Fermi National Accelerator Laboratory

FERMILAB-Pub-87/25-A
RU-87/$B_1$/189

# Vacuum Energy of $M^4 \times S^M \times S^N$ in Even Dimensions

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## ABSTRACT

We obtain an expression for the one loop effective potential coming from quantum fluctuations of scalars and spin-1/2 fermions in a higher-dimensional manifold of product form $M^4 \times S^M \times S^N$ . In contrast to previous calculations, we consider the case in which the total number of dimensions is even, since this is the relevant case for superstring compactification. A detailed calculation for a ten-dimensional spacetime with two internal 3-spheres is performed, and an approximate static solution for the geometry is found, with the two internal radii being of the order of the Planck length for a sufficiently large number of matter fields. We study the stability of this solution and make some remarks on the possible cosmological implications of our results.

*January 1987*

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Operated by Universities Research Association Inc. under contract with the United States Department of Energy

## 1. Introduction

The original motivation for calculating 1-loop quantum effects in higher dimensional theories was to obtain a static configuration for the product manifold with the radius of the internal compact space close to the Planck scale [1]. The internal space was initially taken to be flat, thus requiring the matter potential itself to be stationary and leaving the internal radius as a free parameter of the theory. It was subsequently suggested by Candelas and Weinberg [2] that, if the internal manifold was curved, equilibrium could come, instead, from the balance of classical and quantum contributions to the effective potential.

More explicitly, in this approach the energy-momentum tensor is determined purely by the 1-loop quantum fluctuations of matter fields and not by some "monopole like" topologically-non-trivial configuration of generalized gauge fields. This quantum contribution is to be balanced by the classical curvature term for the internal manifold that appears in Einstein's equations, rendering the internal radius stable.

Candelas and Weinberg argued that, for a sufficiently large number of matter fields, the gravitational contributions to the 1-loop effective action could be neglected. Nevertheless, more recent results by Ordóñez and Rubin [3] and Chodos and Myers [4] indicate that 1-loop graviton contributions are roughly 3 orders of magnitude bigger than scalar or spin-1/2 fermionic contributions. Thus, the number of matter fields necessary to render Candelas and Weinberg's argument valid seems to be unnaturally large. In this connection, it may be interesting to note that these results are obtained when only 1-loop matter (and gravity) fluctuations act as the sources for the energy-momentum tensor. Other effects, like fermionic condensation or monopole terms, may also be important and can be used together with quantum effects in order to obtain a stable geometric configuration for the internal manifold [5,6].

Another important aspect of the 1-loop calculations in the literature is that they have been mostly performed in an odd total number of spacetime dimensions. The general reason for this is that, at 1-loop order, the possible counterterms that can be constructed for the renormalization of the theory are all of even number of derivatives, like squares of the curvature scalar, etc. Due to the general coordinate invariance of the theory, no such terms can be constructed in odd dimensions and we need not worry about arbitrary parameters that can destroy the predictive power of the 1-loop calculation. In particular, if we use the zeta-function regularization method, it has been shown that in odd dimensions the effective potential is indeed independent of the mass parameter that gives the canonical dimensionality to the path integral measure [7].

It so happens that in odd dimensions it is necessary to include a D-dimensional cosmological constant (D is the total number of space-time dimensions) if we want to have a flat 4-dimensional Minkowski spacetime ( $M^4$ ) as a solution to Einstein's equations. This unpleasant feature of odd-dimensional theories can be modified in even dimensions. The extra dimensionful parameter which, in even total number of dimensions comes naturally from the measure of the path integral, plays the role of an effective cosmological constant, and can be used to obtain a flat Minkowski space in 4-dimensions. (In even or odd D, the flatness of $M^4$ is in no sense "natural", but is a consequence of blatant "fine-tuning" !). This procedure was first suggested in ref.3, and was applied by Myers [8] to a $M^4 \times S^N$ model with gravitons.

In this paper we will be interested in calculating the 1-loop effective potential generated by quantum fluctuations of massless scalars and spin- 1/2 fermions with an even-dimensional background given by the product space $M^4 \times S^M \times S^N$ , where $S^M$ is the M-dimensional sphere. A similar calculation has been performed by Kikkawa et al. [9] for an odd number of dimensions. The interest in generalizing their result to even dimensions comes from the recent importance that even-dimensional field theories have acquired as limiting cases of string models. In particular, $N = 1$ $D = 10$ supergravity coupled to $N = 1$ super Yang-Mills has been shown to be the point like limit of type I and heterotic superstring models [10]. It is thus natural to ask if quantum effects will play an important role in stabilizing the internal manifold of these theories. Of course, if the theory is supersymmetric, the quantum contributions from bosons and fermions should cancel exactly, although once we impose boundary conditions to the supersymmetry transformations it is not clear that the cancellation is so straightforward. So, we will keep an open mind and perform the calculation in a nonsupersymmetric background. Our calculations should then be taken as a further step towards the understanding of the role of quantum effects in compactification of theories with a non-trivial internal space. The lessons learned now will certainly be useful in a more realistic context.

Another limitation of our results is the assumption of the background geometry to be $M^4 \times S^M \times S^N$. To be fully consistent with the current ideas in string theories, we should be studying quantum effects in Calabi-Yau manifolds with points identified under some discrete symmetry group, i.e., Calabi-Yau manifolds with noncontractible loops. However, these are manifolds without isometries (and, as of yet, no known metric), so this task is at present intractable. We therefore study $M^4 \times S^M \times S^N$ as an interesting topologically-nontrivial manifold on which we *can* do the harmonic analysis needed for computing quantum effects. Other one-parameter manifolds can be studied with similar techniques.

We would also like to point out that the cosmological stability of manifolds with one or more internal spheres has been recently studied by one of us and collaborators [5]. It was shown that, for certain even-dimensional theories without a cosmological constant, it may be possible to obtain a stable configuration with a 4-dimensional Minkowski spacetime and a static internal space if monopole-like terms are balanced with 1-loop quantum effects. In particular, if gluino condensation is taken into account, a stable compactification seems possible for the $N = 1$ $D = 10$ supergravity model with $M^4 \times S^3 \times S^3$. The caveat is that, in the calculation of the 1-loop matter contribution to the effective potential, not only were the two internal radii taken to be identical (thus occupying only a line in their configuration space ) but also, and most importantly, the modifications coming from even dimensions were neglected. (That is, the form of the quantum potential valid for D odd was *assumed* applicable to D-even). Thus, we would like to consider the present calculation as a natural continuation of this previous paper. Once we obtain the general form for the Casimir energy for arbitrary values of the two internal radii in even dimensions, we can go back to the problem of studying the cosmological stability of the above compactification [11].

The paper is organized as follows. In section 2 we develop the general formalism to be used in the 1-loop calculation. In section 3 we calculate the zeta-function for scalars and fermions in the chosen background. Details of the calculation are given in the two appendices at the end. In section 4 we obtain the 1-loop potential for the particular case of two internal 3-spheres. The potential is exact if the respective radii of the two 3-spheres are equal, and correct to first order in the deviation from equality otherwise. In section 5 we find a static solution of the field equations which follow from this potential and show, by studying its stability, that this solution is a saddle point of the effective potential. We conclude in section 6 with general remarks and with a brief discussion of the cosmological relevance of our results.

## 2. General Formalism

The classical action for minimally coupled massless scalars and spin 1/2 fermions in a D-dimensional space-time is

$$S\left[\phi,\Psi\right]=\int d^{D}Z\sqrt{g^{D}}\left(-\frac{1}{2}\partial_{A}\phi\partial^{A}\phi+i\bar{\Psi}\hat{\gamma}^{A}\Psi_{;A}\right) \tag{1}$$

where $A=0,1,\ldots,D-1$ and the conventions for the metric and the Dirac matrices are those of reference [2], adapted to the case of two internal spheres.

