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ZERO TEMPERATURE QUARK MATTER EQUATION OF STATE

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A B S T R A C T

An equation of state is computed for a plasma of one flavour quarks interacting through some phenomenological potential, in the Hartree approximation, at zero temperature. Assuming that the confining potential is scalar and colour-independent, it is shown that the quarks undergo a first-order mass phase transition. In addition, due to the way screening is introduced, all the thermodynamic quantities computed are independent of the actual shape of the interquark potential. This equation of state is then generalized to a potential with scalar and vector components, Fock corrections are discussed and the case of a several quark flavour plasma is studied.

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1. INTRODUCTION

In order to discuss properties of the primordial quark-hadron phase transition or of condensed stellar objects, it is necessary to know the equation of state of the quark-gluon plasma. At sufficiently high densities or temperatures, when asymptotic freedom holds, perturbation-based equations of state may be used¹. At low densities or temperatures, however, since we do not know how confinement occurs in QCD, one must resort to using phenomenological quark matter equations of state. Various such kinds of phenomenological equations of state have been proposed, each differing in the way confinement has been imposed: through boundary conditions as in the M.I.T. bag equation of state², through the acquisition of a large mass by quarks (coupled to a scalar field) outside hadrons as in soliton-bags³, etc.

In this talk, we will derive a phenomenological equation of state for quarks interacting through some interquark potential. (The idea of describing the interactions between quarks by phenomenological interquark potentials such as those of potential models used to fit experimental data on quarkonia, was first suggested by Wagoner & Steigman⁴). It was later taken up by Olive⁵) and Boal, Schachter & Woloshyn⁶) to describe the quark-gluon plasma in the Thomas-Fermi approximation. Their type of equations of state was applied to studies of the primordial quark-hadron phase transition by Källman⁷) and Schramm & Olive⁸.) In addition, in order to compute this equation of state, we will use one of the simplest non-perturbative approximation, the Hartree approximation. We will restrict ourselves here, to zero-temperatures. In this case, non-virtual gluons -contrarily to photons- may be present because, due to their interactions, they may condensate in momentum space, but they will contribute to the equation of state for zero, so we need not consider them.

This approach is presented in more details in section 2 for scalar colour-independent confining potentials. In section 3, it is extended to confining (colour-independent) potentials with scalar and vector components, Fock corrections are studied and the case of several quark flavours is examined. Finally, we discuss the advantages and drawbacks of this method, as well as possible improvements, in section 4.

2. METHOD

a. The effective Lagrangian and the approximation

As a starting point, let us assume that the quarks interact via the following effective Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i \not{\partial} - m)\psi(x) - i \int d^3z \bar{\psi}(x)\bar{\psi}(z)V(|x-z|)\psi(z)\psi(x) \quad (2.1)$$

i.e. we treat the quarks as Dirac particles and their interactions are accounted for by the phenomenological potential V .

From (2.1), one can derive the energy momentum tensor $T_{\mu\nu}$. On the other hand, one may assume that the medium is uniform and isotropic, in which case one also has

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu} \quad (2.2)$$

where p is the pressure and ϵ is the energy density. From (2.1) and (2.2) it follows that

$$\epsilon = \frac{\langle T^{00} \rangle}{V} = \int \frac{d^4p}{(2\pi)^4} \frac{p_0\gamma_0 + \vec{p}\vec{\gamma} + m}{2} G^<(p) \quad (2.3a)$$

$$p = \frac{\langle T^{ii} \rangle}{3V} = \int \frac{d^4p}{(2\pi)^4} \frac{p_0\gamma_0 - \vec{p}\vec{\gamma}/3 - m}{2} G^<(p) \quad (2.3b)$$

So that all what is needed in order to calculate the equation of state, is the expression of $G^<(p)$ and this is what we now turn to.

The (Dirac) equation of motion which follows from (2.1) is

$$(i \not{\beta} - m)G(x, y) = \delta^4(x - y) - i \int d^3 z G(x, z; y, z^+) V(|x - z|)_{|z_0=x_0} \quad (2.4)$$

where the notation z^+ means that z^{0+} is infinitesimally greater than z_0 .

In order to solve (2.4), one has to specify what is the form of the interquark potential V and how to express the two-particle Green function in terms of known or calculable quantities. First, we will assume the following structure for the interaction potential

$$V(r) = V_V(r) \gamma_\mu^{(1)} \gamma^\mu^{(2)} - V_S(r) 1_D^{(1)} 1_D^{(2)} \quad (2.5)$$

In practice, one takes the vector term V_V to be the one-gluon potential expected to be dominant at short distances and the scalar term V_S to be the confining potential expected to be dominant at large distances¹

In addition to specifying the Lorentz structure of the potential, its colour structure has to be determined. While it is normal to multiply the one-gluon exchange term by a factor of $\lambda^{(1)} \cdot \lambda^{(2)} / 4$, no decisive theoretical or experimental argument can be put forward to decide what to do for the confining part. So in addition to assuming that the confining potential is scalar, we will assume simply that it does not depend on colour². (Note that such assumptions are also made in the MIT and SLAC models.) So

$$\begin{cases} V_S(r) &= V_C(r) 1_C^{(1)} 1_C^{(2)} \\ V_V(r) &= V_G(r) \vec{\lambda}^{(1)} \cdot \vec{\lambda}^{(2)} / 4 \end{cases} \quad (2.6)$$

Finally, the two-particle Green function will be approximated by

$$G(x, y; z, t) \sim G(x, z) G(y, t) \quad (2.7)$$

¹Phenomenologically, a scalar component V_S is necessary because in the non-relativistic expansion of the Bethe-Salpeter equation, the spin-orbit term has opposite sign for scalar or vector potentials, the right sign to get the observed ordering of the 3P_j levels of charmonium being that of scalar potentials⁹. *The confining potential is usually assumed to be this scalar component.* In the next section, we also discuss the possibility that the confining potential has both a scalar and a vector component. As a matter of fact, one does not know whether the confining potential does not have a more complicated structure (i.e. with tensor, axial, etc, components) but potentials of the form (2.5) are the simplest ones leading to good theoretical predictions for quarkonia.

