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Lattice Virasoro Algebra and Corner Transfer Matrices in the Baxter Eight-Vertex Model

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Abstract

A lattice Virasoro algebra is constructed for the Baxter eight-vertex model. The operator L_0 is obtained from the logarithm of the corner transfer matrix and is given by the first moment of the XYZ spin chain Hamiltonian. The algebra is valid even when the Hamiltonian includes a mass term, in which case it represents lattice coordinate transformations which distinguish between even and odd sublattices. We apply the quantum inverse scattering method to demonstrate that the Virasoro algebra follows from the Yang-Baxter relations.

Conformally invariant two-dimensional field theories exhibit an infinite degree of symmetry corresponding to invariance under analytic reparametrizations of the complex coordinate $z = x + iy$ (or $\bar{z} = x - iy$), where x and y are Euclidean coordinates. The algebraic structure of this symmetry is expressed in the Lie algebra of the conformal generators L_n and $\bar{L}_n$ associated with infinitesimal coordinate transformations $z \to z + \epsilon z^{n+1}$ and $\bar{z} \to \bar{z} + \bar{\epsilon}\bar{z}^{n+1}$. The operators L_n and $\bar{L}_n$ are constructed from moments of the stress-energy tensor and constitute a pair of commuting Virasoro algebras. [1]

Another type of infinite dimensional symmetry which occurs in some two dimensional theories is that of complete integrability. At the quantum level, the algebraic structure of complete integrability is most generally expressed in terms of Yang-Baxter relations [2,3] which are either trilinear relations between local interaction "vertices" or commutation relations for local operator-valued "L-matrices." In the quantum inverse method [4], the Yang-Baxter relations are used to determine the algebra of operators extracted from the monodromy matrix. These include a one-parameter set of commuting transfer matrices $T(v)$ (which acts as the generating function for an infinite number of conservation laws) as well as the mode creation and annihilation operators from which the Bethe-type eigenstates of $T(v)$ are constructed [4].

There are certain systems which exhibit both quantum integrability and conformal symmetry, for example, completely integrable lattice models (e.g. Ising or Baxter models) at the critical temperature. For these cases, it is reasonable to expect that the algebraic structure of the Yang-Baxter relations is closely related to that of the conformal Virasoro algebra. In this note, we will discuss the connection between Yang-Baxter relations and Virasoro algebras in the general framework of exactly solvable models and show that it is not restricted to critical cases. We demonstrate this by constructing a Virasoro algebra for the general 8-vertex model. The result is based on Baxter's method of corner transfer matrices [5], which has proved to be a powerful method for obtaining exact results in integrable lattice models. For the general eight-vertex model, we construct the Virasoro algebra in the diagonal representation of the corner transfer matrix (CTM). To trace its connection to the Yang-Baxter relations, we then consider the more restricted case of the six-vertex model with $|\Delta| > 1$ and explicitly construct the Virasoro operators in terms of the elements of the monodromy matrix of the quantum inverse scattering formalism. The interpretation of the CTM as the lattice generalization of a Lorentz boost or rapidity shift operator [6,7]plays a central role in this construction.

An important feature of the lattice Lorentz group is that it is compact due to the Brillouin zone periodicity of momentum space on a lattice. It is this periodicity which leads to the discreteness of the eigenvalues of L_0. Note that in the usual construction of a conformal Virasoro algebra at the critical point[8], operators are defined by integrals over closed contours in the complex z-plane ("radial quantization"), and integer-spaced eigenvalues of L_0 arise from periodicity under Euclidean rotations by 2π. Here we consider operators which act on infinite or semi-infinite rows of spins and are thus necessarily defined in a fixed-time quantization scheme. In the continuum theory, radial and fixed-time quantization are equivalent by a conformal transformation. However, at finite lattice spacing, this equivalence no longer holds. For this reason the Brillouin zone periodicity of the lattice is an essential component of our construction. In the elliptic function parametrization of the Baxter model, periodicity of boosts and Euclidean rotations are associated with real and imaginary values of the rapidity, respectively, and are related to each other by a Jacobi (conjugate-modulus) transformation which interchanges the two elliptic periods. There is thus a kind of dual relationship between the Virasoro algebra we construct here and the more familiar one at the critical point. However, our construction applies to the general case when $T \neq T_c$, i. e. when the spectrum has a mass gap. This occurs because the Baxter model is equivalent to a staggered lattice fermion system [9,10]. An investigation of the transformation properties of the fermion fields reveals that, in the massive case, the Virasoro generators induce coordinate transformations which distinguish between even and odd sublattices. A detailed discussion of the geometrical significance of the algebra, the role of mass, and other related issues will be given in a forthcoming paper.

