



HEAVY QUARKS ON THE LATTICE\*

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**Abstract**

A method of treating heavy quarks is applied to lattice Q.C.D. for heavy quark masses ( $m_H$ ) and lattice spacing ( $a$ ) satisfying the condition  $m_H a > 1$ . Explicit applications to the measurement of heavy-light meson masses, decay constants, and mixing parameters are presented. Numerical results for  $B$  mesons are obtained on a  $8^2 \times 16 \times 24$  lattice with  $\beta = 5.7$ .

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## I. Introduction

Heavy quarks play an important role in studying the dynamics of Q.C.D. For quarks with masses ( $m_H$ ) much larger than the typical Q.C.D. scale of strong interactions ( $\Lambda_{Q.C.D.}$ ), their motion can be treated non-relativistically and the dynamics considerably simplified.

The first systems involving heavy quarks to be observed were systems consisting of mesons with one heavy quark  $Q_H$  and one heavy antiquark  $\bar{Q}_H$ , the charmonium ( $c\bar{c}$ ) and, more recently, the bottomonium ( $b\bar{b}$ ) system. In these systems, the Q.C.D. interaction between the quarks is well represented by a nonrelativistic potential. Phenomenological potential models provide an accurate description of the data for the ( $c\bar{c}$ ) and ( $b\bar{b}$ ) systems and thus give empirical evidence for the use of nonrelativistic dynamics. The form of the potential is determined from the data for distances between .1 and 1 Fermi.<sup>[1]</sup> At shorter distances, perturbative Q.C.D. can be used to directly compute the potential and, furthermore, the potential (or static energy)  $\mathcal{E}(R)$  can be calculated from Q.C.D. on the lattice even in the non-perturbative region.

The relation between Q.C.D. and  $\mathcal{E}(R)$  in Euclidean space is made explicit through the Wilson Loop  $W(R, T)$ <sup>2</sup>.  $W(R, T)$  is defined to be the average over all field configurations of the trace of gauge field links around a rectangular path of spatial extent  $R$  and temporal extent  $T$ .  $\mathcal{E}(R)$  can be defined simply as:

$$\mathcal{E}(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln (W(R, T)) . \quad (1.1)$$

The static energy has been precisely measured on the lattice in the quenched approximation to Q.C.D.<sup>3,4</sup> and recently measured in the full theory as well<sup>4</sup>. The agreement between theory and experiment is excellent.

The leading spin dependent relativistic corrections to the dynamics of ( $Q_H \bar{Q}_H$ ) systems can also be studied. As shown by F. Feinberg and the present author<sup>5</sup>, the static energy arises as the leading term (independent of  $1/m_H$ ) in a systematic expansion in powers of  $1/m_H$  for the heavy quark propagators, while retaining terms to order  $1/m_H^2$  in the expansion allows the determination of the leading spin dependent potentials for ( $Q_H \bar{Q}_H$ ) systems. The evaluation of these potentials on the lattice requires the calculation of correlations of gauge electric and magnetic

plaquettes along the Wilson loop. These calculations have been done by a number of groups<sup>6</sup>.

The static energy and spin dependent potentials measured on the lattice contain the information about the masses and properties of  $Q_H\bar{Q}_H$  states which depends on the dynamics of Q.C.D. These known potential functions then determine the spectrum of excited states as well as the ground state with given ( $J^{PC}$ ) since the Schroedinger equation may be used.

The analysis of baryons ( $Q_H^{(1)}Q_H^{(2)}Q_H^{(3)}$ ) on the lattice is analogous to that for mesons although it is somewhat more involved. For baryon systems, the static energy can be defined by means of a generalization of the Wilson loop formalism to  $W(\vec{R}_1, \vec{R}_2, \vec{R}_3, T)$ . Again,  $W(\vec{R}_1, \vec{R}_2, \vec{R}_3, T)$  is an average over the field configurations of a gauge invariant product of links along a path. However, the path here is more complicated. It consists of three parts, each corresponding to one of the quarks, which are joined at the beginning at a common point by the anti-symmetric tensor  $\epsilon_{ijk}$  to form a gauge invariant object. Then each part has a spatial path  $R_i$  in some direction  $\hat{n}_i$  followed by a timelike path of length  $T$  after which the parts are rejoined at the end by spacelike path to meet a common point using the  $\epsilon$  tensor. The baryon static energy is defined in terms of the resulting construction of  $W$  by

$$\mathcal{E}(\vec{R}_1, \vec{R}_2, \vec{R}_3) = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln [W(\vec{R}_1, \vec{R}_2, \vec{R}_3, T)] . \quad (1.2)$$

Attempts have been made to measure this static energy on the lattices<sup>7,8</sup>. Although most of the spectroscopy of heavy baryons has not yet been observed, there are a number of relevant questions about the nature of the forces between quarks in a three-body system. Some of these questions are:

- 1) What is the relative strength of the three-body to two-body forces?
- 2) How likely are the three quarks to be in a quark-diquark spatial configuration?