As we mentioned before, the energy-momentum tensor will be produced exclusively by 1-loop quantum fluctuations in the matter fields. Accordingly, the effective action $\Gamma$ corresponding to (1) is, to first order in Planck's constant [3]

$$\Gamma\left[\phi,\Psi\right]=S\left[\phi_{c},\Psi_{c}\right]+\Gamma_{Q}\left[\phi_{c},\Psi_{c}\right], \tag{2}$$

with $\phi_c$ and $\Psi_c$ satisfying the appropriate classical equations of motion for $\phi$ and $\Psi$, and where $\Gamma_Q$ is defined as $-i$ times the logarithm of $Z_Q$ , the quantum part of the generating functional,

$$Z_Q = e^{i\Gamma_Q[\phi_c,\Psi_c]} = \int [d\phi]\,[d\bar{\Psi}]\,[d\Psi]\,e^{iS[\phi,\Psi]}. \tag{3}$$

It will prove convenient to define an operator $\hat{S}_2$ in terms of $S$ by the relation

$$iS[\Phi] = \frac{i}{2}\int d^D Z\sqrt{g^D}\,\Phi\hat{S}_2\Phi, \tag{4}$$

where for economy the field $\Phi$ represents both $\phi$ and $\Psi$ and $\hat{S}_2$ is a second-order differential operator (which in our case is simply the Laplacian on $M^4 \times S^M \times S^N$ ).

At this point we have the choice of working with Euclidean or Lorentzian signature. In our case, as we are not including quantum effects coming from the graviton, the differences will not be important. Nevertheless, as pointed out in ref.3, the two approaches may lead to different results once gravity is taken into account. The "Euclideanization" seems to be ambiguous and should be avoided. As gravitational quantum effects should eventually be considered, we will adopt the Lorentzian signature.

We refer the reader to ref.3 for a detailed discussion of the integration in (3). Here, we begin by sketching the bosonic case.

In order to perform the integral, we expand the fields in eigenfunctions of the operator $\hat{S}_2$ . The measure of the path integral will then be given by an infinite product of the coefficients of the expansion, which are dimensionless quantities. We thus introduce the constant $\mu$, with dimensions of mass, to restore the canonical dimensionality of the path integral measure. If $-\Lambda_j$ (a real number) represents the eigenvalues of $\hat{S}_2$ , the 1-loop generating functional is given by

$$Z_Q = \Pi_j\left(\bar{\mu}\Lambda_j^{-1/2}\right), \quad \bar{\mu} \equiv \mu e^{-i\pi/4}\sqrt{2\pi}. \tag{5}$$

Defining the 1-loop effective potential, $V_Q$, by

$$Z_Q = e^{-iV_Q} \tag{6}$$

we obtain

$$V_Q = i(ln\bar{\mu})\sum_j - i\frac{1}{2}\sum_j ln\Lambda_j. \tag{7}$$

In order to regularize the infinite sums in (7), we use the zeta-function method [12]. Define for, $Res \gg 0$ ,

$$\varsigma(s) \equiv \sum_j \Lambda_j^{-s}. \tag{8}$$

For sufficiently large s, $\varsigma(s)$ in (8) converges to an analytical function of s. The effective potential becomes

$$V_Q = i\left(ln\bar{\mu}\varsigma(0) + \frac{1}{2}\varsigma'(0)\right), \tag{9}$$

where $\varsigma(0)$ and $\varsigma'(0)$ are calculated by analytic continuation of (8).

Of course, the effective potential in (7) is a sum of the contributions from scalars and fermions. In what follows we will label these contributions by (0) and (1/2) respectively. Accordingly, we write eq.(9) as [2]

$$V_Q = bV_Q^{(0)} - 4fV_Q^{(1/2)} \tag{10}$$

with,

$$V_Q^{(i)} = i\left(ln\bar{\mu}\varsigma^{(i)}(0) + \frac{1}{2}\varsigma'^{(i)}(0)\right), \quad i = 0, 1/2 \tag{11}$$

and b and f the number of spin 0 and spin-1/2 fields, respectively. Two comments are now in order. The reader may wonder why we have used the same constant $\mu$ for the path integral measure of both scalars and fermions. In principle we should not do so but it turns out that $\varsigma^{(1/2)}(0) = 0$, as we will see later, so this is not a problem. (In an odd number of spacetime dimensions, $\varsigma^{(0)}(0) = 0$ as well [7], reflecting our earlier comment on the absence of counterterms with even number of derivatives in this case.)

We are interested in studying static configurations of the background geometry $M^4 \times S^M \times S^N$ with the vacuum expectation value of the energy-momentum tensor given by the 1-loop matter fluctuations described above. By using the fact that this product space is maximally symmetric and writing the Ricci tensor for the spheres as

$$R_{ij} = -\frac{(M-1)}{\rho_M^2} g_{ij}, \quad R_{mn} = -\frac{(N-1)}{\rho_N^2} g_{mn}$$

where $\rho_M(\rho_N)$ is the radius of the M(N)-sphere, Einstein's equations become simply [2,9]

$$\frac{1}{2}\left[\frac{M(M-1)}{\rho_M^2} + \frac{N(N-1)}{\rho_N^2}\right] = 8\pi G_D \frac{V_Q}{V_4 \Omega_M \Omega_N} \tag{12.1}$$

$$-\frac{(M-1)}{\rho_M^2} + \frac{1}{2}\left[\frac{M(M-1)}{\rho_M^2} + \frac{N(N-1)}{\rho_N^2}\right] = 8\pi G_D \frac{1}{M V_4 \Omega_M \Omega_N} \rho_M \frac{\partial V_Q}{\partial \rho_M} \tag{12.2}$$

$$-\frac{(N-1)}{\rho_N^2} + \frac{1}{2}\left[\frac{M(M-1)}{\rho_M^2} + \frac{N(N-1)}{\rho_N^2}\right] = 8\pi G_D \frac{1}{N V_4 \Omega_M \Omega_N} \rho_N \frac{\partial V_Q}{\partial \rho_N}. \tag{12.3}$$

We have thus 3 equations for 3 unknowns, $\rho_M, \rho_N$ and $\mu$. Once we know the functional form of $V_Q$ (which, contrary to the case with one internal sphere, is now a complicated function of the ratio $\frac{\rho_M}{\rho_N}$), it is possible in principle to obtain a solution of eq.(12) for constant values $\rho_{M0}, \rho_{N0}$ of the internal radii. In particular, we will use eq.(12.1) to fix the value of $\bar{\mu}$ in terms of $\rho_{M0}$ and $\rho_{N0}$.

## 3. Zeta-function for Scalars and Fermions

In order to obtain the 1-loop effective potential (eq.10) we have to calculate the zeta-functions $\varsigma^{(0)}(s)$ and $\varsigma^{(\frac{1}{2})}(s)$ .

Taking into account the degeneracies of the Laplacian operator for scalars on spheres [2], the zeta-function $\varsigma^{(0)}(s)$ is given by

$$\varsigma^{(0)}(s) = \sum_{k=-\infty}^{\infty} \sum_{t=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma(t+2m)\Gamma(\ell+2n)}{\Gamma(2m+1)\Gamma(2n+1)} \frac{(2t+2m)(2\ell+2n)}{t!\ell!} \times \left[\frac{t(t+2m)}{\rho_M^2} + \frac{\ell(\ell+2n)}{\rho_N^2} + k^2\right]^{-s} \tag{13}$$

where $m = \frac{M-1}{2}$ and $n = \frac{N-1}{2}$ . As usual, the summation over $k$ can be replaced by an integral over $d^4k$ , $\sum_k \to \frac{V_4}{(2\pi)^4} \int d^4k$ . This integration can be easily carried out, and we get,

$$\varsigma^{(0)}(s) = i\frac{V_4}{16\pi^2} \left(\frac{1}{s^2-3s+2}\right) \varsigma^{(0)}(\hat{s}), \;\; \hat{s} \equiv s-2 \tag{14}$$

with

$$\varsigma^{(0)}(\hat{s}) = \sum_{t=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma(t+2m)\Gamma(\ell+2n)}{\Gamma(2m+1)\Gamma(2n+1)} \frac{(2t+2m)(2\ell+2n)}{t!\,\ell!} \left[\frac{t(t+2m)}{\rho_M^2} + \frac{\ell(\ell+2n)}{\rho_N^2}\right]^{-\hat{s}} .(15)$$

The effective potential for scalars can be written in terms of $\varsigma^{(0)}(\hat{s})$ as,

$$V_Q^{(0)} = \frac{V_4}{32\pi^2} \left[\frac{1}{2}\varsigma'^{(0)}(-2) + (\frac{3}{4} + ln\bar{\mu})\varsigma^{(0)}(-2)\right] .(16)$$

In order to compute $\varsigma^{(0)}(\hat{s})$ , we closely follow the method developed by Kikkawa et al. [9], which is described in appendix A of their paper. There, the calculations have been done for odd-dimensional space-times. In appendix A of our paper we discuss in some detail the modifications one has to perform in order to calculate $\varsigma^{(0)}(\hat{s})$ in even dimensions, when both $M$ and $N$ are odd. (For $M$ and $N$ both even the method is very similar.) The final expression for $\varsigma^{(0)}(\hat{s})$ is given in eq.(A.24). We notice that, as one expects, $\varsigma^{(0)}(-2)$ is nonvanishing in our case. In fact, the last term in eq.(A.24) is the only one that gives a finite contribution for $\hat{s} = -2$ . This term is proportional to

$$sin(\pi \hat{s}) \frac{\Gamma(a-p-\frac{3}{2})}{\Gamma(a)}, \quad a \equiv \hat{s} - r - \frac{3}{2} \tag{17}$$

which is finite for $\hat{s} = -2$ since $\Gamma(a-p-\frac{3}{2})$ behaves like $\frac{1}{(\hat{s}+2)}$ for $\hat{s} \to -2$ .