We employ standard notation for the Boltzmann weight parameters a, b, c, and d of the eight-vertex model (c.f. Ref. [11]). In the "arrow" representation, a single vertex may be written as a two-spin operator which takes the lower and right arrows of the vertex to the left and upper arrows respectively,

$$V_j = \frac{1}{2}[a + d + (a-d)\sigma_j^z\sigma_{j+1}^z + (b+c)\sigma_j^x\sigma_{j+1}^x + (b-c)\sigma_j^y\sigma_{j+1}^y]. \qquad (1)$$

where σ_j^x, σ_j^y, and σ_j^z are Pauli spin matrices acting on the j^{th} spin. The corner transfer matrix (CTM) is defined by picking an origin at the center of the lattice and considering one quadrant of the lattice with fixed spins along the two edges. The elements of the CTM are labelled by the row and column of edge spins and are given by the partition function with edge spins fixed (weighted sum over all spins interior to the quadrant). Following Baxter [5,11] we denote a row of vertices by

$$G_j^{(n)} = V_n V_{n-1} V_{n-2} ... V_j \tag{2}$$

and write the CTM as

$$A^{(n)} = G_1 G_2 G_3 ... G_n \tag{3}$$

Baxter showed that, in the low temperature regime, the CTM with its largest eigenvalue normalized to unity has a well defined thermodynamic limit,

$$A = \lim_{n\to\infty} A^{(n)}/\lambda_0^{(n)} \tag{4}$$

where $\lambda_0^{(n)}$ is the largest eigenvalue of $A^{(n)}$.

Let us briefly review the properties of the operator A which were discovered by Baxter. We begin by introducing the elliptic function parametrization of the vertex weights as discussed in [11],

$$a = \frac{\operatorname{snh}(\lambda - u)}{\operatorname{snh}\lambda} \tag{5}$$

$$b = \frac{\operatorname{snh} u}{\operatorname{snh}\lambda} \tag{6}$$

$$c = 1 \tag{7}$$

$$d = k \operatorname{snh} u \operatorname{snh}(\lambda - u) \tag{8}$$

where k is the elliptic modulus. Following Baxter[5,11], we work in the "principle regime," $0 < k < 1$ and $0 < u < \lambda < K'$, where K' is a complete elliptic integral (quarter period). In the discussion of the quantum integrability of the Baxter model, the parameter u in (8) plays the special role of the spectral or lattice rapidity variable which parametrizes a set of commuting transfer matrices. Baxter found that the operator A could be diagonalized by a u-independent similarity transformation,

$A_D = PA(u)P^{-1}$, where the diagonal CTM has the remarkably simple form

$$A_D = \begin{pmatrix} 1 & 0 \\ 0 & e^{-u} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{-2u} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{-3u} \end{pmatrix} \otimes \cdots \tag{9}$$

Thus, the eigenvalues of $-\frac{1}{u}\log A$ are all non-negative integers. Baxter also showed that the operator $-\frac{1}{u}\log A$ could be written as the first moment of the XYZ Heisenberg spin chain Hamiltonian density,

$$-\frac{1}{u}\log A = \sum_{j=1}^{\infty} j\mathcal{H}_{XYZ}(j, j+1) \tag{10}$$

where

$$\mathcal{H}_{XYZ}(j, j+1) = J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z \tag{11}$$

and $J_x : J_y : J_z = ab + cd : ab - cd : \frac{1}{2}(a^2 + b^2 - c^2 - d^2)$. (Note that the logarithmic derivative of the row-to-row transfer matrix is the XYZ Hamiltonian, i.e. the zeroth moment of $\mathcal{H}_{XYZ}$.) In its diagonal form, obtained from (9), this may be written as

$$-\frac{1}{u}\log A_D = \sum_{l=1}^{\infty} \frac{l}{2}(\sigma_l^z + 1) \tag{12}$$

where σ_l^z is a Pauli matrix acting on a spin at site l.

By introducing the fermion variables

$$\tilde{\psi}(l) = \left(\prod_{i=1}^{l-1} i\sigma_i^z\right)\sigma_l^+ \tag{13}$$

(where $\sigma^+ = \sigma^1 + i\sigma^2$), the diagonal operator (12) may be written

$$-\frac{1}{u}\log A_D = \sum_{l=1}^{\infty} l\tilde{\psi}^\dagger(l)\tilde{\psi}(l) + const. \tag{14}$$

The boost operator is actually the direct product of an upper-left and a lower-right CTM, which corresponds to extending the sum in (14) from $-\infty$ to ∞ and normal ordering. We also modify this operator by adding a term proportional to $\frac{1}{2}$ the fermion number operator, giving

$$\tilde{L}_0 = \sum_{l=-\infty}^{\infty} (l + \frac{1}{2}) : \tilde{\psi}^\dagger(l)\tilde{\psi}(l) : + const. \tag{15}$$

Here the normal ordering is taken with respect to the lowest eigenstate which has all negative l modes filled.

