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Partial-Wave Unitarity and Closed-String Amplitudes

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ABSTRACT

The high-energy unitarity of a closed-string four-scalar tree amplitude is investigated using partial-wave analysis. A general argument is presented that such an amplitude in any known string theory will violate unitarity at sufficiently large s when perturbing around flat space-time of dimension six or more. The claim is checked explicitly on the four-dilaton tree amplitude of the superstring SST II in ten dimensions. The troublesome domain of interactions is the Regge region, effectively with one-graviton exchange in the t channel. It appears to be challenging to demonstrate explicitly the unitarity of string theories at high energies.



1. Introduction

Although the low-energy behavior of a string theory¹ presumably is entwined in compactification, the limiting high-energy behavior could simply be that of the perturbation expansion developed about the flat space-time of critical dimension. As a preliminary check of this possibility, one can examine the consistency of the perturbation expansion of a string theory about flat space-time at high energies. Partial-wave unitarity, when applicable, is a powerful constraint that a perturbation expansion should satisfy at each order if it is to be well-defined. The constraint is particularly stringent at high energies. For example, a well-known demonstration² that the Higgs and longitudinally-polarized weak gauge boson sector of the standard model is strongly coupled if the Higgs mass is above about 1 TeV follows from the violation of the $J=0$ partial-wave unitarity condition by the relevant tree amplitudes at large s . It will be argued that partial-wave unitarity is a useful tool in studying string perturbation theory.

In this letter, the partial-wave unitarity conditions are formulated in higher dimensions. Infrared divergences in less than five space-time dimensions invalidate the partial-wave expansion in theories of gravitation. Partial-wave unitarity should hold in flat space-time of dimension six or greater; it is not yet determined if it should hold in five dimensions also.

Because the graviton is spin-2 and massless, single graviton exchange in the t channel typically leads to the growth of a tree-level two-to-two elastic amplitude as s^2/t when $s \rightarrow \infty$ and t is fixed near 0. Because of the absence of infrared divergences there is no difficulty in demonstrating that such growth necessarily leads to unitarity violation at large s in six or more dimensions. Indeed, it has long been known that the exchange of a particle with spin greater than 1

can lead to troubles with unitarity at small t ³. Apparently, it has not been appreciated before that the problem is relevant to gravitation, and that it is sensitive seemingly to the number of flat dimensions. As an illustration of the issue, the four-dilaton scattering amplitude of the closed superstring SST II in ten dimensions is examined. It is found that the expected violation of unitarity occurs at sufficiently large s . Note that the absence or decoupling of negative norm states does not imply unitarity at high energies. Evidently, an explicit demonstration of the unitarity of (closed-) string theories is needed.

2. Partial-Wave Unitarity and Graviton Exchange

Consider the scattering of four massless identical scalars in flat D -dimensional space-time. The two incoming (outgoing) momenta are labeled by p_1 and p_2 (p_3 and p_4). With $p_1 + p_2 = p_3 + p_4$, define $s = 2p_1 \cdot p_2$, $t = -2p_1 \cdot p_4$, and $u = -s - t = -2p_1 \cdot p_3$. The four-point elastic amplitude is $T_{el}(s, t)$. Work in the center of mass frame. The kinematics of s -channel scattering are simply described by $s > 0$ and one scattering angle, θ , in terms of which $t = -s \sin^2(\theta/2)$. As illustrated in fig.1 $\cos\theta = \hat{i} \cdot \hat{f}$. The amplitude typically can be expanded in an appropriate set of orthogonal polynomials,

$$T_{el}(s, t) = \lambda_D s^{2-D/2} \sum_{l=0}^{\infty} \frac{1}{N_l^\nu} C_l^\nu(1) C_l^\nu(\cos\theta) a_l(s) \quad (1)$$

where $\nu = (D - 3)/2$, $C_l^\nu(x)$ is the Gegenbauer polynomial of order ν and degree l , and

$$\lambda_D = 2 \Gamma\left(\frac{D}{2} - 1\right) (16\pi)^{\frac{D}{2}-1} \quad (2)$$

is a constant included for later convenience. The Gegenbauer polynomials satisfy⁴

$$\int_{-1}^1 dx (1-x^2)^{\frac{D}{2}-2} C_l^\nu(x) C_l^\nu(x) = N_l^\nu \delta_{ll'} = \frac{2^{1-2\nu} \pi \Gamma(l+2\nu)}{l!(\nu+l)\Gamma^2(\nu)} \delta_{ll'}, \quad (3)$$

and are chosen for the expansion because the phase-space factor for θ in D dimensions is $\sin^{D-3}\theta d\theta$.