Thus, for $\varsigma^{(0)}(-2)$ we get

$$\varsigma^{(0)}(-2) = \frac{-1^{(m+n)}}{2\rho_M^4} \sum_{r=0}^{m-1} a_{mr} \left(\frac{\rho_M}{\rho_N}\right)^{2r+7} \frac{1}{\Gamma(-r-\frac{1}{2})} \sum_{p=0}^{n-1} C_{np} (C^2)^{-r-2+p} \times$$
$$\Gamma(p+\frac{3}{2}) \frac{\pi}{(5+r+p)!} (-1)^{(p+1)} \tag{18}$$

In a similar way, we can calculate the zeta-function for spin-$\frac{1}{2}$ fermions. Taking into account the degeneracies of the Laplacian operator for Dirac fields on spheres [2], $\varsigma^{(\frac{1}{2})}(s)$ is given by

$$\varsigma^{(\frac{1}{2})}(s) = \frac{V_4}{(2\pi)^4} \int d^4k \sum_{t=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{\frac{(M+N+2)}{2}} \frac{\Gamma(t+M)\Gamma(\ell+N)}{\Gamma(M)\Gamma(N) t! \ell!} \times$$
$$\left[ \frac{(t+\frac{M}{2})^2}{\rho_M^2} + \frac{(\ell+\frac{N}{2})^2}{\rho_N^2} + k^2 \right]^{-s} \tag{19}$$

where we have already replaced the $k$ -summation by an integral. The integration can again be performed to give

$$\varsigma^{(\frac{1}{2})}(s) = i \frac{V_4}{16\pi^2} \left( \frac{2^{\frac{(M+N+2)}{2}}}{s^2 - 3s + 2} \right) \varsigma^{(\frac{1}{2})}(\hat{s}) \quad , \hat{s} \equiv s - 2 \tag{20.1}$$

with

$$\varsigma^{(\frac{1}{2})}(\hat{s}) = \sum_{t=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\Gamma(t+M)\Gamma(\ell+N)}{\Gamma(M)\Gamma(N)t!\ell!}\left[\frac{(t+\frac{M}{2})^2}{\rho_M^2} + \frac{(\ell+\frac{N}{2})^2}{\rho_N^2}\right]^{-\hat{s}} . \tag{20.2}$$

The calculation of $\varsigma^{(\frac{1}{2})}(\hat{s})$ is slightly different from the bosonic case. In appendix B we develop it in detail for the particular case of $M$ and $N$ both odd. There we find that $\varsigma^{(\frac{1}{2})}(-2)$ vanishes. The contribution of the fermions to the effective potential is then

$$V_Q^{(\frac{1}{2})} = \frac{V_4}{64\pi^2} 2^{\frac{(M+N+2)}{2}} \varsigma'^{(\frac{1}{2})}(-2), \tag{21}$$

where $\varsigma^{(\frac{1}{2})}(-2)$ is given in eq.(B.11).

Due to the complexity of the expressions for $\varsigma^{(0)}(-2)$ (and $\varsigma'^{(0)}(-2)$ ) and $\varsigma'^{(\frac{1}{2})}(-2)$ , very little can be said, at this level, of the typical properties of the effective potential. (We refer the reader to the paper by Kikkawa et al. for an analysis of the asymptotic properties of $V_Q$ in odd-dimensional spacetimes.) Instead of integrating all the expressions numerically and then constructing tables for various possible products of spheres, we will analyze in detail the particular case $M^4 \times S^3 \times S^3$ . As mentioned in the introduction, our interest in this background comes from its possible relevance for the stability of certain cosmological models [11].

## 4. The effective potential for $M^4 \times S^3 \times S^3$ at one loop level

We start by obtaining the zeta-fuction and its derivative for the scalar fields. By taking $M = N = 3$ we find, from the definitions in appendix A with $m = n = 1$ ,

$$C^2 = 1 + \left(\frac{\rho_N}{\rho_M}\right)^2 \quad , \quad a_{mr} = a_{10} = 1 \quad , C_{np} = C_{10} = 1.$$

Putting this results into eq.(18) we get,

$$\zeta^{(0)}_{3\times 3}(-2) = \frac{\pi}{5!8}\frac{\rho_M^3}{\rho_N^7}\left[1 + \left(\frac{\rho_N}{\rho_M}\right)^2\right]^5 \quad . \tag{22}$$

In order to calculate the derivative of the zeta-function, we first notice that the term with $\left[\frac{sin(\pi\hat{s})}{\Gamma(\hat{s})}\right]$ in eq.(A.24) will not contribute since its derivative with respect to $\hat{s}$ goes to zero with $\hat{s} \to -2$ . Inspection of this equation shows that only the last term will have a pole interfering with $sin(\pi\hat{s})$ .

We can thus write, after careful differentiation of this term

$$\begin{aligned}
\zeta'^{(0)}_{3\times 3}(-2) = &- \frac{\pi}{\rho_M^4}\sum_{L=1}^{L_0} L^2\left[\int_0^\infty dx\; x^2(x^2 + A_L^2)^2\frac{2}{e^{2\pi x}-1} + P\int_0^{A_L} dx\; x^2(A_L^2 - x^2)^2 cot(\pi x)\right] + \\
&- \frac{\pi}{\rho_M^4}\sum_{L=L_0+1}^{\infty} L^2\int_{B_L}^\infty dx\; x^2(x^2 - B_L^2)\frac{2}{e^{2\pi x}-1} + \frac{8\pi}{105}\frac{\rho_M^3}{\rho_N^7}\times \\
&\times\left[-P\int_0^C dx\; cot(\pi x)x^2\;(C^2 - x^2)^{\frac{7}{2}} - \int_0^\infty dx\; x^2\;(x^2 + C^2)^{\frac{7}{2}}\frac{2}{e^{2\pi x}-1}\right] + \\
&+ \frac{\pi}{5!16}C^{10}\frac{\rho_M^3}{\rho_N^7}\left[\frac{47}{30} + 2ln\left(\frac{\rho_N}{C}\right)^2\right].
\end{aligned} \tag{23}$$

For the fermionic case, $\zeta'^{(\frac{1}{2})}_{3\times 3}(-2)$ can be easily obtained from eq.(B.11). By noticing that the only nonvanishing coefficients of the degeneracy factors are

$$A_{30} = -\frac{1}{8}\; , A_{31} = \frac{1}{2}$$

$$C_{30} = \frac{1}{8}\ , C_{31} = \frac{1}{2}$$

we obtain,

$$\begin{aligned}
\varsigma'^{(\frac{1}{2})}_{3\times 3}(-2) =& \frac{\pi}{\rho_M^4}\sum_{L=\frac{5}{2}}^{\infty}\frac{1}{2}(L^2-\frac{1}{4})\int_{B_L}^{\infty}dx\ (x^2+\frac{1}{4})(x^2-B_L^2)^2\frac{1}{e^{2\pi x}+1}+\\
&+\frac{\pi}{\rho_M^4}\Big[\frac{1}{15}\left(\frac{\rho_M}{\rho_N}\right)^5\left(-\frac{1}{8}\tilde{\varsigma}(-5,\frac{3}{2})+\frac{1}{2}\tilde{\varsigma}(-7,\frac{3}{2})\right)+\\
&+\frac{4}{105}\left(\frac{\rho_M}{\rho_N}\right)^7\left(-\frac{1}{8}\tilde{\varsigma}(-7,\frac{3}{2})+\frac{1}{2}\tilde{\varsigma}(-9,\frac{3}{2})\right)\Big]
\end{aligned}\tag{24}$$

where

$$\tilde{\varsigma}(-a,\frac{3}{2}) = \frac{1}{2}(\frac{3}{2})^a - \frac{(\frac{3}{2})^{a+1}}{a+1} - 2\int_0^{\infty}\frac{sin\left[a\ arctg(\frac{2t}{3})\right]\left[t^2+(\frac{3}{2})^2\right]^{\frac{a}{2}}}{e^{2\pi t}-1}dt. \tag{25}$$

Using equations (22)-(24) in equations (16) and (21), we finally obtain an expression for the 1-loop effective potential for the particular geometry $M^4\times S^3\times S^3$ .