In the interpretation of the CTM as a Lorentz boost or Euclidean rotation operator [6], u is the lattice analog of a rapidity or rotation angle parameter. It is this interpretation which provides the key to writing down the operators $\tilde{L}_n$ which are related to $\tilde{L}_0$ by a Virasoro algebra. For this purpose it is illuminating to write the classical conformal generators $l_n = z^{n+1}\frac{\partial}{\partial z}$ (where $z = x + iy$) in terms of an angle (rapidity) variable in momentum space. First define the Euclidean light cone momentum $p = p_x - ip_y$ conjugate to z, and then introduce a rapidity or angle variable α by letting $p = e^{-i\alpha}$. Thus, for $n \geq 0$

$$z^{n+1}\frac{\partial}{\partial z} \to (-i)^n p(\frac{\partial}{\partial p})^{n+1} \to i\frac{\partial}{\partial \alpha}(e^{i\alpha}\frac{\partial}{\partial \alpha})^n \tag{16}$$

The operator l_0, represented by $i\frac{\partial}{\partial \alpha}$, is diagonalized by eigenfunctions of the form $f_l(\alpha) = e^{i(l+\varepsilon)\alpha}$, where l is any integer and the value of ε is determined by boundary conditions connecting α to $\alpha+2\pi$. Choosing them to be antiperiodic, we take $\varepsilon = \frac{1}{2}$. The action of l_n on these functions is given by

$$l_n f_l(\alpha) = (l+\frac{1}{2})(l+\frac{3}{2})...(l+n+\frac{1}{2})f_{l+n}(\alpha) = \frac{\Gamma(l+n+\frac{3}{2})}{\Gamma(l+\frac{1}{2})}f_{l+n}(\alpha) \tag{17}$$

Comparing equations (15) and (17) for $n = 0$, we are led to introduce the following lattice operators in the Baxter model:

$$\tilde{L}_n = \sum_{l=-\infty}^{\infty} \frac{\Gamma(l+n+\frac{3}{2})}{\Gamma(l+\frac{1}{2})} : \tilde{\psi}^\dagger(l)\tilde{\psi}(l+n) : +h\delta_{n,0} \tag{18}$$

where the $\tilde{\psi}'s$ are as defined previously in eq.(13) and the constant h will be chosen below. Note that, although the correspondence between $z^{n+1}\frac{\partial}{\partial z}$ and (17) only holds for $n \geq 0$, we are taking (18) as the definition of $\tilde{L}_n$ for negative n as well. It is at this point that an important property of our Virasoro algebra arises: it does not satisfy the self-adjointness condition $\tilde{L}_n^\dagger \neq \tilde{L}_{-n}$. The unitarity argument which limits the allowed values of the central charge c does not apply because the norm determinant is not given by the Kac formula [12]. (In fact c turns out to be negative.) It is a straightforward exercise to verify that the operators (18) satisfy a Virasoro algebra,

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{19}$$

The nonzero value of the central charge c is an anomaly which comes about from filling the Dirac sea. It may be computed by standard methods [13]. The constant h in (18) is dictated by the form of (19). This gives

$$c = -2 \;\; h = -\frac{1}{8}. \tag{20}$$

It is interesting to note that the values (20) correspond to the first disallowed member of the Friedan, Qiu, Shenker sequence (the $m = 1, p = q = 0$ case in eq. (6) and (7) of Ref. [14]). That point was excluded by a positivity argument [15] which does not apply here because our algebra is not self-adjoint.

Equation (18) expresses the operators $\tilde{L}_n$ for the eight-vertex model in terms of the fermion operators $\tilde{\psi}(l)$ which were defined in the diagonal representation of the CTM. To clarify the connection between the algebra (19) and the Yang-Baxter relations, we would like to express the L_n's in the original arrow representation, i.e. to calculate $L_n = P^{-1}\tilde{L}_n P$. For this purpose it is useful to restrict our consideration to the six-vertex model with $|\Delta| > 1$. For this case, the quantum inverse scattering formalism [16] can be used to explicitly construct the Virasoro operators in terms of elements of the monodromy matrix. (It may be possible to carry out a similar procedure for the general case, but there are additional complications in the quantum inverse formalism for the general Baxter model.) The monodromy matrix $\mathcal{T}_N(v)$ is defined as a row of N vertices with free ends. Taking N to be even, we write

$$\mathcal{T}_N(v) = \tilde{V}_{-\frac{N}{2}}\tilde{V}_{-\frac{N}{2}+1}\ldots\tilde{V}_{\frac{N}{2}-1}\tilde{V}_{\frac{N}{2}} \equiv \begin{pmatrix} A_N(u) & B_N^*(u) \\ B_N(u) & A_N^*(u) \end{pmatrix} \tag{21}$$

