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Gauge Invariant Closed String Field Theory and the Virasoro-Shapiro Amplitude

R. Holman

NASA/Fermilab Astrophysics Center

Fermilab, P.O. Box 500, MS209

Batavia, IL 60510

Siddhartha Sen

School of Mathematics

Trinity College, Dublin 2 Ireland

Abstract

Using a gauge invariant closed string field theory recently proposed by us, we show that the amplitude for 4-tachyon scattering in our theory is given by the Virasoro-Shapiro amplitude.

It is widely believed (or hoped!) that string field theory will give us a handle on possible non-perturbative ground states of the string, analogous to the Schwarzschild solution in general relativity. For this reason, among others, it would be extremely useful to construct gauge invariant, interacting string field theories.

Witten [1] has proposed such a theory for the open string. His construction involves the definition of a non-commutative differential geometry, where string fields are treated as one-forms (as advocated by Banks and Peskin [2]) and the BRST charge Q acts as the exterior derivative (in the critical dimension only!). An integral and wedge product are then defined for these forms by considering string path integrals over appropriately chosen 2-D surfaces. From this formalism a non-linear gauge invariance can be formulated, whose linear piece had been considered in [2,3] and a Chern- Simons action for the string field can be written down.

By generalizing the above constructions, we have recently been able to construct a field theory for *closed* bosonic strings [4]. We showed that, at least formally (i.e. at the level considered in [1]) all the relevant axioms of non-commutative geometry were satisfied.

In this paper, we wish to show that our theory can pass one of the crucial tests of any interacting closed string field theory. We will show that starting from our interaction Lagrangian, the Virasoro-Shapiro amplitude for 4-tachyon scattering can be obtained. In doing this we will follow closely the analysis of Giddings [5], who showed that Witten's theory led to the Veneziano amplitude. That this should be the case is a highly non-trivial result, since it was not at all clear that the correct measure and integration region for the Koba-Nielsen variable need have been obtained.

The fact that our theory does give rise to the Virasoro-Shapiro amplitude is rather surprising in view of the objections raised by Giddings and Martinec [6] about the class of closed string field theories we are considering. At the end of our calculation we will review these objections and show how we avoid them.

First, let us review and elaborate on our field theory. Since we are using the BRST formalism, our states belong to a Hilbert space that includes ghost states. At this point we must now choose a BRST charge Q to define our closed string geometry. One of the criteria that Q must satisfy is that it give rise to the correct physical spectrum. This point was addressed in [7] and we use their notation in what follows. The closed string has two independent BRST charges Q_R, Q_L for the right and left moving sectors respectively. We write:

$$Q_R = c^0{}_R K_R + d_R + \delta_R - 2b_{0R} \Downarrow. \tag{1}$$

and likewise for Q_L. Here $c^0{}_R$ and b_{0R} denote the ghost and antighost zero modes respectively while K_R is basically L_0 and d_R, δ_R and $\Downarrow$ are other operators whose precise forms will not concern us in this paper. If we now consider $Q = Q_L + Q_R$ we have:

$$Q = \tilde{Q} + \hat{c}^0(K_R - K_L) + \hat{b}_0(\Downarrow - \bar{\Downarrow}). \tag{2}$$

$$\tilde{Q} = c^0(K_R + K_L) + d + \delta + b_0(\Downarrow + \bar{\Downarrow}). \tag{3}$$

where we have defined:

$$c^0 \equiv 1/2(c^0{}_R + c^0{}_L), \tag{4a}$$

$$\hat{c^0} \equiv 1/2(c^0{}_R - c^0{}_L), \tag{4b}$$

$$d \equiv d_R + d_L, \tag{4c}$$

and likewise for $\hat{b_0}$ and δ. We now restrict ourselves to the subspace S defined by:

$$(K_R - K_L)A = 0$$
$$\hat{b_0}A = 0. \tag{5}$$

Then Q becomes $\tilde{Q}$ when acting on S and it generates the correct closed string spectrum. It also satisfies $\tilde{Q}^2 = 0$ on S [7]. We can also show that S is stable under the action of $\tilde{Q}$. This follows from the fact that $[\tilde{Q}, K_R - K_L]$ vanishes while $[\tilde{Q}, \hat{b_0}] \propto K_R - K_L$. It can also be checked that the current $\tilde{J}^\alpha$ that gives rise to $\tilde{Q}$ is conserved even on *curved* world-sheets; this is a consequence of the equivalent staement for the original BRST current J^α [1].