The answers to these questions are, in fact, also useful in understanding the phenomenology of the observed light baryons<sup>8,9</sup>.

Of course, it is also possible to compute the spin-dependent splittings of the baryons by using the corrections to order  $1/m_H^2$  in the heavy quark propagator and a simple generalization of the analysis for ( $Q_H\bar{Q}_H$ ) systems. To my knowledge, this has not yet been done.

Finally, I come to the main subject of this work, the application of heavy quark methods to heavy-light systems. These systems consist of one heavy quark ( $Q_H$ ) of mass  $m_H$  and a light antiquark ( $\bar{Q}_L$ ) of mass  $m_L$  bound to a color singlet heavy flavor meson. The dynamical scales for such systems are:

$$m_L \lesssim \Lambda_{Q.C.D.} \ll m_H \quad (1.3)$$

The obvious examples of such mesons are the  $B$  mesons  $-(b\bar{u}), (b\bar{d})$  and  $(b\bar{s})$ . It is also likely that to a fair approximation the  $D$  and  $F$  mesons,  $(c\bar{u}), (c\bar{d})$  and  $(c\bar{s})$ , can be treated as heavy-light systems. Of course, the  $T$  mesons,  $(t\bar{u}), (t\bar{d})$ , and  $(t\bar{s})$ , will be ideal heavy-light systems when discovered.

Measurement of the structure of the heavy-light systems, such as masses, lifetimes, decay modes, flavor changing natural currents, and CP violations, constrain the fundamental parameters of the quark Lagrangian. Therefore, understanding the dynamical effects of Q.C.D. in these systems will help determine, the heavy quark masses and the parameters of the KM matrix, as well as put bounds on any new interactions beyond the standard model.

The crucial difference between heavy-light systems and the purely heavy quark systems is that the heavy-light systems cannot, in general, be treated by non-relativistic potential methods. Because the reduced mass ( $\mu$ ) of such a system is given by:

$$\mu \equiv \frac{m_L m_H}{m_L + m_H} \simeq m_L \lesssim \Lambda_{Q.C.D.}, \quad (1.4)$$

the typical momenta are not small relative to the reduced mass. Also, it is clear that perturbative methods are inappropriate. Furthermore, though these systems can be studied on the lattice, it is not feasible to consider lattices with unit length smaller than  $1/m_H$  for  $b$  (or  $t$  quarks), so that a method to treat heavy-light ( $H-L$ ) systems with

$$m_L \lesssim \Lambda_{Q.C.D.} \ll 1/a \ll m_H \quad (1.5)$$

is required.<sup>†</sup>

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<sup>†</sup>It is not sufficient to simply take  $\kappa$  small because although the leading path for heavy quark propagation will be straight line motion in the time direction the corrections to this path do not have the proper strengths.

An important simplification in analyzing  $H - L$  systems occurs because the typical three momenta  $\langle \vec{p}^2 \rangle^{1/2}$  should be of the order of the Q.C.D. scale

$$m_L \lesssim \langle \vec{p}^2 \rangle^{1/2} \ll m_H \quad (1.6)$$

Therefore, at least the heavy quark can be treated non-relativistically within the  $H - L$  system, and an expansion in powers of  $1/m_H$  is sensible for the heavy quark. The resulting operators can then be calculated on the lattice.

## II. Fermion Propagator in an External Gauge Field

Consider a heavy quark propagating in the presence of the non-Abelian gauge field. Since the momenta transfers  $\langle \vec{p}^2 \rangle^{1/2}$  in a bound ( $Q_H \bar{Q}_L$ ) state are typically much smaller than  $m_H$ , the motion of the heavy quark is only slightly affected by each action of the gauge field. The exceptions to this rule are associated with the gauge field interactions which produce renormalizations of the heavy quark effective action (i.e., mass and wave function renormalizations and vertex corrections). Since  $\langle \vec{p}^2 \rangle^{1/2} \gtrsim m_H$  implies  $\langle \vec{p}^2 \rangle^{1/2} \gg \Delta_{Q.C.D.}$ , these contributions are small and calculable perturbatively. For an explanation of how to treat these terms in a complete and systematic way, see W.E. Caswell and G.P. Lepage<sup>10</sup>. We will ignore these corrections here but they will be treated in a future publication.