Here the vertex $\tilde{V}_j$ is regarded as a 2×2 matrix of one-spin operators, rather than a two-spin operator,

$$\tilde{V}_j = \begin{pmatrix} w_4 + w_3\sigma_j^3 & w_1 - iw_2\sigma_j^2 \\ w_1 + iw_2\sigma_j^2 & w_4 - w_3\sigma_j^3 \end{pmatrix} \tag{22}$$

where

$$\begin{aligned} w_1 &= \tfrac{1}{2}(c+d) \quad & w_2 &= \tfrac{1}{2}(c-d) \\ w_3 &= \tfrac{1}{2}(a-b) \quad & w_4 &= \tfrac{1}{2}(a+b) \end{aligned} \tag{23}$$

For the six-vertex case we take $d = 0$. The row-to-row transfer matrix $T_N(u)$ is the trace of the monodromy matrix, $T_N = Tr\,\mathcal{T}_N = A + A^*$. The elements of the monodromy matrix possess algebraic properties which follow from the Yang-Baxter relations. In order to consider the algebraic relation between the monodromy matrix and the CTM, we need to take the infinite volume limit, $N \to \infty$ (see Ref. [4]. In this limit the algebra of A and B operators simplifies to the following:

$$A(u)B(v) = \frac{\sinh(u-v+\lambda)}{\sinh(u-v)} B(v)A(u) \tag{24}$$

$$A^*(u)B(v) = \frac{\sinh(u-v-\lambda)}{\sinh(u-v)} B(v)A^*(u) \tag{25}$$

$$B(u)B^*(v) = \frac{\sinh(u-v+\lambda)\sinh(u-v-\lambda)}{\sinh^2(u-v)} B^*(v)B(u) \tag{26}$$

$$+A^*(u)A(u)\delta(u-v) \tag{27}$$

$$[A(u), A(v)] = [A(u), A^*(v)] = [B(u), B(v)] = 0 \tag{28}$$

In Ref. [6] it was shown that the CTM acts as a rapidity shift operator when applied as a similarity transformation to the row-to-row transfer matrix. This follows simply from the commutation relation

$$\left[\mathcal{H}_{XYZ}(j,j+1), \tilde{V}_j(u)\tilde{V}_{j+1}(u)\right] = \tilde{V}_j(u)\overleftrightarrow{\partial_u}\tilde{V}_{j+1}(u) \tag{29}$$

which is obtained from the Yang-Baxter relations. From the commutator(29) it is easy to show that L_0 generates a rapidity shift when commuted with the monodromy matrix,

$$[L_0, \mathcal{T}(u)] = \frac{\partial}{\partial u}\mathcal{T}(u) \tag{30}$$

Next we construct the reflection operator

$$R^*(u) = B(u)A^{-1}(u) \tag{31}$$

which is the creation operator for normalized eigenstates of the row-to-row transfer matrix. The commutation relations of the operators R and R^* involve two-body phase shift factors. These factors may be removed by defining [17]

$$\chi(u) = R(u)\exp\left(i\int_{-\pi}^{u} dv\Theta(u-v)R^*(v)R(v)\right) \tag{32}$$

where $\Theta(v)$ is the two-body phase shift,

$$\tan\tfrac{1}{2}\Theta(u) = -i\tanh u \coth\lambda \tag{33}$$

From the algebra of the A and B operators (28), it follows that $\chi(u)$ satisfies canonical anticommutation relations,

$$\{\chi(u), \chi^\dagger(v)\} = \delta(u-v) \tag{34}$$

$$\{\chi(u), \chi(v)\} = \{\chi^\dagger(u), \chi^\dagger(v)\} = 0 \tag{35}$$

From the commutation relation (30) we find that $\chi(u)$ also has the rapidity shift property

$$[L_0, \chi(u)] = \frac{\partial}{\partial u}\chi(u) \tag{36}$$

Finally we introduce the Fourier transformed operators

$$\psi(l) = \int_{-\pi}^{\pi} \frac{du}{2\pi} e^{i(l+\frac{1}{2})u} \chi(u), \tag{37}$$

which have the property

$$[L_0, \psi(l)] = i(l + \frac{1}{2})\psi(l). \tag{38}$$

From this commutation relation, we conclude that L_0 is given in terms of $\psi(l)$ by the expression (18) with the tilde's removed, and that therefore, $\psi(l) = P^{-1}\tilde{\psi}P$. This explicitly defines the transformation which diagonalizes the CTM and allows us to express the Virasoro operators in the arrow representation.

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