Our integral and product can now be defined using $\tilde{Q}$ as a template. The construcion closely resembles that in [1]. Since we treat the closed string as two open strings joined together at two special points ($\sigma = \pm\pi/2$ in our parametrization, in which all strings have σ running from $-\pi$ to $+\pi$) we use two copies of the surfaces in [1] identifying them in such a way so as to represent closed strings. We also allow for the possibility that our surfaces may have boundaries, handles and extrinsic curvatures. This gives us the freedom to choose the ghost insertions required for BRST invariance arbitrarily. Thus, unlike [1], we can write down a non-zero Chern-Simons action for the closed string.

We will first describe Giddings' calculation for the open string and then modify it for the closed string. First a gauge must be fixed. Giddings chooses the Siegel gauge $b_0 A = 0$. Open string physical states then have ghost number $-1/2$ and can be written as $A = \tilde{A} b_0 | \uparrow\rangle$, where $\tilde{A}$ lives in the part of the Hilbert space that omits the ghost zero modes, and $| \uparrow\rangle$ is the c^0 vacuum, with ghost number $+1/2$. We can now reexpress the kinetic operator as a path integral of $b_0\Delta$ over a thin strip ($\Delta = {L_0}^X + {L_0}^{gh}$) with boundary conditions specified by $\tilde{A}| \uparrow\rangle$. The b_0 insertion can then be written as a line integral $\int d\sigma[b_{zz} + b_{\bar{z}\bar{z}}]$ .Thus, aside from this integral, the propagator is:

$$\Delta^{-1} = \int_0^\infty d\tau e^{-\tau\Delta} \qquad (6).$$

We are then led to the following rules for computing the 4-point amplitude: Integrate over the length of the intermediate strip the path integral over the surface in fig.(1) using the bosonic string action with *bosonized* ghosts ϕ with an explicit line integral for the antighost insertion. The boundary conditions are provided by incoming momentum and ghost number wave-functions, and when projected to finite points, they can be represented by vertex operators.

Since the theory is conformally invariant, we can map the region in fig.(1) to the upper half plane (UHP) via Schwartz-Christoffel transform ations with analytic continuations. This then allows Giddings to perform the required functional integrals. These provide him with the correct measure and integrand while SL(2,R) invariance requires him to sum over both the s and t channels, giving rise to the full region of integration of the Koba-Nielsen variable x.

What modifications are necessary for the closed string calculation? First, we fix our gauge as before, $b_0 A = 0$, but now we note that $A \in S$ so that A satisfies $\hat{b}_0 A = 0$ automatically (this restriction will be essential for correctly counting the total ghost number). We may then write A as:

$$A = \tilde{A} b_0 | \uparrow\rangle \otimes \hat{b}_0 |\hat{\uparrow}\rangle, \tag{7}$$

where $| \uparrow\rangle$ and $|\hat{\uparrow}\rangle$ are the ghost states annihilated by c^0 and $\hat{c}^0$ respectively. $\tilde{A}$ has ghost number zero so that A has total (right plus left) equal to -1. As in the open string case we represent the b_0 insertion as a line integral. However no such insertion is required for the $\hat{b}_0$ since the calculation does not involve $\hat{b}_0$. The propagator is $\Delta = (L_0 + \bar{L}_0)^X + (L_0 + \bar{L}_0)^{gh}$ and can be represented as in eq.(6).