²We discussed briefly other possibilities for the colour dependence in ref. 10 and 11

(This in fact the Hartree approximation as will become clear later.)

Insertion of (2.7) into (2.4) then leads to an equation in terms of one-particle Green functions only

$$[i \beta 1_D 1_c - m 1_D 1_c + i \int d^3 z G(z, z^+) V(|x - z|)_{|z_0=z_0} G(x, y) = \delta^4(x - y) 1_D 1_c \quad (2.8)$$

The Hartree potential is defined by the following matrix

$$U_H \equiv -i \int d^3 z G(z, z^+) V(|x - z|)_{|z_0=z_0} \quad (2.9)$$

Using (2.5) and (2.6) in (2.9), one obtains

$$U_H \sim - \int d^3 z V_C(|x - z|) . Tr[-i G(z, z^+) 1_D 1_c] . 1_D 1_c + \int d^3 z V_C(|x - z|) . Tr[-i G(z, z^+) \gamma_\mu \lambda_a] . \gamma^\mu \lambda^a \quad (2.10)$$

The last term in this equation is in fact null at equilibrium because then the Green function is proportional to the unity matrix in colour space. So that

$$U_H = U_S^H . 1_D 1_c \quad (2.11)$$

where

$$U_S^H \equiv - \int d^3 z [V_C(|x - z|) . Tr[-i G(z, z^+) 1_D 1_c]$$

Equation (2.8) may be written in momentum space as below

$$\begin{aligned} [\not{p} - m_H] 1_c G(p) &= 1_D 1_c \\ \text{where} & \\ m_H &= m + U_S^H \end{aligned} \quad (2.12)$$

(m_H is designated thereafter as the effective mass).

This corresponds to the following diagrams in the calculation of $G(p)$

$$iG(k) = \begin{array}{c} \uparrow \\ | \\ iG_0(k) \end{array} + \begin{array}{c} \uparrow \\ | \\ iG_0(k)U_S^H \\ | \\ G_0(k) \end{array} \text{---} \bigcirc$$

where G^0 stands for the noninteracting propagator.

The problem of a given quark having two-body interactions with others has been replaced by that of a free quark in the external field (2.9). The effect of this field is to change the mass m into the effective mass m_H .

b. Finite Hartree equations

(2.12) can be solved by using the methods of temperature-dependent quantum field theory¹²⁾ and we obtain the following expression for the quantity which we want to compute, the retarded propagator

$$\begin{aligned}
-iG(z, z^+) &= \int \frac{d^4 p}{(2\pi)^4} G^<(p) \\
&= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\exp[\beta(\sqrt{p^2+m_H^2}-\mu)+1]} \frac{\gamma_0 \sqrt{p^2+m_H^2} - \vec{\gamma} \vec{p} + m_H}{2\sqrt{p^2+m_H^2}} \cdot 1_c \\
&+ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\exp[\beta(\sqrt{p^2+m_H^2}+\mu)+1]} \frac{-\gamma_0 \sqrt{p^2+m_H^2} - \vec{\gamma} \vec{p} + m_H}{2\sqrt{p^2+m_H^2}} \cdot 1_c \\
&+ \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma_0 p_0 + \vec{\gamma} \vec{p} + m_H}{p_0^2 - p^2 + m_H^2 + i\epsilon} \cdot 1_c
\end{aligned} \tag{2.13}$$

The first two integrals represent the contribution of matter while the last one accounts for vacuum fluctuations -and gives rise to a divergence. However, owing to the fact that our approach is essentially phenomenological, these fluctuations should not show up; the matter part only should be used throughout the calculations (see appendix). Inserting the matter part of (2.13) into the expression of U_S^H in (2.11)

then leads to the following self-consistent equation for U_S^H as a function of the chemical potential, also denoted by μ , at zero temperature:

$$U_S^H = - \int d^3z [V_S(|z-x|)] \cdot \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\} \quad (2.14)$$

where

$$p_F = \begin{cases} \sqrt{\mu^2 - m_H^2} & \text{if } \mu > m_H \\ 0 & \text{otherwise} \end{cases}$$

This equation is rather similar to that of the scalar plasma studied by Kalman¹³⁾ and Diaz-Alonso & Hakim¹⁴⁾ or of the scalar part of the nuclear matter model developed by Walecka and Chin¹⁵⁾

$$U_S^H = - \frac{g_s^2}{m_s^2} \cdot \left\{ 2 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\} \quad (2.15)$$

The part of the ratio $\frac{g_s^2}{m_s^2}$ is played here by $\int d^3z V_C(|z-x|)$.

However contrarily to (2.15) expressions (2.14) may still contain infinities, since $\int d^3z V_C(|x-z|)$ is infrared divergent, for instance in the case of a linear confining potential. But, as advocated by Kogut and Susskind¹⁶⁾, infinitely rising potentials are expected to be screened by the creation of quark-antiquark pairs. As a first approximation, one may suppose that there exist some screening length c_S (to be determined from experiment), fixed whatever the value of μ is, then

$$\begin{aligned} U_S^H &= - \int_0^{c_S} d^3r V_C(r) \\ &\text{or } - \int d^3r V_C(r) \exp(-r/c_S) \\ &\text{or etc} \\ &\times \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\} \end{aligned} \quad (2.16)$$

according to the way screening occurs.