The unitarity condition on the partial wave $a_l(s)$ derives from the unitarity of the S -matrix⁵. Set $S = 1 + iT$, and

$$\langle j | T | k \rangle = \langle j | T | k \rangle (2\pi)^D \delta^D(p_j - p_k). \quad (4)$$

Then,

$$\langle f | T | i \rangle - \langle i | T | f \rangle^* = i \sum_n \langle f | T | n \rangle \langle i | T | n \rangle^* (2\pi)^D \delta^D(p_n - p_i). \quad (5)$$

Note that a conventional S -matrix should exist in higher dimensions even when massless particles are present because of the phase-space suppression of infrared divergences.

To translate eqn.(5) into a statement on the partial waves requires several manipulations. Specializing to the case where $|i\rangle$ and $|f\rangle$ describe elastic scattering specified by s and θ ,

$$\langle f | T | i \rangle - \langle i | T | f \rangle^* = 2i \text{Im} T_{el}(s, t). \quad (6)$$

Furthermore, the sum over intermediate states can be divided in a helpful way. Given a particular intermediate state, there is a set of states related to it just

by an overall rotation of the given intermediate state relative to \hat{i} . Then any member of the set is distinguished by a direction \hat{n} , or equivalently by a rotation from \hat{i} , $R(\Omega_n)$.

$$\begin{aligned} & \sum_n \langle f|T|n\rangle \langle i|T|n\rangle^* \delta^D(p_n - p_i) \\ &= \sum_n' \int d\Omega_n \langle f|T|n; \Omega_n\rangle \langle i|T|n; \Omega_n\rangle^* \delta^D(p_n - p_i) \end{aligned} \quad (7)$$

where the primed sum runs only over intermediate states which are not related by an overall rotation. A partial-wave expansion can be made for each of the matrix elements, e.g.

$$\langle i|T|n; \Omega_n\rangle = \lambda_D s^{2-D/2} \sum_{l=0}^{\infty} \frac{1}{N_l^\nu} C_l^\nu(1) C_l^\nu(\cos\theta_{in}) a_l(s, \{n\}'), \quad (8)$$

where $\cos\theta_{in} = \hat{i} \cdot \hat{n}$, and $\{n\}'$ are invariants independent of θ_{in} which complete the specification of the intermediate state. An analogous formula holds for $\langle f|T|n; \Omega_n\rangle$.

The unitarity equation becomes

$$\begin{aligned} & 2 \sum_{l=0}^{\infty} \frac{1}{N_l^\nu} C_l^\nu(1) C_l^\nu(\cos\theta) \text{Im} a_l(s) = (2\pi)^D \lambda_D s^{2-D/2} \sum_n' \delta^D(p_n - p_i) \\ & \sum_{l,l'=0}^{\infty} \frac{1}{N_l^\nu} C_l^\nu(1) \frac{1}{N_{l'}^\nu} C_{l'}^\nu(1) a_l(s, \{n\}') a_{l'}^*(s, \{n\}') \int d\Omega_n C_l^\nu(\cos\theta_{fn}) C_{l'}^\nu(\cos\theta_{in}). \end{aligned} \quad (9)$$

(The suppression of singularities at $\cos(\theta_{fn}$ or $\theta_{in}) = \pm 1$ by the integration measure justifies, somewhat naively, the interchange of integration and summation for gravitation when $D \geq 6$.) Note that $\cos\theta_{fn} = \cos\theta \cos\theta_{in} + \sin\theta \sin\theta_{in} \cos\phi_{fn}$

as indicated in fig.2. Using the addition formula for Gegenbauer polynomials⁴

$$\begin{aligned} \Gamma^2(\nu) C_l^\nu(\cos\theta_{fn}) &= \Gamma(2\nu - 1) \sum_{m=0}^l 4^m \Gamma(l - m + 1) \Gamma^2(\nu + m) (2\nu + 2m - 1) \\ & [\Gamma(l + 2\nu + m)]^{-1} (\sin\theta \sin\theta_{in})^m C_{l-m}^{\nu+m}(\cos\theta) C_{l-m}^{\nu+m}(\cos\theta_{in}) C_m^{\nu-\frac{1}{2}}(\cos\phi_{fn}) , \end{aligned} \quad (10)$$