We now follow the standard prescription for the 4-point amplitude: integrate over the length of the intermediate strip,τ, *and* over the location of the joins (fig.(2)). In terms of the closed string version of fig.(1) (that is, two copies joined back to back) this requires integrating from E to F and then on the other side back to E again. In terms of the variable z we will define later, this is equivalent to integrating over the entire imaginary axis. The external states can be expressed as vertex operators

$$V(p) = e^{ip\cdot x} \times e^{-i\phi}, \tag{8}$$

where the external states have total ghost number (left + right) -1. We can check that total ghost number is conserved when the insertions due to the $\oint$ and $*$ operations for each vertex are taken into account. The four incoming states contribute $g = -4$, while the antighost insertions give $g = -2$ (one each for b_0 and $\hat{b}_0$). The curvature singularities for the two vertices contribute $2[2g(*) + g(\oint)]$, where $g(\oint)$ and $g(*)$ are the ghost numbers of $\oint$ and $*$ respectively. For a theory with action given by:

$$S = \oint [A * \tilde{Q} A + \frac{2}{3} A * A * A], \tag{9}$$

we must have $g(*) = 2, g(\oint) = -1$ [4]. Thus the vertices contribute total ghost number $+6$ so that ghost number is conserved.

The amplitude of interest is given by:

$$A_S = \int d\tau \langle exp[i\sum_j P_j \cdot X(j)]\rangle_X \times \langle exp[i\sum_j J_j \cdot \phi_j](\int \frac{dw}{2\pi i} b_{ww} + \frac{d\bar{w}}{2\pi i} b_{\bar{w}\bar{w}})\rangle_\phi, \quad (10)$$

where we have defined:

$$\langle ... \rangle_X \equiv \int DX(...) e^{-S_X}, \quad (11)$$

and likewise for $\langle ... \rangle_\phi$. P_j and J_j (j=1,..4) are the external state momentum and ghost number sources at $(A, \acute{A}, \acute{B}, B)$ in fig.(1). Note that in eq.(10) $d\tau$ is now a two real dimensional integral since we are integrating over the length of the intermediate cylinder and where it joins.

We have mentioned previously that our theory treats closed strings as two open ones with the proper identifications between them. This allows us to use the same maps as in [5] to map our geometry to the *whole* complex plane (as opposed to the half plane as in [5]) by letting the variable z in the map $w(z)$ ([5] eq.(7)) to range over the entire complex plane. Imposing the width constraints required by the scattering geometry, we arrive at

$$\Lambda_0(\theta_1, k) - \Lambda_0(\theta_2, k) = 1/2 \quad (12)$$

$$K(\acute{k})[Z(\theta_2, \acute{k}) - Z(\theta_1, \acute{k})] = \frac{\tau}{2} \quad (13)$$

where Λ is Heumann's lambda function, Z is the Jacobian zeta function and K is the complete elliptic integral of the first kind . We also have: $k^2 = \frac{\gamma^2}{\delta^2}, \acute{k}^2 = 1 - k^2, sin^2\theta_1 = \frac{\beta^2}{\beta^2+\gamma^2}, sin^2\theta_2 = \frac{\alpha^2}{\alpha^2+\gamma^2}$. From these equations it follows that:

$$d\tau = \frac{2\pi}{K(\gamma^2)} \frac{d\alpha}{\sqrt{(1+(\alpha\gamma)^2)(\alpha^2+\gamma^2)}}. \quad (14)$$

At this point we let α become complex, thus rendering τ complex. It is then the real part of τ that satisfies eqns.(12, 13).

It is important to stress that the map taking the scattering geometry to the complex plane takes it, in fact to a *cut* plane. This is due to the ghost insertions required for BRST invariance (see [5] for more details) . The cuts run along

$(i\gamma, i\delta), (-i\gamma, -i\delta), (i\delta, i\infty), (-i\infty, -i\delta)$ (see fig.(3)). The cuts represent the fact that a Riemann surface with non-zero curvature can only be represented in the plane if singular curvature points are present with cuts joining them. In order to complete our calculation, we need the Green's function for the cut plane. However this is equivalent to using the standard Green's, function for the plane and then carrying out the relevant integrations taking the cuts into account. In our problem the singularities are pushed to $\pm i\infty$ so that we are able to use the normal Green's function for the plane.