The full fermion propagator in the external gauge field

$$-i \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \equiv S_H(x, y; A) \quad (2.1)$$

satisfies the Dirac equation

$$(\gamma^\mu D_\mu - m_H) S_H(x, y) = \delta^4(x - y) \quad (2.2)$$

where

$$D_\mu \equiv i\partial_\mu + g t_a A_\mu^a. \quad (2.3)$$

In the extreme limit,  $m_H \rightarrow \infty$ , the full propagator can be calculated. The fermion propagator in this limit

$$\lim_{m_H \rightarrow \infty} S_H(x, y) \equiv S_0(x, y) \quad (2.4)$$

satisfies the equation

$$\begin{aligned} (\gamma^0 \mathcal{D}_0 - m_H) S_0(x, y) = \\ (i\gamma^0 \partial_0 + g t_a \gamma^0 A_0^a - m_H) S_0(x, y) = \delta^4(x - y) \end{aligned} \quad (2.5)$$

The formal solution to this equation is given by

$$\begin{aligned} S_0(x, y) = -iP \left( \begin{array}{c} x^0 \\ y^0 \end{array} \right) \delta(\vec{x} - \vec{y}) \left\{ \left[ \theta(x^0 - y^0) e^{-im_H(x^0 - y^0)} \right. \right. \\ \left. \left. \left( \frac{1 + \gamma^0}{2} \right) \right] + \left[ \begin{array}{cc} x^0 & \leftrightarrow & y^0 \\ \frac{1 + \gamma^0}{2} & \leftrightarrow & \frac{1 - \gamma^0}{2} \end{array} \right] \right\} \end{aligned} \quad (2.6)$$

where  $P \left( \begin{array}{c} x^0 \\ y^0 \end{array} \right)$  is the path ordered (**P**) exponential

$$\mathbf{P} \exp \left[ ig \int_{y^0}^{x^0} dz^0 A_{a0}(\vec{x}, z^0) t^a \right] \quad (2.7)$$

along the straight line path in the time direction from time  $y^0$  to time  $x^0$ . The heavy quark is static and simply gains an Eikonal phase as it propagates in time. The operators  $\frac{1+\gamma^0}{2} (\frac{1-\gamma^0}{2})$  are projections onto the positive (negative) energy states for the heavy quark propagation for  $(x^0 - y^0) > 0$ .

Given the solution in the limit  $m_H \rightarrow \infty$ , the general propagator  $S_H(x, y)$  can be solved perturbatively in powers of  $1/m_H$  using  $S_0$  as the lowest order solution.

For  $(x^0 - y^0) > 0$ , the positive energy full propagator to order  $1/m_H^2$  is given by<sup>5</sup>:

$$\begin{aligned} \left[ 1 + \frac{1}{4m_H^2} (\vec{D}^2 - g\vec{\sigma} \cdot \vec{B}_a t^a) \right]_x S_H(x, y; A) = S_0^+(x, y) \\ - \int d^4 w S_0^+(x, w) \left[ \frac{1}{2m_H} (\vec{D}^2 - g\vec{\sigma} \cdot \vec{B}_a t^a) \right. \\ \left. + \frac{i}{4m_H^2} (\delta_{ij} - i\epsilon_{ijk} \vec{\sigma}^k) g E_a^i t^a \cdot D^j \right]_w S_0^+(w, y) \\ + 0(1/m_H^3) \end{aligned} \quad (2.8)$$

where  $S_0^+(x, y)$  is simply the positive energy ( $\frac{1+\gamma^0}{2}$ ) part of Eq. (2.6).

