form; i.e., the regions where the minimum value of $\tilde{H}_\sigma$ is negative (and occurs for $\sigma_0^2 > 0$). Those regions are shown in Fig. 2. Roughly speaking, they form a wedge, beginning at $\alpha \simeq 1/7$ and for large α, β bounded by the lines: $\beta = \alpha$ and $\beta = 0.5\,\alpha^{1.93}$. (Note that for $\alpha = \beta \to 0$ it can be shown analytically that solutions also exist[2,6].) The solution space mapped by our truncated *ansatz* differs only slightly from the one mapped by our full *ansatz*. Except near the upper boundary of solution space the energies found by the full and truncated *ansätze* agree very well (better than 10%).

By simultaneously determining all the variational parameters (μ, κ, and σ_0) we have studied the validity of our 'concrete vortex approximation'. We find that the only region of α-β space where there is significant 'back reaction' of the σ condensate upon the vortex is near the line $\beta = \alpha$, and only if **constraint 1** is saturated, i.e., $m_\Phi^4/\lambda_\Phi \approx m_\sigma^4/\lambda_\sigma$; otherwise the 'concrete vortex approximation' is a very good one. That the back reaction should be significant when **constraint 1** is saturated should come as no surprise, as in this case the two fields have comparable vacuum energies.

Critical Currents and the Quench Transition. Now we allow the BSCS to carry a current. Currents on the string arise due to the twisting of the phase of the σ field along the string. Taking the phase to vary as $\phi = 2\pi Nz/L$, where L is the length of the wire and N is the net topological phase twist, one can directly solve Maxwell's equations[2,6]. The current is $I = 2\pi N/eL\omega$, where $\omega = [1 + \frac{e^2K}{2\pi}\ln(\kappa L)]/e^2K$ is essentially an inductance per unit length, and $K = 2\pi\int r dr \sigma_0(r)^2 = \pi\sigma_0^2/2\kappa^2$.

Due to the current there are additional contributions to the Hamiltonian per length: the KE of the charge carriers and the magnetic field energy, which together are given by $\tilde{H}_I = \omega I^2/2$. The ratio of the magnetic field energy to the KE of the charge carriers is $e^2K\ln(\kappa L)/2\pi$, and except for the region near the upper boundary of solution space this ratio $\gg 1$—in this respect a BSCS is like an ordinary current carrying wire.

The persistence of the current in a BSCS loop owes to the conservation of the topological phase twist N. The σ field can only untwist itself in a process where σ locally becomes zero and the phase of σ becomes undefined. A BSCS loop cannot, however, carry an arbitrarily large current. For a sufficiently high current it becomes energetically favorable for the σ field to locally relax to zero allowing the phase to untwist. Consider the pieces of $\tilde{H}$ associated with the σ condensate: the noncurrent piece $\tilde{H}_\sigma$ is < 0, while the current piece $\tilde{H}_I$ is > 0; note too that for $\sigma_0^2 = 0$, $\tilde{H}_\sigma + \tilde{H}_I = 0$. Thus the critical current is achieved when $\tilde{H}_\sigma + \tilde{H}_I$ becomes nonnegative. In the limit that $K \gg 1$ the critical current is

$$I_{crit} = \pi\mu\sqrt{\frac{12}{\lambda_\sigma \ln(\kappa L)}}(\alpha x - \beta x F(x) - x^{-1})$$

where $F(x) = (x^4 + 6x^3 + 6x^2)/(x^4 + 6x^3 + 13x^2 + 12x + 4)$. The factor involving x, α, and β is easily computed and in general is of the order of the parameter α.

For $I > I_{crit}$, the σ pieces of $\tilde{H}$ have their global minimum at $\sigma_0 = \tilde{H}_I + \tilde{H}_\sigma = 0$; however, because of the logarithm in the expression for the current, over a substantial region of solution space $\tilde{H}_\sigma + \tilde{H}_I$ still has a local minimum for $\sigma_0 \neq 0$ with $\tilde{H}_\sigma(\sigma_0) + \tilde{H}_I(\sigma_0) > 0$, indicating that the transition associated with the loss of superconductivity is *first order*. Presumably the quench transition proceeds through the nucleation of bubbles of true vacuum (within which $\sigma_0 = 0$); depending upon the bubble action, supercritical currents may be metastable with significant lifetimes.