and the property⁴

$$\int_0^\pi d\phi \sin^{D-4}\phi C_m^{\frac{D}{2}-2}(\cos\phi) = 2^{-2\nu+1} \pi \frac{\Gamma(2\nu)}{\Gamma^2(\nu + \frac{1}{2})} \delta_{m0}, \quad (11)$$

the angular integral is

$$\int d\Omega_n C_l^\nu(\cos\theta_{fn}) C_l^\nu(\cos\theta_{in}) = (4\pi)^\nu \frac{\Gamma(l+1)\Gamma(\nu)}{\Gamma(l+2\nu)} N_l^\nu C_l^\nu(\cos\theta) \delta_{ll}. \quad (12)$$

After using the identity⁴

$$C_l^\nu(1) = \frac{\Gamma(l+2\nu)}{\Gamma(l+1)\Gamma(2\nu)}, \quad (13)$$

and identifying terms proportional to $C_l^\nu(\cos\theta)$ in eqn.(9), one finds

$$Im a_l(s) = 8(2\pi)^{2D-2} \left(\frac{s}{4}\right)^{2-D/2} \sum_n' \delta^D(p_n - p_i) |a_l(s, \{n\}')|^2. \quad (14)$$

The right side of eqn.(14) is a sum of positive terms. The restriction of the sum to the elastic contribution yields the basic partial -wave unitarity conditions,

$$Im a_l(s) \geq |a_l(s)|^2. \quad (15)$$

It is straightforward to check that when equality holds for all l , $\sigma_{\text{total}} = \sigma_{\text{elastic}}$.

As an immediate corollary for any l , $(Im a_l - \frac{1}{2})^2 + (Re a_l)^2 \leq \frac{1}{4}$ or

$$|a_l(s)| \leq 1 . \quad (16)$$

Convenient forms for $a_l(s)$ are

$$a_l(s) = \frac{s^{\frac{D}{2}-2}}{\lambda_D C_l^\nu(1)} \int_0^\pi d\theta \sin^{D-3}\theta C_l^\nu(\cos\theta) T_{el}(s, t) , \quad (17)$$

$$a_l(s) = \frac{2^{D-3} s^{\frac{D}{2}-3}}{\lambda_D C_l^\nu(1)} \int_{-s}^0 dt \left(-\frac{t}{s} - \frac{t^2}{s^2} \right)^{\frac{D}{2}-2} C_l^\nu(1 + 2t/s) T_{el}(s, t). \quad (18)$$

Dimensional analysis requires that the factor of $s^{\frac{D}{2}-2}$ be present in eqn.(17) because $a_l(s)$ is dimensionless (if e.g. eqn.(16) is to be sensible) and the dimension of a four-point amplitude in D dimensions is $[mass]^{4-D}$.

It is now possible to discuss a general feature of the unitarity of theories of gravitation in higher dimensions. Given that the graviton is spin-2 and massless, the tree approximation to $T_{el}(s, t)$ is expected to behave as s^2/t as $s \rightarrow \infty$ for t fixed near 0 (due to one graviton exchange in the t -channel). This behavior typically holds over an interval from $t = 0$ to a lower limit, denoted by $-\epsilon(s)$, which is not strongly dependent on s . Then, from eqn.(18), in tree-level about flat space-time

$$a_l(s) \sim s \int_{-\epsilon(s)}^0 dt (-t)^{\frac{D}{2}-3} + \dots . \quad (19)$$

In particular, strings are essentially Reggeized even at tree-level. The graviton Regge trajectory is dominant at tree-level for closed strings. As long as the graviton Regge trajectory is differentiable at $t = 0$, $\epsilon(s)$ is proportional to $\frac{1}{\ln s}$ and the exhibited power behavior will be modified only by logarithms.