For  $H - L$  systems, only the corrections to order  $1/m_H$  will be needed. So Eq. (2.8) can be simplified to:

$$\begin{aligned}
S_H(x, y; A) &= S_0^+(x, y) - \int d^4w S_0^+(x, w) \\
&\quad \frac{1}{2m_H} [\bar{D}^2 - g\vec{\sigma} \cdot \vec{B}_a t^a]_{\underline{w}} S_0^+(w, y) \\
&= e^{-im(x^0 - y^0)} \left( \frac{1 + \gamma^0}{2} \right) [-i\theta(x^0 - y^0)] \\
&\quad \left\{ P \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} + i \int_{y^0}^{x^0} dw^0 P \begin{pmatrix} x^0 \\ w^0 \end{pmatrix} \right. \\
&\quad \left. \frac{1}{2m_H} (\bar{D}^2 - g\vec{\sigma} \cdot \vec{B}_a t^a)_{(\underline{x}, w^0)} \right. \\
&\quad \left. P \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \right\} \delta(\vec{w} - \vec{y}) . \tag{2.9}
\end{aligned}$$

The term  $\bar{D}^2/2m_H$  is a perturbative correction for the heavy quark motion and the  $\frac{\vec{\sigma} \cdot \vec{B}}{2m_H}$  is a spin-dependent correction coupled to the gauge magnetic field.

Finally, it should be noted that by using the non-Abelian version of Stokes' Theorem<sup>5</sup>:

$$\begin{aligned}
\bar{D}_z P \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} &= \int_{y^0}^{x^0} dw^0 \left\{ P \begin{pmatrix} x^0 \\ w^0 \end{pmatrix} [g\vec{E}_a(w^0, \vec{x}) t^a] \right. \\
&\quad \left. P \begin{pmatrix} w^0 \\ y^0 \end{pmatrix} \right\} + P \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \bar{D}_z \tag{2.10}
\end{aligned}$$

for  $\vec{x} = \vec{y}$ , the  $\bar{D}^2$  term can be reexpressed in terms of  $\partial_z^i, \partial_y^i$  and the gauge electric field  $E_a^i$  integrated along the path of heavy quark propagation.

We now have all the tools needed to study heavy-light systems on the lattice.

### III. Applications to Heavy-Light Systems

To apply this formalism<sup>11</sup>, consider any interpolating field for a heavy-light meson

$$\begin{aligned} \mathcal{O}(x_1, x_2) \Big|_{x_1^0=x_2^0} &\equiv \bar{\psi}_L(x_1) \Gamma^A P \\ \exp \left\{ ig \oint_{C_{x_2}}^{x_1} dz^\mu A_\mu^a(z) t_a \right\} \psi_H(x_2) \end{aligned} \quad (3.1)$$

with the quantum numbers of the ground state of interest  $|n\rangle$ . Here a nonlocal operator was used with  $\Gamma^A$  denoting spin and angular momentum structure required to couple to the state  $|n\rangle$ .  $C$  denotes any path between  $x_2$  and  $x_1$ . The matrix element

$$\langle 0 | \mathcal{O}(x_1, x_2; C) | n \rangle \quad (3.2)$$

can be thought of as a (path-dependent) wave function for a heavy quark at  $x_2$  and a light antiquark at  $x_1$ . Of course, this analogy cannot be carried very far. Since the light quark is fully relativistic light quark-antiquark pairs are unsuppressed.

The basic matrix element needed to study heavy-light mesons is the correlation function

$$\langle 0 | \mathcal{T} \mathcal{O}(x_1, x_2) \mathcal{O}^\dagger(y_1, y_2) | 0 \rangle \quad (3.3)$$

with  $x_1^0 = x_2^0 \gg y_1^0 = y_2^0$  and  $x_1^0 - y_1^0 \equiv T$ . In the limit that  $T$  becomes large, the correlation is dominated by the lowest mass state which couples to  $\mathcal{O}(x_1, x_2)$ , i.e.,  $|n\rangle$ ; and Eq. (3.3) becomes

$$\begin{aligned} \sum_{n_i} \langle 0 | \mathcal{O}(\vec{x}_1, \vec{x}_2) | n_i \rangle e^{-m_{n_i} T} \langle n_i | \mathcal{O}^\dagger(\vec{y}_1, \vec{y}_2) | 0 \rangle \\ \xrightarrow{T \rightarrow \infty} \langle 0 | \mathcal{O}(\vec{x}_1, \vec{x}_2) | n \rangle e^{-m_n T} \langle n | \mathcal{O}^\dagger(\vec{y}_1, \vec{y}_2) | 0 \rangle. \end{aligned} \quad (3.4)$$

The simplest example of this relation is for the pseudoscalar mesons. Suitable operators for all other low-lying ( $Q_H \bar{Q}_L$ ) systems are given in Reference 11.