We could now proceed by integrating numerically all terms appearing in the derivatives of the $\varsigma^{(i)}(-2)$ in order to obtain a final expression for $V_Q$ , as it is usually done in the literature. Instead, we will focus our attention on small deviations from the case where the two internal radii are the same. The motivations to do so are twofold; first, by inspecting the numerical results obtained by Kikkawa et al. [9], we can see that the equilibrium values for the radii are not for $\rho_{M0}=\rho_{N0}$ , but that $\frac{\rho_{M0}}{\rho_{N0}}$ (in their paper defined as $\hat{k}$ ) typically deviates from unity by a factor not bigger than 30% , at least for the lower dimensional cases. Second, in order to study the cosmological stability of the compactified solution, it is extremely convenient to reexpress Einstein's equations in terms of the small perturbation to analyze its time evolution. Later on we will see that our approximation is reasonable.

Accordingly, we define $\varepsilon$ , the deviation from equality of the radii ,as

$$\frac{\rho_N}{\rho_M} \equiv 1 + \varepsilon \ , \tag{26}$$

and retain, in what follows, terms of no higher than *linear* degree in $\varepsilon$ . (In this approximation, the variable $L_0$ introduced in appendix A is $L_0 \simeq \sqrt{2}(1 + \frac{\varepsilon}{2})$ and the closest integer to it is then 1.) The effective potential will be a function of $\bar{\mu}, \rho_M$ and $\varepsilon$ .

As an example we obtain $\varsigma^{(0)}(-2)$ given by eq.(22),

$$\varsigma^{(0)}(-2) \simeq \frac{\pi}{5!8}(1 - 7\varepsilon)\frac{2^5}{\rho_M^4}(1 + 5\varepsilon) = \frac{1.047 \times 10^{-1} - 2.094 \times 10^{-1}\varepsilon}{\rho_M^4} \ , \tag{27}$$

where we have used $(\frac{\rho_N}{\rho_M})^n \simeq 1 + n\varepsilon$ .

For the calculation of $\varsigma'^{(0)}(-2)$ we note that most integrals can be performed analytically, or at least can be expressed in terms of simple functions that can be evaluated numerically. After a tedious calculation we find,

$$\varsigma'^{(0)}(-2) = \frac{1}{\rho_M^4}\left[0.0729 + 0.2096 ln\rho_M - \varepsilon(0.0681 + 0.4192 ln\rho_M)\right] \ . \tag{28}$$

Similarly, for the fermionic case we obtain

$$\varsigma'^{(\frac{1}{2})}(-2) = \frac{-1}{\rho_M^4}(3.610 \times 10^{-5} - \varepsilon 5.344 \times 10^{-4} \ ). \tag{29}$$

Using equations (27)-(29) we obtain the expression for the 1-loop effective potential,

$$V_Q = \frac{V_4}{\rho_M^4}[b(3.639 \times 10^{-4} - \varepsilon 6.053 \times 10^{-4} + 3.315 \times 10^{-4}(1 - 2\varepsilon) ln(\rho_M \bar{\mu})) +$$
$$+ f(3.657 \times 10^{-6} - \varepsilon 5.414 \times 10^{-5})]. \tag{30}$$

(This potential does not exhibit symmetry under interchanging the respective radii of the 3-spheres, due to our choice of an asymmetric approximation scheme, equation (26)).

## 5.Approximate Solution and Stability Analysis of the Effective Potential

In this section we will insert the 1-loop quantum potential into the right hand side of Einstein's equations (12.1-3) in order to solve for the critical values of $\bar{\mu}, \rho_{M0}$,and $\varepsilon_0$ .These solutions are equivalent to imposing the following conditions on the total effective potential $V_{eff}$ [2],

$$V_{eff}(\bar{\mu}, \rho_{M0}, \varepsilon_0) = 0 \tag{31.1}$$

$$\left.\frac{\partial V_{eff}}{\partial \rho_M}\right|_{\rho_{M0}} = 0 \quad , \left.\frac{\partial V_{eff}}{\partial \varepsilon}\right|_{\varepsilon_0} = 0 \qquad .(31.2)$$

Once we obtain the critical point we will discuss the stability of $V_{eff}$ by evaluating its second derivatives.

For the $M^4 \times S^3 \times S^3$ case we can write Einstein's equations, within our approximations, as,

$$\frac{6(1-\varepsilon)}{\rho_M^2} = \frac{8\pi G_4}{V_4} V_Q \tag{32.1}$$

$$\frac{6(2-3\varepsilon)}{\rho_M^2} = \frac{8\pi G_4}{V_4} \rho_M \frac{\partial V_Q}{\partial \rho_M} \tag{32.2}$$

$$\frac{6(2-3\varepsilon)}{\rho_M^2} = \frac{8\pi G_4}{V_4} \frac{\partial V_Q}{\partial \varepsilon} \quad , \tag{32.3}$$

where we used that $G_D = G_4 \Omega_M \Omega_N$ , and $V_Q$ is given by eq.(30).

The procedure now is very simple. We use eq.(32.1) to solve for $ln(\rho_{M0}\bar{\mu})$ in terms of $\rho_{M0}$ and $\varepsilon_0$ and substitute this expression into the two remaining equations to find a solution for $\rho_{M0}$ and $\varepsilon_0$ parametrized only by the number of matter fields. After some algebra we obtain

$$\varepsilon_0 = 1.447 \times 10^{-1} + 6.871 \times 10^{-2} \frac{f}{b} \tag{33.1}$$

$$\frac{\rho_{M0}^2}{8\pi G_4} = b10^{-6}(8.14 - 5.26 \times 10^{-1}\frac{f}{b}) \tag{33.2}$$

In particular, if we take $b \sim f \sim 10^4$ , we find

$$\rho_{M0} \simeq 1.428 L_P \quad , \tag{34}$$

where $L_P$ is the Planck length.

Thus, for a sufficient number of matter fields, the internal scale is sufficiently large to justify the use of the 1-loop approximation [1]. Note also that $\varepsilon_0 \sim 0.15$ , reflecting the fact that the critical value for $V_{eff}$ has both internal radii differing by only 15% ; so the approximation of working only to first order in $\varepsilon$ is also reasonable.

Next we obtain the condition necessary to test the stability of the effective potential $V_{eff}$ . From eq.(32.1) $V_{eff}$ is given by,

$$V_{eff} = -\frac{3}{8\pi G_4}\left(\frac{1}{\rho_M^2} + \frac{1}{\rho_N^2}\right) + \frac{V_Q}{V_4} \tag{35}$$

Note that we have restored $\rho_N$ to the definition of $V_{eff}$ . One has to be careful with the differentiation of this expression since $G_4$ depends on the internal radii.

The nature of the critical point is determined by the expression,

$$I = \frac{\partial^2 V_{eff}}{\partial \rho_M^2}\frac{\partial^2 V_{eff}}{\partial \rho_N^2} - \frac{\partial^2 V_{eff}}{\partial \rho_M \partial \rho_N}\frac{\partial^2 V_{eff}}{\partial \rho_N \partial \rho_M} \quad , \tag{36}$$

evaluated at $\rho_{M0}$ , $\varepsilon_0$ . If $I > 0$ and $\frac{\partial^2 V_{eff}}{\partial \rho_M^2} > 0$ the critical point is a relative minimum, if $I > 0$ and $\frac{\partial^2 V_{eff}}{\partial \rho_M^2} < 0$ it is a relative maximum; if $I < 0$ it is a saddle point; and, for $I = 0$ , nothing can be said using the determinant $I$ alone.