If the original expansion, eqn.(1), is valid, then the integral over t in eqn.(19) is convergent and partial-wave unitarity embodied in eqn.(16) will be violated at large s roughly by a power of s . Counting powers, the full amplitude should be no more singular than $1/t$ at $t = 0$, corresponding to massless particle exchange. Clearly, the partial-wave expansion fails for $D = 4$ and below due to infrared divergences. The conditions of unitarity must be formulated differently in this case. The expansion, eqn.(1), is certainly valid for $D > 6$ because an elastic amplitude no more singular than $1/t$ at small t is square-integrable with respect to the appropriate weight function. In fact, the expansion is also valid for $D = 6$; it is $D = 5$ which is the marginal case. The issue of whether the partial-wave expansion and the inequality, eqn.(16), hold in five-dimensional flat space-time is beyond the scope of this letter. In general, then, it is extremely difficult for any theory of gravity in six or more dimensions to avoid unitarity violation at large s in a four-scalar tree amplitude calculated about flat space-time.

Tachyon-free string theories provide a good testing ground for the general conclusions above since the integral which projects out the l th partial wave is well-defined. As an illustration, the four-dilaton tree amplitude of the type II closed superstring will be analyzed in ten-dimensional flat space-time. The amplitude can be computed from results in the literature. The four-point tree amplitude for massless bosons in the type II superstring is given in ref.7 in terms of the polarization tensor $\zeta^{\mu\nu}$. The four-dilaton amplitude is obtained by setting $\zeta^{\mu\nu}(k)$ equal to $\eta^{\mu\nu} - (k^\mu \bar{k}^\nu + k^\nu \bar{k}^\mu)/k \cdot \bar{k}$ where $\eta^{\mu\nu}$ is the ten-dimensional flat-space metric tensor and $\bar{k}^\mu = (k^0, -\vec{k})$ if $k^\mu = (k^0, \vec{k})$.

$$T_{et}(s, t) = \frac{\kappa^2}{64} (s^4 + t^4 + u^4) \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} \quad (20)$$

The coupling constant is κ , and α'_{closed} has been set equal to $1/4$, half the open string value. Using familiar properties of the gamma function, eqn.(20) becomes

$$T_{el}(s, t) = \frac{\kappa^2}{64} [s^4 + t^4 + (s+t)^4] \frac{\sin(\pi \frac{t}{8}) \sin[\pi(\frac{s}{8} + \frac{t}{8})]}{\pi \sin(\pi s/8)} \left\{ \frac{\Gamma(-\frac{t}{8})\Gamma(\frac{s}{8} + \frac{t}{8})}{\Gamma(1 + s/8)} \right\}^2. \quad (21)$$

The tree amplitude has poles at $s = 8N$, where N is any nonnegative integer. This reflects the existence of massless and massive states coupling as s -channel resonances of zero width. The presence of zero-width poles at large s is inconsistent with Regge behavior and, indeed, unitarity. As in field theory, it is expected that loop corrections will give the massive states widths effectively. The widths should eliminate this source of unitarity violation as well as yield good Regge behavior. Actually, the amplitude already exhibits Regge behavior along the ray $s = \tilde{s}(1 + i\epsilon)$, where \tilde{s} is real⁸. The limiting procedure, which is adopted below, defined by taking \tilde{s} to be large and then $\epsilon \rightarrow 0^+$ gives the replacement $exp(-i\pi t/8)$ for $\sin[\pi(s+t)/8]/\sin(\pi s/8)$ in eqn.(21). The resulting amplitude has Regge behavior with a specific signature factor.

$$T_{el}(s, t) \rightarrow \kappa^2 \frac{128}{\pi} \sin(\pi t/8) e^{-i\pi t/8} \Gamma^2(-t/8) \left(\frac{s}{8}\right)^{2+t/4} \quad (22)$$

It should be remarked that this subtlety has no real bearing on the final result for $a_l(s)$ at large s .