The obvious interpolating operator for the ground state pseudoscalar meson ( $Q_H \bar{Q}_L$ ) is given by Eq. (3.1) with  $x_1 = x_2$  and  $\Gamma^A = \gamma_5$ . However, it is more useful in this discussion to take the zeroth component of the axial vector current as the interpolating operator so that  $\Gamma^A = \gamma^0 \gamma_5$ . For definiteness, choose the  $b$  quark

system. Then the pseudoscalar ( $Q_H \bar{Q}_L$ ) systems are ( $b\bar{u}$ ), ( $b\bar{d}$ ) and ( $b\bar{s}$ ). The matrix element of the zeroth component of the axial vector current  $\chi(x)$  which couples the  $b$  quark pseudoscalar meson state  $|B\rangle$  to the vacuum  $|0\rangle$  is related to the decay constant  $f_B$  by

$$-i \langle 0 | \chi(x) | B \rangle = f_B m_B. \quad (3.5)$$

Therefore, the correlation relation, Eq. (3.3), become

$$\begin{aligned} & \langle 0 | \mathcal{T} \chi(x) \chi^\dagger(y) | 0 \rangle_{\text{Euclidean Space}} \\ & \xrightarrow{T \rightarrow \infty} \frac{f_B^2 m_B^2}{2m_B} e^{-m_B T} \delta(\vec{x} - \vec{y}) \end{aligned} \quad (3.6)$$

In terms of the functional integral over the gauge fields, Eq. (3.6) is given by

$$\int [dA] \text{Tr} \left( \gamma^0 \gamma_5 S_H(x, y; A) \gamma^0 \gamma_5 S_L(y, x; A) \right) \frac{1}{[\det S_L]} e^{-S_{YM}} \quad (3.7)$$

where  $S_{YM}$  is the Euclidean action for the pure gauge field. In the quenched approximation, the  $[\det S_L]$  is replaced by  $\mathbf{1}$ .

Substituting our expression for  $S_H$  from Eq. (2.9), we have to lowest order in  $1/m_H$

$$\begin{aligned} & \int [dA] \text{Tr} \left[ \left( \frac{1 - \gamma^0}{2} \right) S_L(y, x; A) P \begin{pmatrix} x \\ y \end{pmatrix} \right] \delta(\vec{x} - \vec{y}) \\ & e^{-m_b T} e^{-S_{YM}} \\ & + \text{order}(1/m_H) \end{aligned} \quad (3.8)$$

Hence for the ground state pseudoscalar ( $Q_H \bar{Q}_L$ ) state (for example  $B$ ) the lattice quantity which must be computed is:

$$\begin{aligned} & \langle \text{Tr} \left[ \begin{pmatrix} \frac{1 - \gamma^0}{2} S_L(y, x) & x \\ & \Pi & U_{z, x+\hat{0}} \\ & z=y & \end{pmatrix} \right] \rangle_{\vec{x}=\vec{y}} \\ & \xrightarrow{T \rightarrow \infty} \frac{1}{2} f_B^2 m_B e^{-T(m_B - m_b)} \end{aligned} \quad (3.9)$$

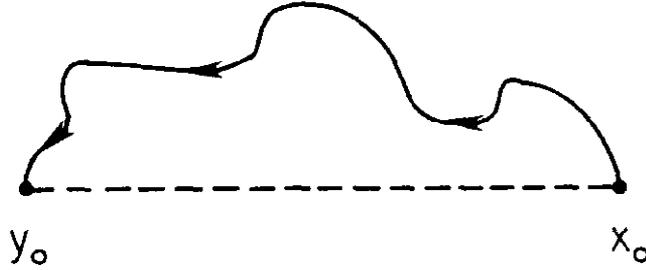


Figure 1: Correlation which determines for  $B$  mesons  $m_B - m_b$  and  $f_B$  on the lattice. The dotted line denotes a product of links along the time direction from  $y^0$  to  $x^0$  ( $x^0 > y^0$ ), and the solid line denotes the negative energy projection  $\frac{(1-\gamma^0)}{2}$  of the light quark propagator from  $x^0$  back to  $y^0$ .