Taking the derivatives of $V_{eff}$ in eq.(35) and using the values of $\rho_{M0}$ *and* $\varepsilon_0$ from eq.(33) with $b \sim f \sim 10^4$ we find,

$$I = (8\pi G_4)^{-3}(-9.23 \times 10^3). \tag{37}$$

Thus, for $10^4$ matter fields, the critical point is a saddle point of the effective potential. The number $10^4$ may be seen as *ad hoc* but it must be realized that this number is about the smallest possible choice for which the 1-loop approximation is valid.

### 6. Conclusion

We have obtained the 1-loop potential that arises from quantum fluctuations of scalars and spin-$\frac{1}{2}$ fermions in the background of the even- dimensional product manifold $M^4 \times S^M \times S^N$. In particular, we have looked for a static configuration with the internal space given by the product of two 3-spheres. Working to first order in the deviation from equality of the two radii, we found that a solution exists for a sufficiently large number of matter fields that has the internal radii differing by 15% , but that this solution is a saddle point of the effective potential. Thus, quantum effects alone may be insufficient to balance the internal space: In the light of previous work on this subject, this conclusion would not be a very surprising one to reach.

Nevertheless, we would like to emphasize that other effects may also play an important role in dictating the dynamics of higher dimensional theories. For example, vacuum expectation values of antisymmetric tensor fields are a necessary ingredient for the compactification of supersymmetric theories and will also give a contribution to the effective potential. So, although we may perhaps have to abandon the idea that quantum effects alone are sufficient to stabilize the internal manifold, we must also realise that this approach will be modified in more realistic calculations; the full potential to be stabilized is certainly much more complex.

Some steps have already been taken to try to include other contributions to the effective potential (see refs.5 and 6 and references therein). This is necessary if we want to study the cosmological evolution of the compactified spacetime to see if it is possible to reproduce the known features of our apparently four-dimensional universe starting from a higher-dimensional theory. For example, we must obtain solutions that exhibit a Friedmann-like behaviour in the four-dimensional physical spacetime while the geometry of the internal space at late epoch is constant, or nearly so.

Thus, obtaining the correct quantum potential is an important step towards the proper understanding of compactified theories, both from a microphysical and from a cosmological point of view.

We would like to thank Dr. K. Kikkawa for useful comments and suggestions. Two of us, (M. G. and P. J.) would like to thank Prof. A. Salam and the ICTP, where part of this work was done, for their kind hospitality. P. J. would like to thank Dr. E. Kolb and the Theoretical Astrophysics Group at Fermilab for their warm hospitality during the completion of most of this work. He would also like to thank Prof. H. Ruegg for his support and A. Malaspinas for helping with the numerical calculations. M. A. R. would like to thank J. Marsie-Hazen for a helpful discussion.

This work was supported in part by the Department of Energy and the National Aeronautics and Space Administration. P. J. is supported by a grant from the Swiss National Science Foundation. M. A. R. is supported in part under the Department of Energy contract grant number DE-AC02-81ER40033B.000.

**Appendix A**

In this appendix we calculate $\varsigma^{(0)}(\hat{s})$ for a total even number of dimensions, when both M and N are odd.

We follow the method and the notations of Kikkawa et al. [9], and will focus on the differences that arise in even dimensions. In order to get a finite value for $\varsigma^{(0)}(\hat{s})$ , one has to perform an analytic continuation with respect to $\hat{s}$ . We write eq.(15) as follows, using the running variables $T = t + m$ and $L = \ell + n$ :

$$\varsigma^{(0)}(\hat{s}) = \sum_{L=n}^{\infty} D_N^{(0)}(L) \sum_{T=m}^{\infty} D_M^{(0)}(T) \left[ \frac{\Lambda_N^{(0)}(L)}{\rho_N^2} + \frac{\Lambda_M^{(0)}(T)}{\rho_M^2} \right]^{-\hat{s}} , \tag{A.1}$$

with

$$D_M^{(0)}(T) = \frac{2T^2}{(2m)!}[T^2 - (m-1)^2]\dots[T^2 - 1^2] \tag{A.2}$$

$$D_N^{(0)}(L) = \frac{2L^2}{(2n)!}[L^2 - (n-1)^2]\dots[L^2 - 1^2] \tag{A.3}$$

$$\Lambda_M^{(0)}(T) = T^2 - m^2 \tag{A.4}$$

$$\Lambda_N^{(0)}(L) = L^2 - n^2 \tag{A.5}$$

The infinite sum over $T$ can be replaced by an integral representation,

$$\begin{aligned} \varsigma^{(0)}(\hat{s}) = \sum_{L=n}^{\infty} D_N^{(0)}(L)(\rho_M)^{2\hat{s}}\left(\frac{-1}{2i}\right) \int_{C_1} dz \, cot(\pi z) D_M^{(0)}(z) \\ \times \left[ z^2 + \left(\frac{\rho_M}{\rho_N}\right)^2 \Lambda_N^{(0)}(L) - m^2 \right]^{-\hat{s}} \end{aligned} \tag{A.6}$$

where the contour $C_1$ in the $z$ -plane is indicated in fig.1. The singularities in $z$ of the integrand depend on $L$ . For this reason it turns out to be convenient to divide the sum over $L$ into two parts,

$$\varsigma^{(0)}(\hat{s}) = Z^{(0)}(\hat{s}) + W^{(0)}(\hat{s}) \tag{A.7}$$

where

$$Z^{(0)}(\hat{s}) = \sum_{L=n}^{L_0} D_N^{(0)}(L)(\rho_M)^{2\hat{s}}(\frac{-1}{2i}) \int_{C_1} dz\ cot(\pi z) D_M^{(0)}(z)(z^2 - A_L^2)^{-\hat{s}} \tag{A.8}$$

$$W^{(0)}(\hat{s}) = \sum_{L=L_O+1}^{\infty} D_N^{(0)}(L)(\rho_M)^{2\hat{s}}(\frac{-1}{2i}) \int_{C_1} dz\ cot(\pi z) D_M^{(0)}(z)(z^2 + B_L^2)^{-\hat{s}} \tag{A.9}$$

with

$$A_L^2 = m^2 - (\frac{\rho_M}{\rho_N})^2 \Lambda_N^{(0)}(L) \tag{A.10}$$

$$B_L^2 = -A_L^2 \tag{A.11}$$

which are positive for $n \leq L < L_0$ and $L_0 \leq L$ respectively. $L_0$ is the largest integer smaller than or equal to

$$[n^2 + m^2(\frac{\rho_N}{\rho_M})^2]^{\frac{1}{2}} \quad .$$

The result for $Z^{(0)}(\hat{s})$ is identical to the one in ref.[9],

$$\begin{aligned} Z^{(0)}(\hat{s}) =& (\rho_M)^{2\hat{s}} \sum_{L=n}^{L_0} D_N^{(0)}(L) [\frac{sin(\pi\hat{s})}{\Gamma(\hat{s})}(-1)^m \frac{1}{2} \times \\ & \sum_{k=0}^{m-1} a_{mk}(A_L^2)^{(k+\frac{3}{2}-\hat{s})}\Gamma(k+\frac{3}{2})\Gamma(\hat{s}-k-\frac{3}{2}) + \\ & sin(\pi\hat{s}) \int_0^\infty dx\ D_M^{(0)}(ix)(x^2 + A_L^2)^{-\hat{s}} \frac{2}{e^{2\pi x}-1} + \\ & - sin(\pi\hat{s}) P \int_0^{A_L} dx\ D_M^{(0)}(x)(A_L^2 - x^2)^{-\hat{s}} cot(\pi x)] \end{aligned} \tag{A.12}$$

where $P$ means the principal value prescription and

$$\begin{aligned} D_M^{(0}(ix) =& (-1)^m \frac{2x^2}{(2m)!}[x^2 + (m-1)^2] \dots (x^2 + 1^2) \\ \equiv& (-1)^m \sum_{k=0}^{m-1} a_{mk} x^{2k+2} \end{aligned} \tag{A.13}$$

The coefficients $a_{mk}$ are defined through eq.(A.13).