Now introduce $z \equiv (\frac{1-\alpha^2}{1+\alpha^2})^2$, remembering that α is complex now. Then, as in [5], A_S can be written as:

$$A_S = -1/4 \int d^2z |z|^{p_1 \cdot p_2} |1 - z|^{p_2 \cdot p_3} \tag{15}$$

where the integral is taken over some region Γ of the complex plane. If Γ is the whole plane then A_S will be the V-S amplitude. Let us first consider the range of $Re(z) \equiv x$. By cosidering what happens as $Re(\tau)$ approaches zero and infinity we find that $x \in [1/2, 1]$ [5]. This corresponds to part of the s-channel. To get the entire range we use the following Sl(2,R) maps:

$$\begin{gathered} x \to 1 - x \\ x \to 1/x \\ x \to \frac{1}{1-x} \\ x \to \frac{x}{x-1} \\ x \to \frac{x-1}{x} \end{gathered} \tag{16}$$

This shows that x really ranges over the real line with no overlapping regions. Note that while these maps are in Sl(2,R), their use only makes sense if we embed Sl(2,R) as a subgroup of Sl(2,C), since it is only then that a continous transformation taking $(0, x, 1, \infty)$ to any of the other six exists. Another way to say this is that if all we had was Sl(2,R) invariance, we could only permute the sources cyclically, which would *not* cover the entire reals.

Now we turn to the range of integration of $Im(z) \equiv y$. This integration corresponds to integrating along the joining point of the propagator to the cylinders representing the incoming and outgoing states. This corresponds to integrating overthe entire imaginary axis. We can neglect the effect of the cuts by integrating along a vertical path infinitesimally displaced from the cut axis. Thus we obtain the V-S amplitude.

Giddings and Martinec [6] have made the following objections to this class of theories:

(a) The four strings can never be placed symmetrically;

(b) The τ going to zero and infinity limits really represent the same physical situation so that if both limits are required to obtain the correct integration region, we would be double counting in moduli space.

To answer (a), we note that the 3-string vertex of Gross and Jevicki [8] does *not* have a symmetric form in terms of the three strings. It is only after the relevant Neumann functions are used that the symmetry appears. This is similar to the situation treated here. It is only after *all* channels have been taken into account that the V-S amplitude emerges. Work is in progress to determine the form of the three closed string vertex in our theory [9].

Regarding point (b), we note that the cut structure actually prevents the point $\tau = 0$ from being present (the relevant cut is the one that runs from $-i\gamma$ to $+i\gamma$). The Green's functions have no discontinuities across the cut due to the use of the reflection principle in identifying the two sides of the cut .

Thus, we conclude that our theory seems to avoid double counting intermediate states contributing to the amplitude. However, it would be more convincing if we could show that our Feynman rules provided a single cover of moduli space, as was shown in [10] for the open string theory.

In summary, we have shown that an analysis similar to that in [5] shows that the four point amplitude can recovered from the candidate field theory proposed in [4]. Our next tasks construct the three-string vertex for our model and then generalize it to supersymmetric case following [11]. Having the 3-string vertex will allow us to make more concrete statements about the symmetries of our theory (i.e. are the

$K_n \equiv L_n - (-)^n L_{-n}$ symmetries?), and in particular, on the nature of the insertions required. While on this topic, we note that the identification of $\pm i\infty$ implicit in our calculation requires that the insertions of $2g(*) + g(\mathit{f})$ at $\pm\frac{\pi}{2}$ be equal.

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Figure Captions

Fig. 1: The 4-tachyon scattering world-sheet for open strings.

Fig. 2: A schematic representation of the closed string 4-tachyon scattering world-sheet.

Fig. 3: The effect of Giddings' map on the closed string world-sheet of fig.2. Note the cut structure.