The brackets  $\langle \rangle$  indicate an average over all gauge configurations weighted by the pure gauge action

$$\langle O \rangle \equiv \frac{\int [dU] O e^{-S_{YM}}}{\int [dU] e^{-S_{YM}}} \quad (3.10)$$

and the path ordered exponential  $P \left( \begin{smallmatrix} x^0 \\ y^0 \end{smallmatrix} \right)$  has been replaced with its lattice equivalent, the product of links in the time ( $\hat{0}$ ) direction from site  $y$  to site  $x$ . Graphically the quantity to be computed numerically on the lattice (Eq. (3.9)) is depicted in Figure 1. Since we are only working here to leading order in  $1/m_H$ , the pseudoscalar and vector ( $Q_H \bar{Q}_L$ ) are unsplit and the lattice calculation for  $m_B$  of Eq. (3.9) should be compared to the experimental center of gravity of these states. For the  $B_{u,d}$  mesons

$$m_B = \left( \frac{3m_{B^*} + m_B}{4} \right) = 5.320 \text{ GeV}/c^2 \quad (3.11)$$

The two physical quantities which can be calculated from Eq. (3.9) from fitting the results to an exponential function of time are:

1.  $m_B - m_b$  which can be determined from constant in the exponent.
2.  $f_B \sqrt{m_B}$  which can be determined from the square root of the prefactor of this

exponential.

These two quantities are independent of the heavy quark mass in the heavy quark limit, e.g., they would be the same as measured in the top system. One immediate consequence of this is that for  $(Q_H \bar{Q}_L)$  systems

$$f_{(HL)} \sim \text{constant} / \sqrt{m_{H\bar{L}}} \quad (3.12)$$

There is one additional complication in computing the physical decay constant  $f_B$ . In order to compare the lattice value of the decay constant to the physical value we must know the relation of the the local operator (in Eq. (3.5)) used on the lattice to evaluate the decay constant and the corresponding operator in the continuum. The physical decay constant is then given by

$$f_B^{\text{Physical}} = Z_A f_B^{\text{Lattice}} \quad (3.13)$$

where  $Z_A$  is the overall multiplicative renormalization constant for the local operator. Of course in lowest order of Q.C.D.  $Z_A = 1$ . In first order it is only necessary to compute the complete one loop corrections perturbative corrections to the operator on the lattice and in some continuum renormalization scheme. These corrections may be reasonably large for the value  $\beta = 5.7$  used in the Monte Carlo calculations presented below. The one-loop renormalization for  $f_\pi$  has been calculated by G. Martinelli and Zhang Li-Cheng<sup>12</sup>. A similar calculation can be performed for heavy-light systems.

Other quantities of interest in heavy-light system may also be computed using the heavy quark expansion<sup>11</sup>. One simple example is the mixing of neutral systems due to flavor changing neutral currents induced in the standard electroweak model. In particular, one needs to evaluate some operators,  $\mathcal{F}_i$ , which connect  $(Q_H \bar{Q}_L)$  and  $(\bar{Q}_H Q_L)$  neutral states. In the  $b$  quark systems, there are two such systems:  $(b\bar{d})$  to  $(\bar{b}d)$  mixing and  $(b\bar{s})$  to  $(\bar{b}s)$  mixing. I will illustrate how one would calculate matrix elements of the mixing operation in the example of the  $(b\bar{s})$  system  $B_s$ . A parameter  $\beta$  contains the strong interaction effects and is defined as the ratio of the expectation

$$\langle B_s | \mathcal{F}_{LL} | \bar{B}_s \rangle$$

of the left-left operator

$$\mathcal{F}_{LL} = \bar{\psi}_b \Gamma^\mu \psi_s \bar{\psi}_b \Gamma_\mu \psi_s \quad (3.14)$$

to the quantity obtained by assuming vacuum insertion. Here  $\Gamma^\mu = \gamma^\mu(1 - \gamma^5)$ . We will again work in lowest order; of course in one loop there is finite renormalization of the operator  $\mathcal{F}_{LL}$  defined on the lattice to convert it to the continuum. These one loop corrections to four Fermi operators have been investigated (for the standard fermion methods) by a variety of authors<sup>13</sup>.

To determine the  $\mathcal{B}$  parameter, we compute the correlation function

$$\langle 0 | \mathcal{T} \chi(x) \mathcal{F}_{LL}(0) \chi(y) | 0 \rangle_{x=y=0}, \quad (3.15)$$

where  $\chi(x) \equiv \bar{\psi}_s(x) \gamma^0 \gamma_5 \psi_b(x)$ . As  $x^0, -y^0$ , and  $x^0 - y^0 \equiv T$  become large, this correlation becomes

$$\xrightarrow{T \rightarrow \infty} \frac{8}{3} \left( \frac{m_B f_B^2}{2} \right)^2 \mathcal{B} e^{-m_B T} \quad (3.16)$$

and the  $\mathcal{B}$  parameter can be extracted. Substituting the expression for all the propagators involved in evaluating Eq. (3.15) and using the heavy quark expansion for the  $b$  quark propagator one can obtain the quantities which require calculation on the lattice to determine the  $\mathcal{B}$  parameter.