We consider now the second term of eq.(A.7). One can again perform an analytic continuation with respect to $\hat{s}$ . The branch points are now on the imaginary axis, i.e., $z = \pm i B_L$ . The contour $C_1$ is replaced by $C_2$ as in fig.2. We obtain,

$$W^{(0)}(\hat{s}) = \sum_{L=L_0+1}^{\infty} D_N^{(0)}(L)(\rho_M)^{2\hat{s}}(\frac{-1}{2i})[I_L^{(1)}(\hat{s}) + I_L^{(2)}(\hat{s})] \tag{A.14}$$

with

$$\begin{aligned} I_L^{(1)}(\hat{s}) =& e^{-i\pi\hat{s}} \int_0^{\infty} dx\ D_M^{(0)}(ix + \Delta)(x + [-i\Delta - B_L])^{-\hat{s}}(x + [-i\Delta + B_L])^{-\hat{s}} + \\ & - e^{i\pi\hat{s}} \int_0^{\infty} dx\ D_M^{(0)}(ix - \Delta)(x + [i\Delta - B_L])^{-\hat{s}}(x + [-i\Delta + B_L])^{-\hat{s}} \end{aligned} \tag{A.15}$$

$$\begin{aligned} I_L^{(2)}(\hat{s}) =& e^{-i\pi\hat{s}} \int_{B_L}^{\infty} dx\ (\frac{2}{e^{2\pi(x-i\Delta)} - 1}) D_M^{(0)}(ix + \Delta)(x + [-i\Delta - B_L])^{-\hat{s}}(x + [-i\Delta + B_L])^{-\hat{s}} + \\ & - e^{i\pi\hat{s}} \int_{B_L}^{\infty} dx\ (\frac{2}{e^{2\pi(x+i\Delta)} - 1}) D_M^{(0)}(ix - \Delta)(x + [i\Delta - B_L])^{-\hat{s}} \times \\ & \times (x + [i\Delta + B_L])^{-\hat{s}} \end{aligned} \tag{A.16}$$

In $I_L^{(2)}(\hat{s})$ the contribution from $-B_L$ to $B_L$ to the integral cancels because the integrand is an odd function. The limit $\Delta \to 0$ can be easily carried out in eq.(A.16) and gives

$$I_L^{(2)}(\hat{s}) = -2isin(\pi\hat{s}) \int_{B_L}^{\infty} dx\ \frac{2}{e^{2\pi x} - 1} D_M^{(0)}(ix)(x^2 - B_L^2)^{-\hat{s}}. \tag{A.17}$$

To take $\Delta \to 0$ in eq.(A.15) is more involved; one has to replace $D_M^{(0)}(ix \pm \Delta)$ using eq.(A.13), carry out the integration and at the end take $\Delta \to 0$ . Doing it this way, one finds a domain for $\hat{s}$ in which $I_L^{(1)}$ is well defined, which allows us to perform the analytic continuation. We obtain,

$$I_L^{(1)}(\hat{s}) = -2isin(\pi\hat{s})\Gamma(-\hat{s}+1)(-1)^m \frac{1}{2} \sum_{r=0}^{m-1} a_{mr}(B_L^2)^{(r-\hat{s}+\frac{3}{2})} \frac{\Gamma(\hat{s} - r - \frac{3}{2})}{\Gamma(-r - \frac{1}{2})} \quad . \tag{A.18}$$

In eq.A.(14) an infinite sum over $L$ remains; as far as the second term $I_L^{(2)}(\hat{s})$ is concerned, the infinite sum over $L$ is convergent since the integral in A.(17) decreases exponentially as $L$ goes to infinity. For the term containing $I_L^{(1)}(\hat{s})$ it is still necessary to regularize the $L$ summation,

$$\sum_{L=L_0+1}^{\infty} D_N^{(0)}(L)(B_L^2)^{r-\hat{s}+\frac{3}{2}} = \left(\frac{\rho_M}{\rho_N}\right)^{2r-2\hat{s}+3} \sum_{L=L_0+1}^{\infty} D_N^{(0)}(L)[L^2 - C^2]^{-a} \qquad (A.19)$$

where

$$C^2 = n^2 + \left(\frac{\rho_N}{\rho_M}\right)^2 m^2 \quad , \quad a = \hat{s} - r - \frac{3}{2} \qquad (A.20)$$

Again the infinite sum is replaced by an integral representation,

$$\begin{aligned} S &\equiv \sum_{L=L_0+1}^{\infty} D_N^{(0)}(L)[L^2 - C^2]^{-a} \\ &= \frac{-1}{2i}\int_{C_3} dz\ D_N^{(0)}(z)(z^2 - C^2)^{-a} cot(\pi z) \end{aligned} \qquad (A.21)$$

We note that in eq.(A.21) we had to use the $cot(\pi z)$ since, for $N$ odd $L$ is an integer, contrary to the case with $N$ even analysed in ref.9. Using

$$D_N^{(0)}(ix) = (-1)^n \sum_{p=0}^{n-1} C_{np} x^{2p+2} \qquad (A.22)$$

and by displacing the contour $C_3$ to $C_4$ as in fig.3, we obtain

$$\begin{aligned} S = &- cos(\pi a) \sum_{L=n}^{L_0} D_N^{(0)}(L)(C^2 - L^2)^{-a} + \\ &- sin(\pi a) P\int_0^C dx\ D_N^{(0)}(x)(C^2 - x^2)^{-a} cot(\pi x) + \\ &+ sin(\pi a)\int_0^{\infty} dx\ D_N^{(0)}(ix)(x^2 + C^2)^{-a} \frac{2}{e^{2\pi x} - 1} + \\ &+ sin(\pi a)(-1)^n \frac{1}{2}\sum_{p=0}^{n-1} C_{np}(C^2)^{(-a+p+\frac{3}{2})} \frac{\Gamma(p+\frac{3}{2})\Gamma(a-p-\frac{3}{2})}{\Gamma(a)} \end{aligned} \qquad (A.23)$$

Putting all results together, we finally obtain the expression for the regularized zeta-function, which is well defined in the region $s \leq 0$ ,

$$
\begin{aligned}
\varsigma^{(0)}(\hat{s}) =&(\rho_M)^{2\hat{s}} \sum_{L=n}^{L_0} D_N^{(0)}(L)\big[sin(\pi\hat{s})\int_0^\infty dx\; D_M^{(0)}(ix)(x^2+A_L^2)^{-\hat{s}}\frac{2}{e^{2\pi x}-1}+\\
&- sin(\pi\hat{s})P\int_0^{A_L} dx\; D_M^{(0)}(x)(A_L^2-x^2)^{-\hat{s}}cot(\pi x)+\\
&+\frac{sin(\pi\hat{s})}{\Gamma(\hat{s})}(-1)^m\frac{1}{2}\sum_{k=0}^{m-1} a_{mk}(A_L^2)^{k+\frac{3}{2}-\hat{s}}\Gamma(k+\frac{3}{2})\Gamma(\hat{s}-k-\frac{3}{2})\big]+\\
&+(\rho_M)^{2\hat{s}}\sum_{L=L_0+1}^{\infty} D_N^{(0)}(L)\big[sin(\pi\hat{s})\int_{B_L}^\infty dx\; D_M^{(0)}(ix)(x^2-B_L^2)^{-\hat{s}}\frac{2}{e^{2\pi x}-1}\big]+\\
&+sin(\pi\hat{s})\Gamma(-\hat{s}+1)(\rho_M)^{2\hat{s}}(-1)^m\frac{1}{2}\sum_{r=0}^{m-1} a_{mr}(\frac{\rho_M}{\rho_N})^{(2r-2\hat{s}+3)}\times\\
&\times\frac{\Gamma(\hat{s}-r-\frac{3}{2})}{\Gamma(-r-\frac{1}{2})}\times S
\end{aligned}
\qquad (A.24)
$$

where $S$ is given in eq.(A.23).