The expression, eqn.(18), is a convenient form with which to begin the partial-wave analysis. Change variables to $y = -\frac{1}{8}t$; use the symmetry⁴ $C_l^\nu(z) = (-)^l C_l^\nu(-z)$ and the crossing symmetry of the amplitude to write

$$a_l(s) = -\frac{\kappa^2}{\lambda_{10}} \frac{1 + (-)^l}{C_l^\nu(1)} \frac{8^6}{\pi} 2s \int_0^{s/16} dy y \sin(\pi y) e^{i\pi y} \Gamma^2(1+y) \frac{\Gamma^2(\frac{s}{8} - y)}{\Gamma^2(\frac{s}{8})} \quad (23)$$

$$\left(1 - \frac{8}{s}y\right)^3 \left[1 + \left(\frac{64}{s^2}\right)^2 y^4 + \left(1 - \frac{8}{s}y\right)^4\right] C_l^\nu\left(1 - \frac{16}{s}y\right).$$

The amplitude has been modified as discussed above after crossing has been used because Regge behavior should be consistent with crossing.

The leading behavior in s of the integral can be found readily. The gamma function has the representation for $x > 0$ ⁶

$$\Gamma(x) = \sqrt{2\pi} [1 + r(x)] e^{(x-\frac{1}{2})\ln x - x} \quad (24)$$

where

$$r(x) = e^{\omega(x)} - 1, \quad (25)$$

$$\omega(x) = \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) e^{-tx}.$$

The inequality $|\omega(x)| \leq \frac{1}{12x}$ holds. Throughout the region of integration the two gamma functions $\Gamma(\frac{s}{8} - y)$ and $\Gamma(\frac{s}{8})$ can be replaced by their asymptotic forms. The other gamma function is written conveniently as

$$\Gamma(1 + y) = \left\{ 1 + \frac{\sqrt{2\pi}}{e} [r(1 + y) - r(1)] \right\} e^{(y+\frac{1}{2})\ln(1+y) - y}. \quad (26)$$

Using eqn.'s (23), (24), and (26), for large s

$$a_l(s) = -\frac{\kappa^2}{\lambda_{10}} \frac{1 + (-)^l}{C_l^\nu(1)} \frac{8^6}{\pi} 2s \int_0^{s/16} dy y \sin(\pi y) f_l^\nu(y) e^{-2y \ln s} [1 + O(1/s)] \quad (27)$$

where

$$f_l^\nu(y) = e^{i\pi y} \left(1 - \frac{8}{s} y \right)^2 \left\{ 1 + \frac{\sqrt{2\pi}}{e} [r(1 + y) - r(1)] \right\}^2$$

$$\left[1 + \left(\frac{64}{s^2} \right)^2 y^4 + \left(1 - \frac{8}{s} y \right)^4 \right] C_l^\nu \left(1 - \frac{16}{s} y \right) \quad (28)$$

$$\exp \{ 2[(y + 1/2)\ln(y + 1) - (y - s/8)\ln(1 - \frac{8}{s}y) + y \ln 8] \}.$$

The integrand is damped exponentially in the upper end of the integration range, corresponding to fixed-angle scattering. Integration by parts, then, in the factor

$e^{-2y \ln s}$ gives a good expansion in $\frac{1}{\ln s}$ when l is not too large. Dropping terms exponentially suppressed in s ,

$$\int_0^{s/16} dy y \sin(\pi y) f_l^\nu(y) e^{-2y \ln s} = \frac{\pi}{4 \ln^3 s} f_l^\nu(0) + \frac{1}{8 \ln^3 s} \int_0^{s/16} dy e^{-2y \ln s} \frac{d^3}{dy^3} [y \sin(\pi y) f_l^\nu(y)] . \quad (29)$$

The remaining integral can be bounded by dividing the region of integration into two pieces, about say $y = 1$. In bounding the remainder it is helpful to bear in mind inequalities which may be proven from eqn.(25) :

$$\left| \frac{d^n}{dx^n} r(x) \right| \leq \frac{k_n}{x^{n+1}} e^{\frac{1}{12x}} \quad \text{for } x \geq 1 \text{ and } n = 1, 2, 3. \quad (30)$$

The magnitude of the integral over the interval $[0,1]$ is less than $K/(\ln s)$. The integrand in the interval $[1, s/16]$ is easily bounded by a positive function largest in magnitude at $y = 1$ where it is proportional to $1/s^2$; therefore, the magnitude of the integral over this interval is less than \tilde{K}/s . The positive constants k_n , K , and \tilde{K} need not be specified.