Static, or Floating Loops. It has been suggested that BSCS loops might be able to achieve a static state where electromagnetic stresses balance the loop string tension[11,12]. That this might occur is easy to understand. Neglecting numerical factors the energy of a loop of length L is: $E \sim \bar{v}^2 L + LI^2$, the first term is the mass of the loop and the second term is the electromagnetic field energy. As the loop oscillates, it radiates both electromagnetic and gravitational radiation, and shrinks. Conservation of the phase twist N means that both the current $I \propto N/L$ and the magnetic field energy $\sim LI^2 \propto 1/L$ must increase as the loop shrinks. Assuming that the loop remains superconducting, it eventually shrinks to a size $L_{static} \sim N/\bar{v}$ where its total energy achieves its minimum. In this floating state electromagnetic stresses balance the string tension. The crucial question is whether or not the static state is achieved before the critical current is exceeded.

The total energy of a loop of length L is: $E(L) = L(\tilde{H}_\Phi + \tilde{H}_\sigma + \tilde{H}_I)$. (Note that in general the energy of a superconducting loop with subcritical current is *less* than that of an ordinary loop since $\tilde{H}_\sigma + \tilde{H}_I$ is necessarily negative.) It is straightforward to solve for the length L_{static} which minimizes the above energy. Since $I \propto 1/L$, we can also solve for the length L_{crit} at which the current reaches the critical current. The existence of a stable, floating state then requires that $L_{static} > L_{crit}$.

We have studied our variational solutions and find that such states only exist when

$$m_\Phi^4/\lambda_\Phi \approx m_\sigma^4/\lambda_\sigma \qquad \text{and} \qquad \beta \approx \alpha \qquad (\Rightarrow\ f \approx \sqrt{\lambda_\Phi \lambda_\sigma}/3)$$

This occurs in the region of solution space where back reaction is significant (not surprising) and so in our analysis we have solved for all the variational parameters simultaneously. For the global case only floating states with sizes less than some maximum size exist, as (for fixed parameters) the energy per length of the loop depends logarithmically upon the size of the loop and sufficiently large loops will require supercritical currents.

In sum, we can say that the region of solution space where the floating state can be achieved with subcritical current is very tiny: in the space of solutions BSCS's which can float are very much the exception. If there are floating BSCS's, they can be a cosmological disaster, as we now describe. A current can be induced in a BSCS if it moves through a region of magnetic field in the early Universe. The existence and origin of primeval magnetic fields is an interesting and unresolved question. Irrespective of primeval fields, currents will develop in BSCS loops in a manner analogous to how strings and monopoles are produced in the early Universe[13]. When the σ condensate forms (which for simplicity we assume occurs when the vortices form) the phase of the σ field cannot smoothly align itself on distance scales larger than the horizon at that time, i.e., its correlation length $\xi_0 \lesssim t_0$ (t_0 is the time of the phase transition). Because of this a length of string L_0 will necessarily have a phase twist of the order $(L_0/\xi_0)^{1/2}$, and hence a current $I \sim 2\pi(L_0/\xi_0)^{1/2}/eL\omega$, which one might refer to as the 'Kibble current'. Loops of size L_0 form by breaking off from long strands of string when the age of the Universe is about $t \sim L_0$. Once such loops shrink to their floating state their mass remains constant and they behave like nonrelativistic matter with a mass density which decreases only as a^{-3} ($a(t)$ is the scale factor of the Universe). Using the birthrate[9] of loops of size L_0 ($\simeq t^{-4}|_{t\simeq L_0}$) it is simple to compute the mass density contributed by floating loops at time t: $\rho_{float} \simeq \bar{v}^{11/2}t^{-3/2}/m_{Pl}^3$ (valid when the Universe is radiation dominated). If floating loops are not to overdominate the mass density of the Universe today, $\bar{v}$ must be less than $\sim 5 \times 10^{13}$ GeV. Of course, if they saturate this bound floating loops contribute $\Omega \simeq 1$ and would be the dark matter.