**Appendix B**

The calculation of $\varsigma^{(\frac{1}{2})}(\hat{s})$ (eq.(19)) follows closely the one for $\varsigma^{(0)}(\hat{s})$ given in Appendix A. We rewrite eq.(19) in the following way,

$$
\varsigma^{(\frac{1}{2})}(\hat{s}) = \sum_{L=\frac{N}{2}}^{\infty}\sum_{T=\frac{M}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)D_M^{(\frac{1}{2})}(T)\left[\frac{T^2}{\rho_M^2}+\frac{L^2}{\rho_N^2}\right]^{-\hat{s}} \qquad (B.1)
$$

where we have put $L=\ell+\frac{N}{2}$ and $T=t+\frac{M}{2}$ ($L$ and $T$ are half-integer), and defined

$$
D_N^{(\frac{1}{2})}(L) = \frac{\Gamma(L+\frac{N}{2})}{\Gamma(N)(L-\frac{N}{2})!} = \frac{[L^2-(\frac{N}{2}-1)^2]\dots[L^2-(\frac{1}{2})^2]}{(N-1)!} \qquad (B.2)
$$

$$
D_M^{(\frac{1}{2})}(T) = \frac{\Gamma(T+\frac{M}{2})}{\Gamma(M)(T-\frac{M}{2})!} = \frac{[T^2-(\frac{M}{2}-1)^2]\dots[T^2-(\frac{1}{2})^2]}{(M-1)!} \qquad (B.3)
$$

Both $D_N^{(\frac{1}{2})}(L)$ and $D_M^{(\frac{1}{2})}(T)$ are even functions of $L$ and $T$. Again we replace the infinite sum over $T$ by an integral representation and we obtain,

$$\varsigma^{(\frac{1}{2})}(\hat{s}) = \sum_{L=\frac{N}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)(\rho_M)^{2\hat{s}}(\frac{-i}{2})\int_{B_1} dz\, tan(\pi z) D_M^{(\frac{1}{2})}(z)\times$$
$$\times\left[z^2 + (\frac{\rho_M}{\rho_N})^2 L^2\right]^{-\hat{s}} \qquad (B.4)$$

Here we use $tan(\pi z)$ since $T = t + \frac{M}{2}$ is half-integer. The contour $B_1$ is indicated in fig.4. The poles of $[z^2 + (\frac{\rho_M}{\rho_N})^2 L^2]$ are located at $z = \pm i B_L$ with $B_L = (\frac{\rho_M}{\rho_N})L$. The contour $B_1$ is then replaced by the contour $B_2$ in fig.4. Again one has to carry out the integration along the path $B_2$ with $\Delta \neq 0$, and take $\Delta \to 0$ only at the end. Due to the fact that the integrand is an odd function, the integral vanishes between $\pm i B_L$. We obtain

$$\varsigma^{(\frac{1}{2})}(\hat{s}) = \sum_{L=\frac{N}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)(\rho_M)^{2\hat{s}} sin(\pi\hat{s})\Big[\int_{B_L}^{\infty} dx\, D_M^{(\frac{1}{2})}(ix)\frac{-2}{e^{2\pi x}+1}(x^2 - B_L^2)^{-\hat{s}} +$$
$$+ (-1)^{(\frac{M-1}{2})}\frac{1}{2}\sum_{p=0}^{(\frac{M-1}{2})} C_{Mp}(B_L^2)^{(p+\frac{1}{2}-\hat{s})}\frac{\Gamma(1-\hat{s})\Gamma(\hat{s}-p-\frac{1}{2})}{\Gamma(\frac{1}{2}-p)}\Big] \qquad (B.5)$$

with

$$D_M^{(\frac{1}{2})}(ix) = (-1)^{(\frac{M-1}{2})}\sum_{p=0}^{(\frac{M-1}{2})} C_{Mp}x^{2p} \qquad . \qquad (B.6)$$

The second term in eq.(B.5) has still to be regularized. To do this we consider

$$Q = \sum_{L=\frac{N}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)(B_L^2)^{(p+\frac{1}{2}-\hat{s})}$$
$$= (\frac{\rho_M}{\rho_N})^{(2p+1-2\hat{s})}\sum_{L=\frac{N}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)(L^2)^{(p+\frac{1}{2}-\hat{s})} \qquad , \qquad (B.7)$$

where we have used the definition of $B_L$. With $D_N^{(\frac{1}{2})}(L) = \sum_{r=0}^{(\frac{N-1}{2})} A_{Nr}L^{2r}$, eq.(B.7) becomes,

$$Q = \sum_{r=0}^{(\frac{N-1}{2})} A_{Nr}(\frac{\rho_M}{\rho_N})^{2p+1-2\hat{s}} \sum_{L=\frac{N}{2}}^{\infty} L^{2r+2p+1-2\hat{s}} . \qquad (B.8)$$

Now we define $Z = \sum_{L=\frac{N}{2}}^{\infty} L^{2r+2p+1-2\hat{s}}$ , which can be written as,

$$Z = \sum_{\tilde{L}=0}^{\infty} (\tilde{L} + \frac{N}{2})^{2r+2p+1-2\hat{s}} \qquad (B.9)$$

$Z$ is the generalized zeta-function, $Z = \tilde{\varsigma}(2\hat{s} - 2p - 2r - 1 \ , \ \frac{N}{2})$ , for which we can use the Hermite representation. We thus get,

$$\begin{aligned} \tilde{\varsigma} =& \frac{1}{2}(\frac{N}{2})^{2r+2p+1-2\hat{s}} + \frac{(\frac{N}{2})^{2+2p+2r-2\hat{s}}}{2(\hat{s}-p-r-1)} + \\ & 2\int_0^{\infty} \frac{sin[(2\hat{s}-1-2p-2r)arctan(\frac{2t}{N})]}{[(\frac{N}{2})^2+t^2]^{\frac{(2\hat{s}-1-2r-2p)}{2}}} \frac{dt}{e^{2\pi t}-1} \quad . \end{aligned} \qquad (B.10)$$

This representation is well behaved for all $\hat{s}$ . The final result is thus

$$\begin{aligned} \varsigma^{(\frac{1}{2})}(\hat{s}) =& \sum_{L=\frac{N}{2}}^{\infty} D_N^{(\frac{1}{2})}(L)(\rho_M)^{2\hat{s}} sin(\pi\hat{s}) \int_{B_L}^{\infty} dx \ D_M^{(\frac{1}{2})}(ix) \frac{-2}{e^{2\pi x}+1}(x^2 - B_L^2)^{-\hat{s}} + \\ & + \Big[(\rho_M)^{2\hat{s}} sin(\pi\hat{s}) \frac{1}{2}(-1)^{\frac{(M-1)}{2}} \sum_{p=0}^{\frac{(M-1)}{2}} C_{Mp} \frac{\Gamma(1-\hat{s})\Gamma(\hat{s}-p-\frac{1}{2})}{\Gamma(\frac{1}{2}-p)} \times \\ & \times (\frac{\rho_M}{\rho_N})^{2p+1-2\hat{s}} \sum_{r=0}^{\frac{(N-1)}{2}} A_{Nr} \ \tilde{\varsigma}(2\hat{s}-1-2p-2r, \ \frac{N}{2})\Big] \quad , \end{aligned} \qquad (B.11)$$

where $\tilde{\varsigma}(2\hat{s} - 1 - 2p - 2r \ , \ \frac{N}{2})$ is given by eq.(B.10).

From eq.(B.11) it follows that $\varsigma^{(\frac{1}{2})}(-2)$ vanishes. We notice that eq.(B.11) is valid for $M$ and $N$ odd. For $M$ odd and $N$ even or vice-versa one can derive in the same way an analogous formula. The case with $M$ and $N$ even is more difficult to handle since the integral does not vanish between $-iB_L$ and $+iB_L$ . The same problem arises when computing $\varsigma^{(0)}(\hat{s})$ .

**Figure Captions**

Figure 1. The contour $C_1$ in eq.(A.6) is replaced by the contour $C_2$ .

Figure 2. The contour $C_1$ in eq.(A.9) is replaced by the contour $C_2$ . Note that the contour runs parallel to the imaginary axis, over $\Delta > 0$ . The limit $\Delta \to 0$ should be taken at the end.

Figure 3. The contour $C_3$ in eq.(A.21) is replaced by the contour $C_4$ . Note that some poles of $cot(\pi z)$ are on the cut associated with the branch point $C$ .

Figure 4. The contour $B_1$ in eq.(B.4) is replaced by the contour $B_2$ . Note that the contour runs parallel to the imaginary axis, over $\Delta > 0$ . The limit $\Delta \to 0$ should be taken at the end.