However, one expects c_S to depend on μ and it is possible to compute this dependence: U_S^H is an energy felt by a given quark, so that it should not exceed twice the Fermi energy $\epsilon_F = \mu$, else pair creation would occur (one must use the Fermi energy as a threshold, and not the quark mass, because in dense matter pairs can

only be created above the Fermi sea). Hence one has

$$\begin{aligned}
2\mu &= \left| - \int_0^{c_S} d^3r V_C(r) \right. \\
&\quad \text{or } - \int d^3r V_C(r) \exp(-r/c_S) \\
&\quad \text{or etc} \\
&\quad \times \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\}_{|U_S^H = \pm 2\mu}
\end{aligned} \tag{2.17}$$

In other words, the finite values of $\int d^3z V_C(|(x-z)|)$ is given by the following formula

$$\begin{aligned}
& - \int_0^{c_S} d^3r V_C(r) \\
& \text{or } - \int d^3r [\alpha V_C(r) \exp(-r/c_S)] \\
& \text{or etc} \\
& = \frac{\pm 2\mu}{\left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\}_{|U_S^H = \pm 2\mu}}
\end{aligned} \tag{2.18}$$

Eq. (2.18) holds for a given μ , whatever the value of U_S^H is, not just for $\pm 2\mu$, and so (2.14) can be replaced by

$$U_S^H = \frac{\pm 2\mu}{\left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\}_{|U_S^H = \pm 2\mu}} \times \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\} \tag{2.19}$$

Note that in the equation (2.19), V_C has completely disappeared -because of the way screening was introduced- so the shape of the confining potential or the way it is damped (i.e. is it exponentially damped or does it stop sharply at a certain distance or etc) does not matter. Note also that only the minus sign is allowed in (2.9) is allowed, else the denominator would have $p_F = 0$ and vanish, so the finite value of $\int d^3z V_C(|(x-z)|)$ is positive.

c. Numerical results (for one flavour)

Equation (2.19) may be solved numerically. In order to get a first insight into the solution, let us assume that the screening length -and so the finite value of $\int d^3z V_C(|(x-z)|)$ as well- is constant. In figure 1a, we show the solution of the equation for the effective mass (2.16) for various positive values of $\int d^3z V_C(|(x-z)|)$

(multiplied by $3m^2/\pi^2$ for computational commodity, and labeled as Γ). In the general case (2.19), the curve for the effective mass is presented in figure 1b. It is rather similar to 1a, except for the fact that, due to screening, it terminates abruptly.

Once (2.19) is solved, the equation of state may be computed easily. By inserting the matter part of (2.13) into (2.3a-b), one gets at zero-temperature

$$\epsilon_Q = 3 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \left\{ \sqrt{\vec{p}^2 + m_H^2} + \frac{\vec{p}^2 + m_H m}{\sqrt{\vec{p}^2 + m_H^2}} \right\} \quad (2.20a)$$

$$p_Q = 3 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \left\{ \sqrt{\vec{p}^2 + m_H^2} - \frac{\vec{p}^2/3 + m_H m}{\sqrt{\vec{p}^2 + m_H^2}} \right\} \quad (2.20b)$$

The density of quarks minus antiquarks is simply given by

$$n_Q = 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \quad (2.20c)$$

In figure 2a, the pressure has been plotted as a function of the chemical potential: it shows that as the density increases a *first-order phase transition* takes place. This transition corresponds to the passage of a state of massive quarks to a state of quarks of decreasing mass (dashed-dotted line in figure 1b), as one would expect from perturbative QCD. In Fig. 2b, the energy per particle as a function of particle density is represented. At low density, it is smaller than m , so the quarks are in a collective bound state. This may be interpreted as the fact that the quark just start to go out of the hadrons. Thus the overall picture which is obtained is satisfying to describe quark matter at low density (where confinement is expected) and medium density (where the quark mass should start to decrease).

3. SOME EXTENSIONS

a. Colour-independent confining potential with scalar and vector

components

As mentioned in the beginning of this section, it is possible that the confining potential has a more complicated Lorentz structure than just scalar. One may wonder how the description of the quark-hadron transition that we just obtained would then be affected. So let us assume that the interquark potential is still given by (2.5) but that

$$\begin{cases} V_S(r) = \alpha V_C(r) 1_c^{(1)} 1_c^{(2)} \\ V_V(r) = V_C(r) \vec{\lambda}^{(1)} \vec{\lambda}^{(2)} / 4 - \beta V_C(r) 1_c^{(1)} 1_c^{(2)} \end{cases} \quad (3.1)$$

Proceeding as above, one obtains the following equation for the interacting propagator

$$\begin{aligned} [\not{p}_H - m_H] 1_c G(p) &= 1_D 1_c \\ \text{where} & \\ m_H &= m + U_S^H \\ p_\mu^H &= p - U_\mu^H \end{aligned} \quad (3.2)$$

i.e.

$$iG(k) = iG_0(k) + iG_0(k) U_S^H G_0(k) + iG_0(k) \cdot i\gamma^\mu U_\mu^H \cdot G_0(k)$$

Again after solving (3.2), one obtains expressions for the components of the Hartree field

$$U_S^H = - \int d^3z [\alpha V_S(|z-x|)] \cdot \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}} \right\} \quad (3.3a)$$

$$U_0^H = \pm \int d^3z [-\beta V_S(|z-x|)] \cdot \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \right\} \quad (3.3b)$$

$$U_i^H = 0 \quad (3.3c)$$

where

$$p_F = \begin{cases} \sqrt{(\mu - U_0^H)^2 - m_H^2} & \text{if } (\mu - U_0^H)^2 > m_H^2 \\ 0 & \text{otherwise} \end{cases}$$

and

there is a plus (resp. minus) sign in the expression of U_0^H if $\mu - U_0^H > m_H$ (resp. $<$).

In order to render (3.3a-b) finite, one has to introduce two screening lengths. The method presented above to compute their density dependence may be used again, one obtains the following set of coupled self-consistent equations

$$U_S^H = \frac{\pm 2\epsilon_F}{\left\{6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}}\right\}_{|U_S^H = \pm 2\epsilon_F}} \times \left\{6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{p^2 + m_H^2}}\right\} \quad (3.4a)$$

$$U_0^H = \frac{\pm 2\epsilon_F}{\left\{6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3}\right\}_{|U_0^H = \pm 2\epsilon_F}} \times \left\{6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3}\right\} \quad (3.4b)$$

which may be solved numerically. It may be shown that the mass transition will survive¹⁰).