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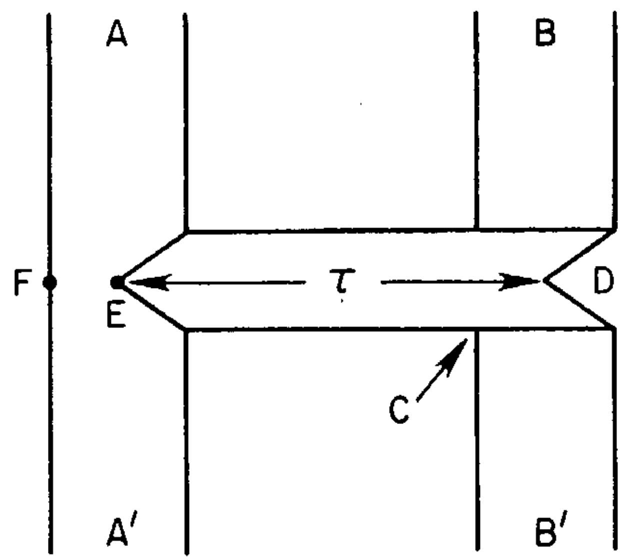

FIGURE 1

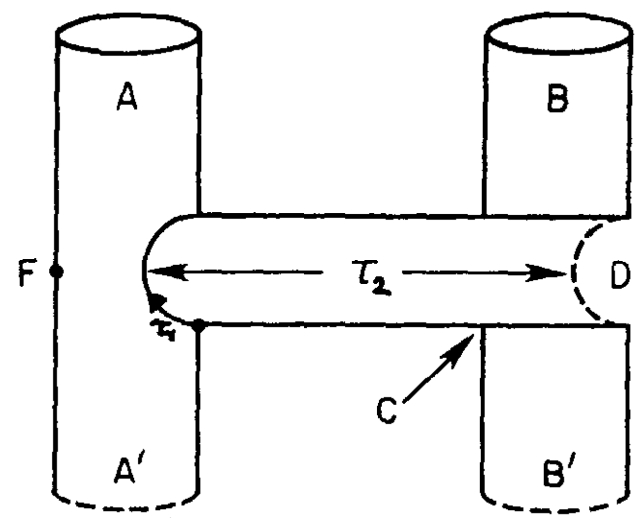

FIGURE 2

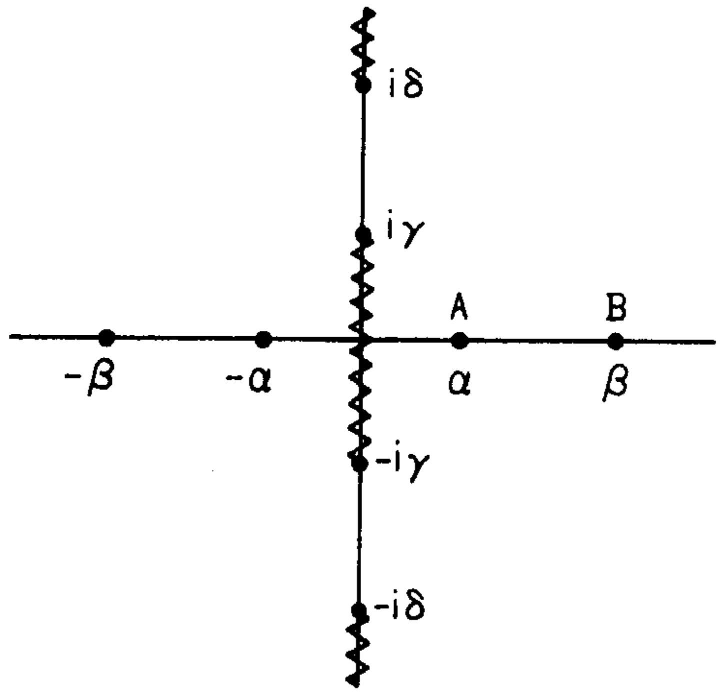

FIGURE 3