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Fig. 1

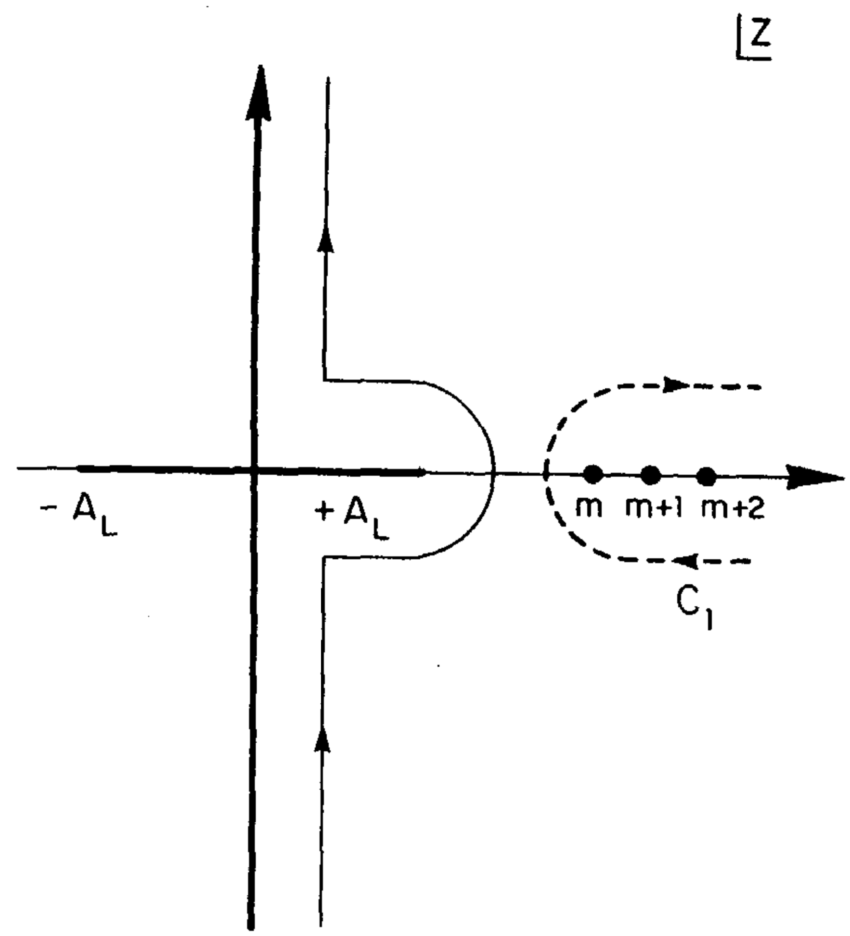

Fig. 2

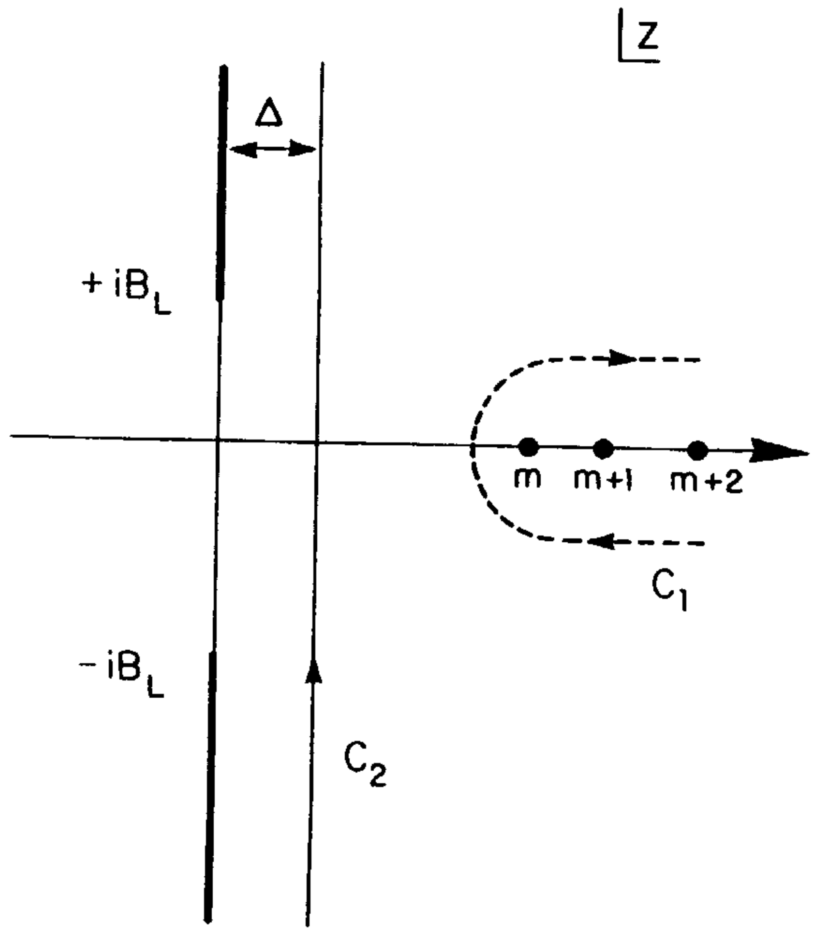

Fig. 3

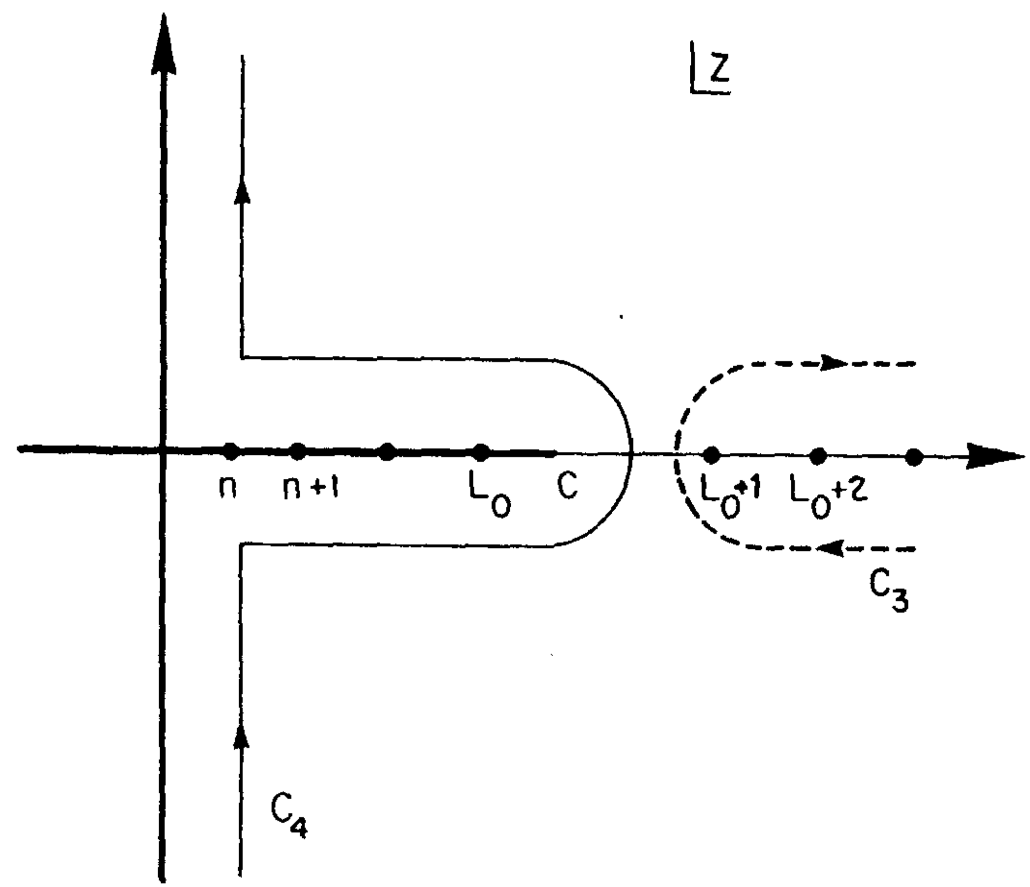

Fig. 4