b. Fock corrections

In the Hartree approximation, the fermionic - anticommutating - character of quark operators is not taken into account. In order to do that, one has to include the Fock terms, so let us replace (2.7) by

$$G(x, y; z, t) = G(x, z)G(y, t) - G(x, t)G(y, z) \quad (3.5)$$

By using similar methods to those of section 2, one can show that the equation of motion (2.4) becomes

$$[\gamma_0 p_{HF}^0 - \vec{\gamma} \vec{p}_{HF} - m_{HF}] G(p) = 1 \quad (3.6)$$

where

$$\begin{aligned} p_{HF}^0 &\equiv p_0 - U_0^H - U_0^F(|\vec{p}|) \\ \vec{p}_{HF} &\equiv \vec{p} + \hat{p} U_V^F(|\vec{p}|) \\ m_{HF} &\equiv m + U_S^H + U_S^F(|\vec{p}|). \end{aligned}$$

which corresponds to the following diagrams

$$iG(k) = \begin{array}{c} \uparrow \\ | \\ iG_0(k) \end{array} + \begin{array}{c} \uparrow \\ | \\ \text{---} \\ | \\ iG_0(k) U_S^H G_0(k) \end{array} \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \uparrow \\ | \\ \text{---} \\ | \\ iG_0(k) \cdot i\gamma^\mu U_\mu^H \cdot G_0(k) \end{array} \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \uparrow \\ | \\ \text{---} \\ | \\ iG_0(k) \cdot U_S^F(k) \cdot G_0(k) \end{array} + \begin{array}{c} \uparrow \\ | \\ \text{---} \\ | \\ iG_0(k) \cdot i\gamma^\mu U_\mu^F(k) \cdot G_0(k) \end{array}$$

The solution of this equation leads to the following components for the Hartree-Fock field

$$U_S^H = \left\{ \int d^3r [-\alpha V_C(r)] \right\} \times 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_{HF}}{\sqrt{p_{HF}^2 + m_{HF}^2}} \quad (3.7a)$$

$$U_0^H = \pm \left\{ \int d^3r [-\beta V_C(r)] \right\} \times 6 \int \frac{d^3p}{(2\pi)^3} \quad (3.7b)$$

$$U_S^F = - \int \frac{d^3p}{(2\pi)^3} \frac{m_{HF}}{2\sqrt{p_{HF}^2 + m_{HF}^2}} \left\{ -\alpha \tilde{V}_C(|\vec{p} - \vec{q}|) + 4[\tilde{V}_G(|\vec{p} - \vec{q}|) - \beta \tilde{V}_C(|\vec{p} - \vec{q}|)] \right\} \quad (3.7c)$$

$$U_0^F = \mp \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left\{ -\alpha \tilde{V}_C(|\vec{p} - \vec{q}|) - 2[\tilde{V}_G(|\vec{p} - \vec{q}|) \right. \quad (3.7d)$$

$$\left. U_V^F = - \int \frac{d^3p}{(2\pi)^3} \frac{3\vec{p}}{2\sqrt{p_{HF}^2 + m_{HF}^2}} \left\{ -\alpha \tilde{V}_C(|\vec{p} - \vec{q}|) - 2[\tilde{V}_G(|\vec{p} - \vec{q}|) - \beta \tilde{V}_C(|\vec{p} - \vec{q}|)] \right\} \right\} \quad (3.7e)$$

($\alpha=1$ and $\beta=0$ correspond to the pure scalar confining potentials of section 2 and $\alpha \neq 0$ and $\beta \neq 0$ to the confing potentials of paragraphe 3.a).

As in the Hartree approximation, the (infinitely rising) confining potential gives rise to infrared divergences. But because of the convolutions, the *potential terms*

in the Fock components (3.7c-e) cannot be factorised as in the Hartree components (3.7a-b). The fact that the potential terms $\int d^3r [\alpha V_C(r)]$ and $\int d^3r [-\beta V_C(r)]$ in the Hartree approximation could be factorised, had been used to compute their (finite) physical value *and* avoid having to specify the actual shape of the potential. In the Fock approximation, this trick cannot be applied: both the shape of the potential and the explicit way to remove infinities have to be specified. As for the shape of the potential, we will assume for simplicity

$$V_G(r) = -a/r \text{ and } V_C(r) = -(kr - U) \quad (3.8)$$

and for the screening, we will assume that, due to pair creation, the interquark potential is exponentially smoothed, i.e. the scalar and vector components of V_C will respectively be multiplied by $e^{-c_1 r}$ and $e^{-c_2 r}$. This procedure was followed for instance in reference 17. Ideally one would like to know the density dependence of k, U, a, c_1 and c_2 to implement and solve (3.7a-e) but we have no way to compute it. However, it seems reasonable to think that since our purpose is to study the effect of the Fock corrections to the Hartree approximation, we will get, *for densities of order some times the normal nuclear matter density, a qualitative understanding by keeping k, U, a, c_1 and c_2 constant. For these low-medium densities, it may then be shown that the mass first order phase transition survives and that the thermodynamical functions can be approximated by their Hartree expression*¹¹.

c. Generalization to several quark flavours

Proceeding as for the one flavour case, it is easy to show that in the case of several quark flavours

$$U_S^H = \frac{2\mu_L}{\sum_{q=u,d,s,\dots} 6 \int_0^{p_F^q} \frac{d^3p}{2\pi^3} \frac{m_H^q}{\sqrt{p^2 + m_H^{q2}}} \Big|_{U_S^H = -2\mu_L}} \cdot \sum_{q=u,d,s,\dots} 6 \int_0^{p_F^q} \frac{d^3p}{(2\pi)^3} \frac{m_H^q}{\sqrt{p^2 + m_H^{q2}}} \quad (3.9)$$

where

$$p_F^q(U_S^H, \mu_q) = \begin{cases} \sqrt{\mu_q^2 - m_H^q} & \text{if } \mu_q \geq m_H^q \\ 0 & \text{otherwise} \end{cases}$$

In the above expressions, μ_L designates the smallest among the Fermi energies of the various quark flavours; pairs of this flavour are created preferably to screen the interquark potential.

It can be seen that the Hartree field U_S^H is the same whatever the flavour, and the effective masses $m_H^q \equiv m_q + U_S^H$, which play a similar part to running masses, will all have a similar decrease.