To expose the different physical contributions to the  $\mathcal{B}$  parameter, it is convenient to define a spin averaged propagation function in an external field  $A$ ,  $M_j^{i\ \mu}$  ( $i, j$  being color indices,  $\mu$  a Lorentz index) as follows:

$$M_j^{i\ \mu}(x^0, 0) \equiv \frac{1}{2} \text{Tr} \left[ \frac{1 - \gamma_0}{2} \Gamma^\mu S_L(0, x^0) P \begin{pmatrix} x^0 \\ 0 \end{pmatrix} \right]_j \quad (3.17)$$

for  $x_0 > 0$  and the same expression except for  $\frac{1-\gamma_0}{2} \rightarrow \frac{1+\gamma_0}{2}$  for  $x_0 < 0$ . Then  $\mathcal{B}$  is given by

$$\mathcal{B} = B_1 + B_2 \quad (3.18a)$$

where

$$B_1 = \lim_{\substack{x^0 \rightarrow \infty \\ y^0 \rightarrow -\infty}} \frac{\sum_\mu \langle M_i^{i\ \mu}(x^0, 0) M_j^{j\ \mu}(y^0, 0) \rangle}{\frac{4}{3} \langle M_i^{i\ 0}(x^0, 0) \rangle \langle M_j^{j\ 0}(y^0, 0) \rangle} \quad (3.18b)$$

and

$$B_2 = \lim_{\substack{x^0 \rightarrow \infty \\ y^0 \rightarrow -\infty}} \frac{\sum_{\mu} \langle M_j^i{}^{\mu}(x^0, 0) M_i^j{}_{\mu}(y^0, 0) \rangle}{\frac{4}{3} \langle M_i^i{}^0(x^0, 0) \rangle \langle M_j^j{}_0(y^0, 0) \rangle} \quad (3.18c)$$

where repeated indices  $i, j, \mu$  are summed and the expectation value is as defined in Eq. (3.10). The denominator is found by evaluating numerator in the vacuum saturation limit.

In the large  $N$  limit  $B_1 = 3/4$  plus a  $O(1/N^2)$  correction.  $B_1$  corresponds to an  $s$ -channel singlet-singlet operator.  $B_2$  equals  $1/4$  in the vacuum saturation limit, but Bardeen, Buras, and Gerard<sup>14</sup> have argued that the physical value of this term in the large  $N$  limit need not be close to  $1/4$ . Of course, the results of Eqs. (3.18) generalizes to any neutral heavy-light pseudoscalar system.

## IV. Numerical Results

For the heavy-light systems, in order to find numerically the quantities which determine the mass and decay constant of the pseudoscalar meson (Eq. (3.9)) and the  $B$  parameter (Eq. (3.15)), calculations were performed on an  $8^2 \times 16 \times 24$  lattice at  $\beta = 5.7$  for various values of  $\kappa$  near  $\kappa_{critical} = .170$ . The measurements were done in collaboration with J. Sexton and H. Thacker<sup>15</sup>. Gauge configurations were generated using the Cabibbo-Marinari Heat Bath method and the fermion propagators were inverted using the Gauss-Seidel method. There were 1500 initialization sweeps and 500 sweeps between configurations. In total, 30 fermion propagators were used to obtain the data. The result for the correlation of Eq. (3.9) is shown in Figure 2 for  $\kappa = .160$ . Using data at  $\kappa = .15, .16$  and  $.165$  and fitting the data to a exponential form  $Ae^{-CT}$  we can extract measurements of the value of  $m_B - m_b$  and  $f_B \sqrt{m_B}$  for the  $(Q_H \bar{Q}_L)$  systems with  $Q_L = u$  or  $d$  ( $\kappa \simeq \kappa_{critical}$ ) and  $Q_L = s$ . For example, using the experimental value of  $m_B$  (Eq. (3.11)), and the lattice as unit length,  $a = .2fm$  the value of the  $b$  quark mass is

$$m_b = 4.64 \pm .10 \text{ GeV}/c^2. \quad (4.1)$$

The error reported here is extracted by a single elimination jackknife method and is purely statistical.