The limiting behavior for $a_l(s)$ at large s when l is not too large follows upon combining eqn.'s (27),(28), and (29).

$$a_l(s) \rightarrow -\kappa^2 \frac{1}{3\pi^4} [1 + (-)^l] \frac{s}{\ln^3 s} [1 + O(\frac{1}{\ln s})] \quad (31)$$

One power of s divided by $\ln^3 s$ is the form argued for previously on general grounds. The leading correction dependent on l comes from a further integration by parts of the remainder in eqn.(29). It is of order $\frac{s}{\ln^4 s} \frac{C_l^{\nu'}(1)}{s C_l^\nu(1)}$, or approximately

suppressed by $\frac{l^2}{s \ln s}$ relative to the leading term in eqn.(31). For $l \gtrsim (s \ln s)^{\frac{1}{2}}$ eqn.(31) is no longer accurate.

Although unitarity in its most stringent form, eqn.(16), is violated at large s by the lowest order result, perturbative unitarity undoubtedly holds. Using eqn.'s (15) and (31), perturbative unitarity implies that the imaginary part of $a_l(s)$ at one-loop should grow as s^2 (or faster). The implied growth of s^2 in $Im a_l(s)$ at one-loop (at least from the elastic contribution) due to two-graviton exchange in the t channel is consistent with the one-loop amplitude behaving roughly as s^3 for t fixed near 0. This is precisely the behavior which should arise from the two-graviton Regge cut^{3,9}. In the language of Regge theory, the graviton is supercritical; the expected behavior for large s of the $(n-1)$ -loop amplitude (effectively with n -graviton exchange in the t channel) is roughly s^{n+1} when t is fixed near 0. In fact, the original motivation for this work was the belief¹⁰ that unitarity would be violated if an amplitude grows so rapidly in the Regge limit.

3. Conclusions

Although a detailed calculation has been performed only for one closed-string theory, a general argument has been presented that the four-point tree amplitudes of any known closed-string theory formulated about flat space-time of dimension six or more will violate unitarity at sufficiently large s . It is then safe to conclude that the string perturbation expansion about the flat space-time of critical dimension is strongly coupled at high energies. The troublesome domain of interactions, the Regge region, is not short-distance dominated so that compactification can complicate the interpretation of the result. The result suggests that any examination of quantum gravitational effects in string theories (e.g.

high-temperature properties) should be carefully studied to check whether the calculations presently possible are subject to large higher-order corrections.

In any case, the unitarization of string perturbation theory at high energies should be understood. Making a comparison to weak-interaction theory, the unitarity violation in the perturbation expansion by powers of s in string theory is not analogous to that in Fermi theory where the fixed-angle scattering behavior is bad. String amplitudes fall exponentially at fixed angles; correspondingly,¹¹ the amplitudes apparently are finite in the ultraviolet. Unitarization in the present context probably should be associated with long-distance physics. It is possible that unitarity in flat space-time of critical dimension can be restored by an appropriate summation of the perturbation series. A more interesting though quite speculative possibility is that the unitarity conditions are more easily satisfied when the number of uncompactified dimensions is less than or perhaps equal to five because of infrared divergences. Then the compactification of at least five spatial dimensions in a string theory with critical dimension ten might be part of the unitarization of the theory. Further work should elucidate the issue.

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FIGURE CAPTIONS

1. Kinematics of elastic scattering.
2. Illustration of angles specifying \hat{n} .

Fig. 1

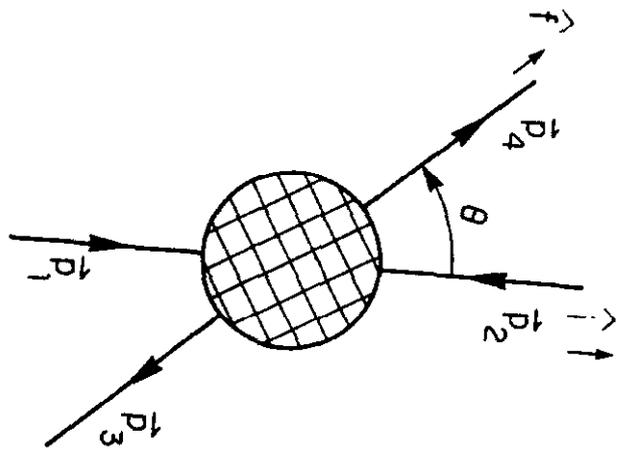


Fig. 2