Example 1: conversion of neutron matter to two flavour quark matter

Let us first study the case of neutron matter undergoing a phase transition to quark matter. Weak interactions do not have time to settle and charge neutrality simply reads

$$e\left(\frac{2}{3}n_u - \frac{1}{3}n_d\right) = 0 \quad (3.10)$$

Hence,

$$\mu_u \equiv \mu \quad (3.11a)$$

and

$$\mu_d = \begin{cases} [2^{2/3}\mu^2 + (1 - 2^{2/3})m_H^2]^{1/2} & \text{if } \mu \geq m \equiv m_u \equiv m_d \\ \mu & \text{otherwise} \end{cases} \quad (3.11b)$$

One sees that μ_d is greater or equal to μ_u , so $u\bar{u}$ pairs will be created rather than $d\bar{d}$ pairs, i.e. in (3.9), $L=u$. Note that quantities will now be plotted not as a function of μ_u or μ_d but as a function of the Gibbs energy per particle $G \equiv \sum_{q=u,d,s,\dots} n_q/n_Q = \mu_u + 2\mu_d$.

Equation (3.9) may be solved numerically. Its solution as a function of G is quite similar to that of the one flavour case in figure 1b, so we do not show it. Then

one can plot $p(G)^3$ and see that the quark plasma will undergo a first order phase transition ; again since it is very much alike figure 2.a, we do not show it. The phase transition takes place at the value $G/m = 1.575$. This value of G corresponds to a density of $n_t/m^3 = 0.0085$.

In what precedes, all the quantities have been computed in unit of the quark mass, m , and indeed this is the only parameter of the two flavour model. It is in fact possible to find a lower bound for this parameter: n_t must be greater or equal to the nuclear matter density, so we must have

$$n_t \geq n_{nuc.matt.} = 1.28 \cdot 10^6 \text{ Mev}^3/m^3 \text{ which implies } m \geq 532. \text{ Mev} \quad (3.12)$$

The fact that we obtain 532.Mev as a lower bound for the constituent mass -usually thought to be of order 340. Mev- is an indication that our model is reasonable but crude. In what follows, we are going to see that it is possible to get an upper bound for m as well.

Example 2: three flavour quark matter in chemical equilibrium

Once the transition is accomplished, the quarks will establish chemical equilibrium via the weak interactions

$$d \longleftrightarrow u + e + \bar{\nu}_e \quad (3.13a)$$

$$s \longleftrightarrow u + e + \bar{\nu}_e \quad (3.13b)$$

$$s + u \longleftrightarrow d + u \quad (3.13c)$$

The weak interactions (3.13a-c) imply that

$$\begin{cases} \mu_d = \mu_e \equiv \mu \\ \mu_u + \mu_e = \mu \end{cases} \quad (3.14)$$

³The quark contribution to the various thermodynamical functions is a sum of terms like (2.20a-c). If electrons must be added to maintain charge neutrality, their contribution to the equation of state may be taken to be that of a free Fermi gas.

and overall charge neutrality requires that

$$e\left(\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_c\right) = 0 \quad (3.15)$$

Thus there is only one independent chemical potential, which we choose as being μ . One sees that $\mu_u = \mu - \mu_s$ is smaller or equal to μ_u , so again $u\bar{u}$ pairs will be created preferably.

Note that in the three flavour case there are two parameters: $m \equiv m_u$ and $r \equiv m_u/m_s$. Since m_u and m_s are constituent masses, we expect that $r \approx 500./340. = 1.47$.

Equations (3.9) for U_S^H and (3.15) for say, μ_s , can be solved simultaneously numerically for various values of μ , with the input (3.14). Once this is done, the behaviour of the various thermodynamical functions can be obtained, it is rather similar to that of one flavour quark matter -so again we do not show them. (This does not mean that there will be another phase transition, once u and d quarks start to appear with a given density, they will be gradually depleted, the pressure needs not vary abruptly).

As can be seen in figures 2a-b, the curves for the various thermodynamical functions terminate suddenly. Here this corresponds to a value of G equals to 1.7, which we will denote by G_{max} . Up to G_{max} , the equation of state is very non-perturbative. On the other side, one may compute the approximate value of the Gibbs energy per particle G_{pert} at which quark matter should start to be describable with a perturbative equation of state. Let us suppose that this happens when the coupling constant equals one ⁴. We can get a rough approximation of G_{pert} by solving

$$\alpha_s(\mu^2) = \frac{6\pi}{(33 - 2N_f) \ln(\mu/\Lambda)} = 1 \quad (3.16)$$

⁴We could just as well take a half or any other value smaller than one, this would increase the upper bound we are looking for. So taking $\alpha_s = 1$ is more restrictive

(This is the expression of the running coupling constant in the case of quarks of mass much smaller than μ .)

If we take Λ to be 200. Mev for instance and $N_f = 3$, the solution of (3.16) is $\mu=401.$ Mev. So, if $r \approx 1.47$, the following constraint should be satisfied

$$G_{maz} \leq G_{pert} = 3 \times 401./m \text{ which implies } m \leq 708.Mev \quad (3.17)$$

This upper bound is compatible with the lower bound (3.12). It corresponds to a transition density $n_t \sim 2.4n_{nuc.matt.}$.

4. CONCLUSION

The methods of relativistic quantum many-body theory have been applied to the study of quark matter interacting through phenomenological potentials at zero temperature. It was shown that if the chosen confining potential has a scalar (or scalar-vector) Lorentz structure and is colour-independent, the quark plasma undergoes a first order mass transition from a state of massive particles at low density to a gas of particles of decreasing mass at high density -as one would expect from QCD. Moreover, at low densities, quarks are in a collective bound state, thus suggesting that they just started to go out of the hadrons. Finally, all the computed quantities are independent of the shape of the interquark potential -because of the way screening through pair creation has been implemented- and Fock corrections may be neglected (at low-medium density). This one-flavour model was then generalized to several flavours and applied to the hadron-quark phase transition in a cold plasma. It was shown that the u-d constituent quark mass had to be in the interval [532.Mev,704.Mev] -which is reasonable for such a simple model. Thus a satisfying description for quark matter seems to emerge from this approach.