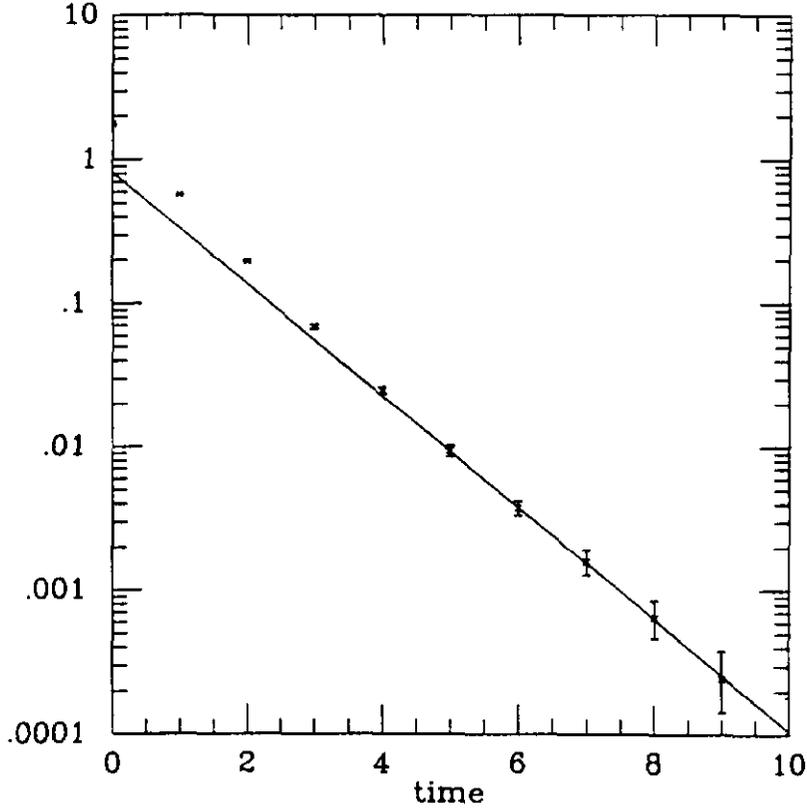


Figure 2: Correlation of Eq.(3.9) for  $\kappa = .160$  as a function of time in lattice units. The solid curve is the best fit to the form  $Ae^{-CT}$  for  $T > 4a$ .

Similarly, an analysis of the  $\beta$  parameter for  $B_d\bar{B}_d$  and  $B_s\bar{B}_s$  mixing can be extracted from the correlation of Eq. (3.18).

The result for the  $\beta$  parameter for the  $B_d$  system is

$$\begin{aligned}
 B_1 &= .70 \pm .06, & B_2 &= .20 \pm .04, \\
 & \text{and} & & \\
 \beta &= B_1 + B_2 & &= .90 \pm .10
 \end{aligned}
 \tag{4.3a}$$

and for the  $B_s$  system:

$$\begin{aligned}
 B_1 &= .73 \pm .06, & B_2 &= .22 \pm .04, \\
 & \text{and} & & \\
 \beta &= B_1 + B_2 & &= .95 \pm .10
 \end{aligned}
 \tag{4.3b}$$

Again the errors quoted here are purely statistical. A full description of our results and error analysis will be reported in Reference 15.

Of course, it must be remembered that these results are for  $\beta = 5.7$  and thus rather far from the continuum limit and that the one loop corrections to these quantities have not been included.

## V. Summary and Outlook

A method of treating heavy-light mesons by making a  $1/m_H$  expansion for the propagation of the heavy quark (as was done previously for heavy-heavy systems) and then computing the resulting terms in the expansion using lattice Monte Carlo methods has been investigated. The systematic expansion for heavy quarks ( $1/a \ll m_H$ ) in powers of  $1/m_H$  should allow the determination of all the physically relevant quantities for low-lying heavy light mesons to at least the same level of accuracy as the corresponding light quark states. In particular, the masses and decay constants can be calculated. An initial numerical analysis confirms the relevance of this method.

The systematics of the effective action for heavy quarks can be improved in perturbation theory to the needed multi loop accuracy. One method to do these perturbative calculations has been already proposed by Caswell and Lepage<sup>10</sup>. The required one loop renormalizations for the quantities discussed here are presently under calculation.

Finally, we have seen that this method is applicable to a variety of other calculations involving heavy-light systems. In particular, all the strong corrections to electroweak processes for these systems are amenable to simplified calculations. In particular, the  $\bar{B}$  parameter for  $B - \bar{B}$  mixing has been explicitly formulated in this method and an estimation been made using Monte Carlo methods.

By next year at this time I hope to be able to report more complete results for these as well as other physical observables of heavy-light systems and to have calculated the one loop corrections to the relation between the lattice measurements and associated physical quantities.

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