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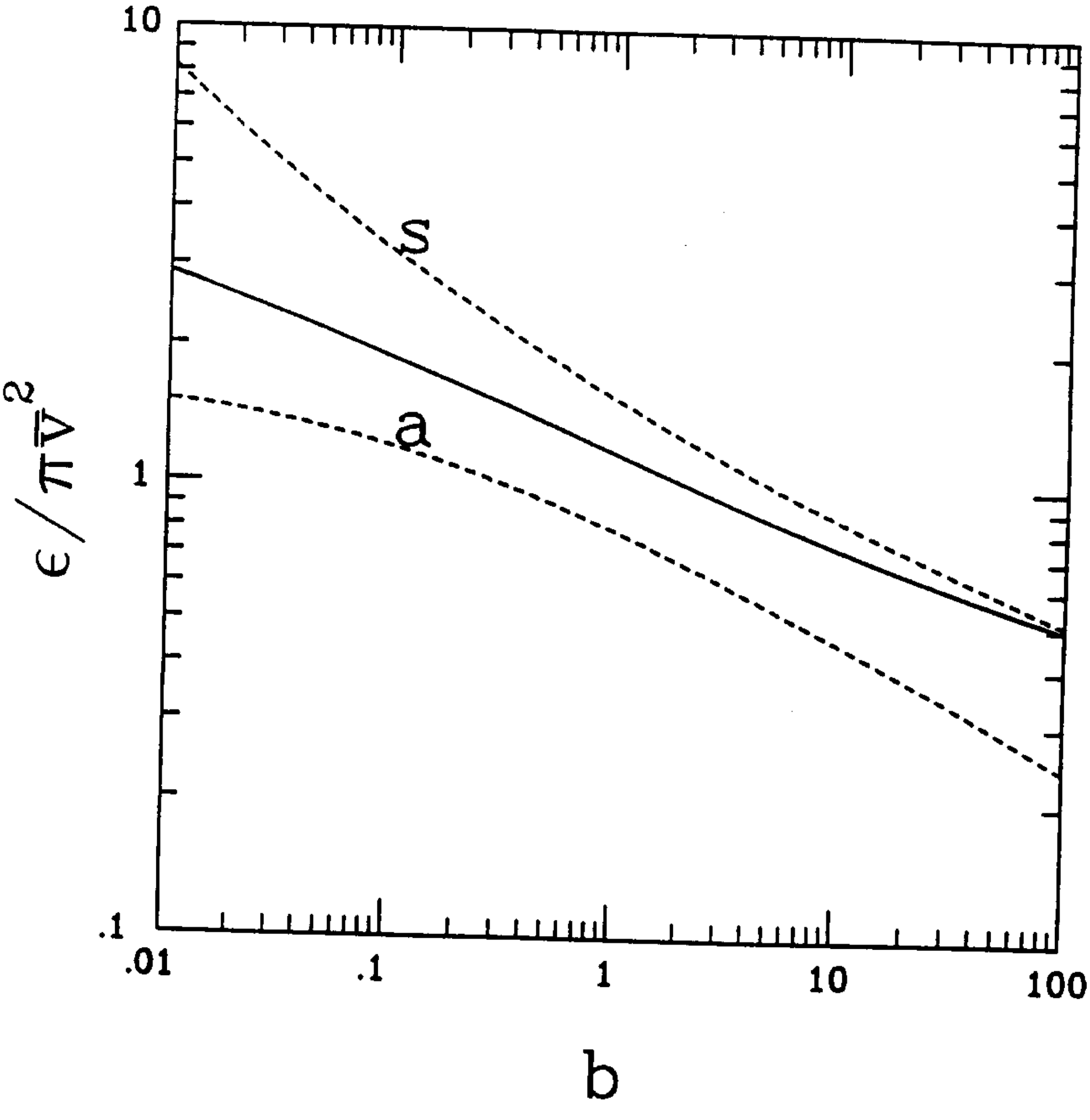
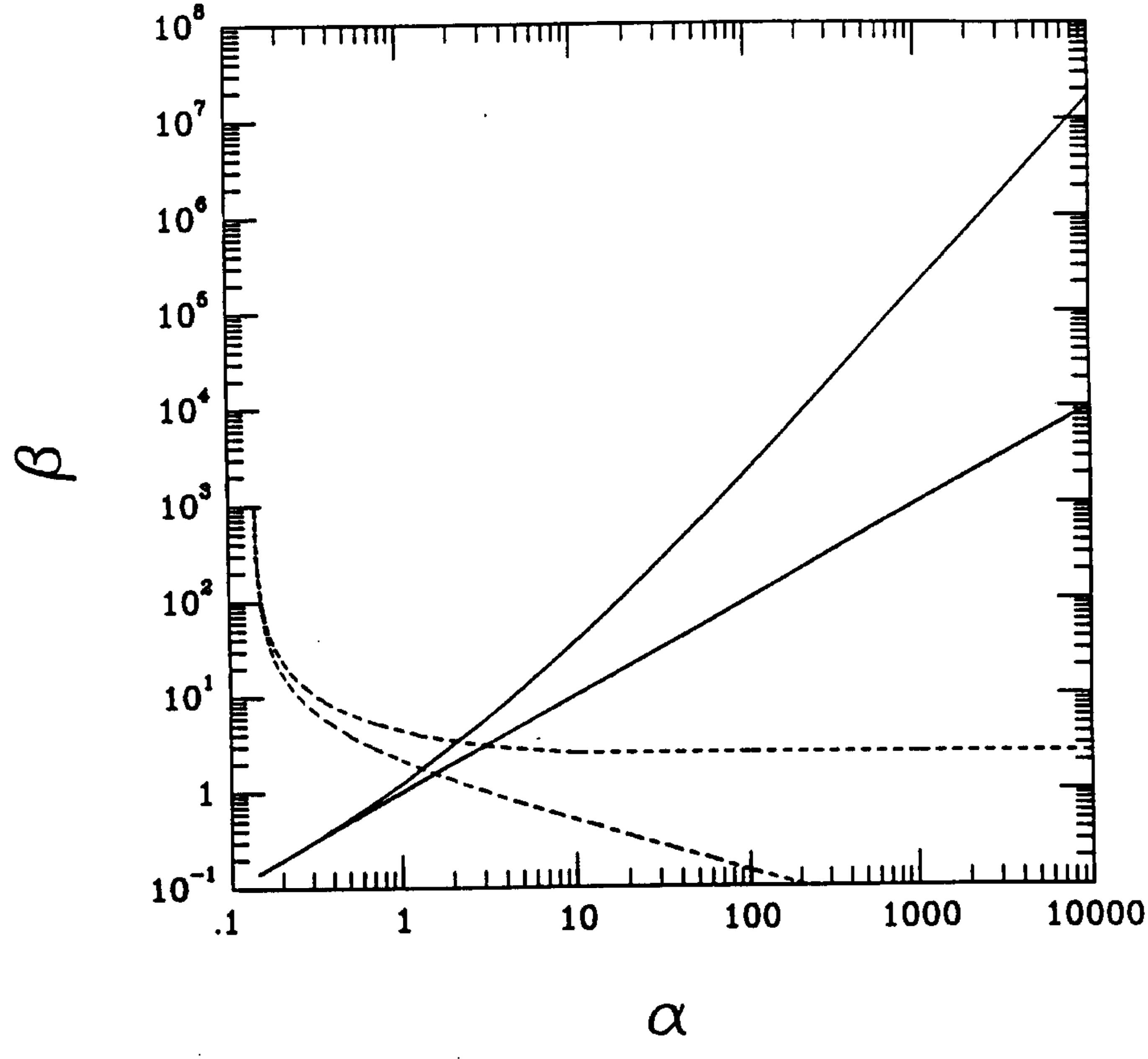
Figure Captions

Figure 1—The energy per length (solid curve), vortex size a, and flux tube size s for a gauged string over the natural range of b. Results over a much extended range are given in Ref. 6.

Figure 2—The α–β parameter space of solutions (region between solid curves), and the range of x ($\equiv \mu/\kappa$) vs. α (region between broken curves).



-FIGURE 1-


10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
1
10^{-1}
β
.1
1
10
100
1000
10000
α

- FIGURE 2 -