However, some reservations must be made. First, instantaneous interquark potentials were used. At low-medium densities where quarks are massive, this should be a reasonable approximation. (Instantaneous potentials have been used widely to study nuclear matter in exactly the same range of densities we are interested in, namely at nuclear density $\sim 1.28 \cdot 10^6 \text{ Mev}^3$ and slightly higher.) But at high densities where quarks become more and more relativistic, such an approach should not be used. As a matter of fact, our equation of state is only computable for low-medium densities (this is reflected by the fact that the curves for the thermodynamical functions, like 2a-b, terminate abruptly); at high densities, it should be matched with a perturbative equation of state. (Note that other phenomenological equations of state such as the M.I.T. one, cannot be used either -for other reasons- at high densities.)

Second, the confining potential may have a more complicated structure than just scalar (or scalar-vector) and colour-independent. (It might be actually interesting to develop the interquark confining potential with Gell-Mann matrices -as can be done in the Clifford algebra with Dirac matrices- and study other more complex colour structures.)

Third and finally, it is not completely obvious which kind of mass must be used for m . Looking at Eq. (2.1) for instance, one might favour a current mass - indeed this is what Olive⁵⁾ did for example - because one would expect that, as a result of interactions with other quarks, this current mass is increased to the value of a constituent mass. However, it is not so: the Hartree field contribution to the quark mass is negative and $m_H/m \rightarrow 1$ as μ decreases as can be seen in figure 1.b. Also it may be shown that if one introduces non-zero temperatures in a gap equation such as (2.19), at a given density, m_H/m decreases as the temperature increases¹⁴⁾, so that, in order that chirality be restored at high density or temperature, the choice

of a constituent mass for m is most reasonable . In addition, since potential models of quarkonia and hadrons are fitted with constituent masses, and that our approach makes use of such potentials, it is also more self-consistent to use constituent masses (this is what was done as well for instance in reference 18 .)

It is also easy to imagine some possible improvement to this model. First, in the Hartree approximation, one replaces the interaction of a number of quarks on a given quark, by an exterior field acting on this quark. As a consequence of this, we get a collective bound state at low densities and not a soup of colour singlets (hadrons). In order to get , if not baryons, at least mesons, an approximation of a different type should be used. (It would also be interesting to study the effects of other approximations on our results.)

Second, one can think of other ways of computing screening lengths in a self-consistent way. For instance, one can assume some value for the screening length, study the plasma oscillations of the quark gas, then from this compute the associated Debye shielding length and see if it agrees with the value initially assumed for the screening length -if not iterate. It would be interesting to compare this approach with the one followed here.

Third and last, this equation of state could be generalized to non-zero temperature. Non-virtual gluons could be considered as modes propagating in the quark plasma -in much the same way that a solid emits phonons- and their energy spectrum obtained by computing the plasma oscillations.

In summary, the equation of state presented here should be easy to improve. In addition, its use in astrophysics does not present any difficulty¹⁹. In our mind, the main interest of this approach is that it allows one to utilize a new source of information: data from quarkonia spectroscopy, and perhaps in the future, results on the interquark potential obtained in lattice gauge theory simulations.

ACKNOWLEDGMENTS

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APPENDIX

In a phenomenological model, one does not expect the vacuum to show up. The fact that we had a vacuum contribution in the expression of $G(z, z^+)$ is due to the formalism that we used. Had we use for instance a particle operator approach rather than a field operator approach, we could have avoided this problem. Precisely, one could start from the many-body hamiltonian

$$H = \sum_{i=1}^N \{-i\vec{\alpha}(i) \cdot \vec{\nabla}_i + \beta(i)m\} + \sum_{i \neq j} \mathcal{V}_{ij} \quad (\text{A.1})$$

and minimize its expectation value in the space of Slater determinants of the form

$$\Psi = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_N) \\ \psi_2(x_1) & \psi_2(x_2) & \dots & \psi_2(x_N) \\ \dots & \dots & \dots & \dots \\ \psi_N(x_1) & \psi_N(x_2) & \dots & \psi_N(x_N) \end{vmatrix} \quad (\text{A.2})$$

where the ψ_i are plane waves of the form

$$\psi_i(x_j) = \frac{1}{\text{volume}} u(\vec{p}_j) e^{-i\vec{p}_j x_j}$$

and

$u(\vec{p})$ is a Dirac spinor.

After doing the same reasoning for antiquarks as well, one would then be lead to⁵⁷⁾

$$U_S^H = - \int d^3z [\alpha V_S(|z-x|)] \cdot \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \frac{m_H}{\sqrt{\vec{p}^2 + m_H^2}} \right\} \quad (\text{2.15a})$$

$$U_0^H = \pm \int d^3z [-\beta V_S(|z-x|)] \cdot \left\{ 6 \int_0^{p_F} \frac{d^3p}{(2\pi)^3} \right\} \quad (\text{2.15b})$$

$$U_i^H = 0 \quad (\text{2.15c})$$

where

$$p_F = \begin{cases} \sqrt{(\mu - U_0^H)^2 - m_H^2} & \text{if } (\mu - U_0^H)^2 > m_H^2 \\ 0 & \text{otherwise} \end{cases}$$

and

there is a plus (resp. minus) sign in the expression of U_0^H if $\mu - U_0^H > m_H$ (resp. $<$).

So equations (2.15a-c) have been obtained without vacuum complications.

It may be worth pointing out at this point, that the usual renormalization procedures such as normal ordering or addition of counter-terms, cannot be used in our approach. First, there is a risk of double counting: when one uses an interquark potential derived from experiment, to compute it theoretically one would already have included counter-terms, so using such a potential in (2.1) and later adding counter-terms would not be very consistent. Next, in order to renormalize the Hartree gap equation (2.15a), one needs to add counter-terms with coefficients of the form¹⁵⁾

$$\begin{aligned}
\alpha &= g_* \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[G_{vac}^{U_s^H=0}] \\
\beta &= -g_*^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[G_{vac}^{U_s^H=0}]^2 \\
\gamma &= 2!g_*^3 \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[G_{vac}^{U_s^H=0}]^3 \\
\lambda &= 3!g_*^4 \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[G_{vac}^{U_s^H=0}]^4
\end{aligned} \tag{A.3}$$

In our case, $\int d^3 r [\alpha V_C(r)]$ plays the part of g_*^2/μ^2 , and it depends on density, so the coefficients given by (A.3) would as well.

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FIGURE CAPTIONS

- Fig. 1a** Plot of the effective mass as a function of the chemical potential for different values of constant positive Γ . A phase transition occurs if $\Gamma \geq 2.5$.
- Fig. 1b** Plot of the effective mass as a function of the chemical potential. Due to screening, it terminates more abruptly than in figure 1a. A glance at Fig. 2a allows one to determine which states are not mechanically stable (dotted line) and which states are physically accessible (dashed line).
- Fig. 2a** Plot of the pressure as a function of the chemical potential. The transition is seen to be first order and is associated to a change of mass (dashed-dotted line in figure 1b). For curve designation, see figure 1b.
- Fig. 2b** Plot of the energy per particle as a function of particle density. The plasma is in a collective bound state at low densities.

Figure 1a

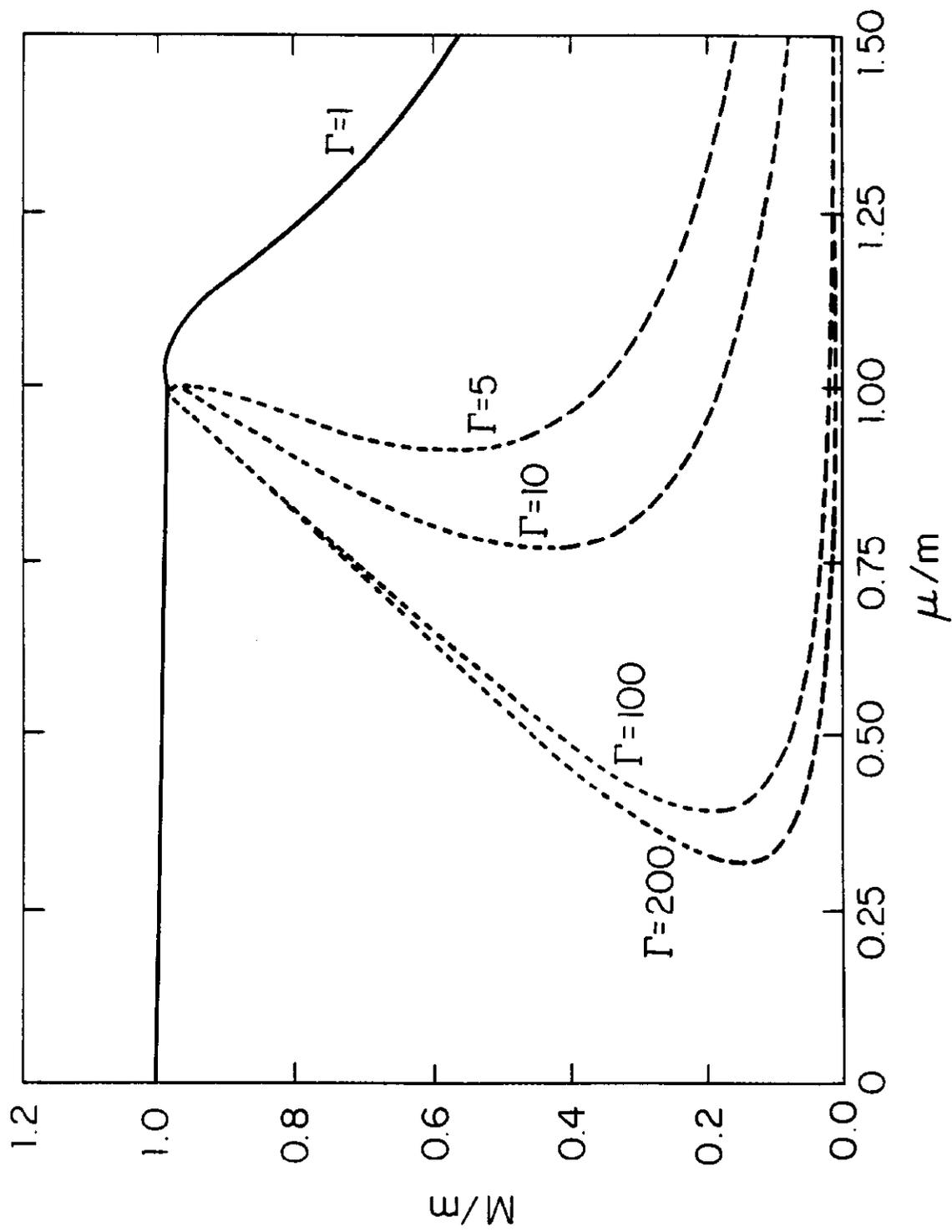


Figure 1b

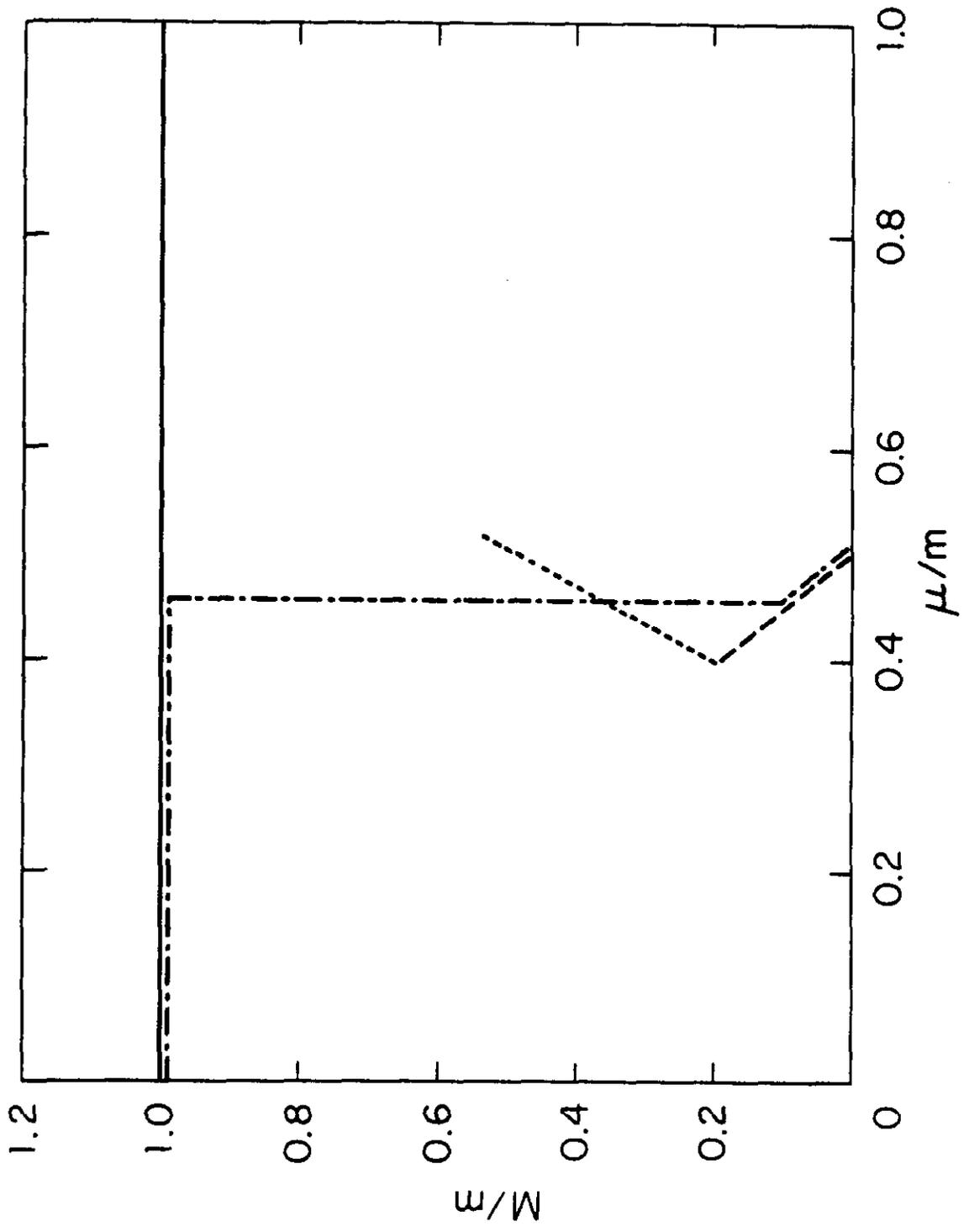


Figure 2a

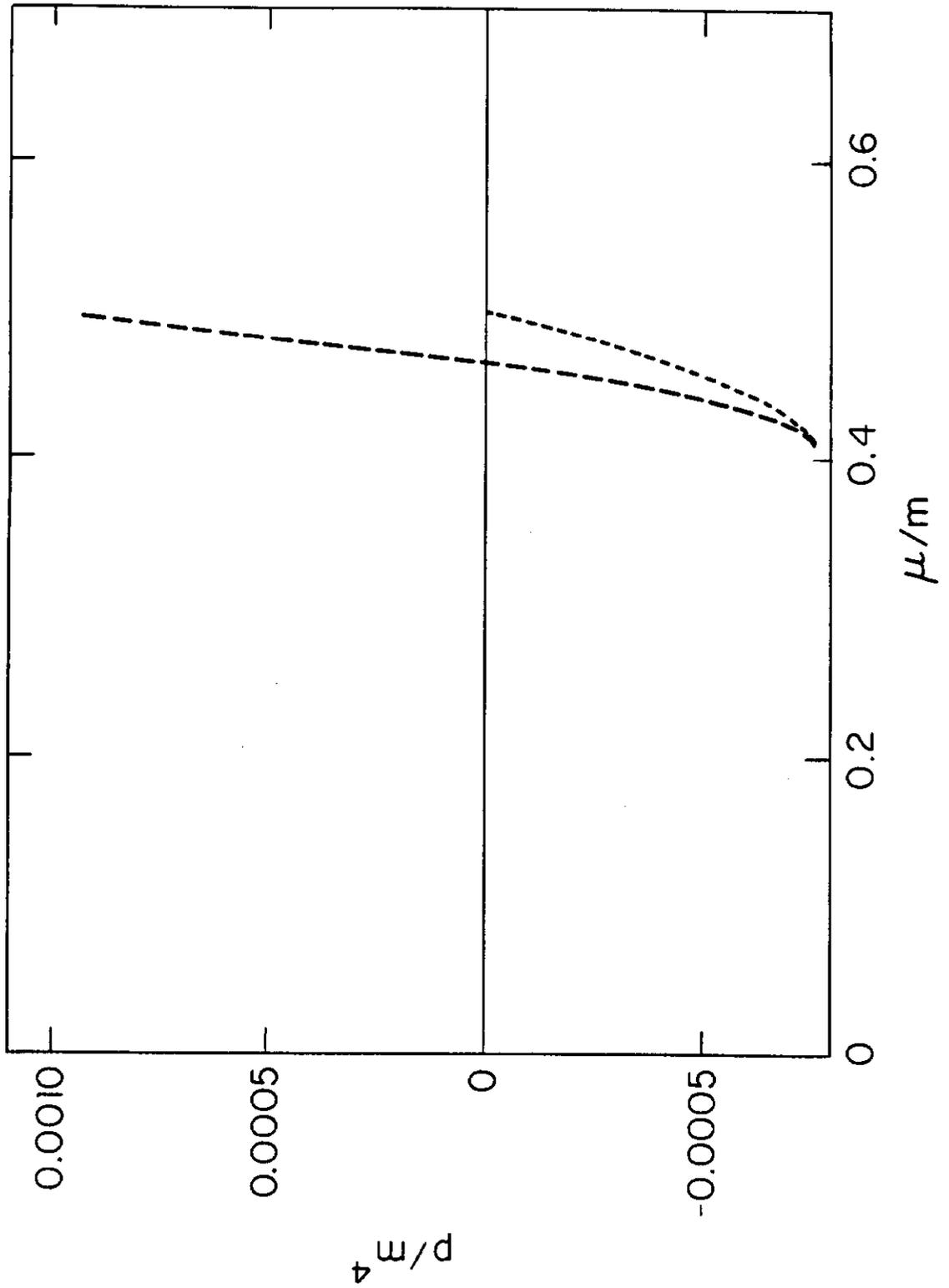


Figure 2b

