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The Fate of the m_b/m_τ Mass Relation in Superstring Theories

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ABSTRACT

We consider the low energy limit of the $E_8 \times E_8'$ heterotic superstring theory compactified on non-simply connected Calabi-Yau manifolds. We then determine what the consequences are of requiring that the m_b/m_τ mass relation be due to a Clebsch-Gordan relation as it is in traditional grand unified models.

(I) -- Introduction

One of the tests that superstring theory must pass is that it give rise to a reasonable "low energy" phenomenology. In principle, this would mean that we must be able to compute all of the couplings such as those in the superpotential (yielding Yukawa couplings, scalar couplings, etc...) from stringy first principles. However, some of these problems are far too hard to be completely solved with the technology currently at hand. Thus, we must limit ourselves, for the immediate present, to asking more general questions about the phenomenology implied by superstrings.

In this paper we consider fermion mass relations in the $E_8 \times E_8'$ heterotic string model. In particular we ask the following question: when can the m_b/m_τ mass relation still be expressed as a ratio of Clebsch-Gordan coefficients as in traditional grand unified models [1]?

Recall that the greatest successes of grand unification were the predictions of $\sin^2\theta_W$ and m_b/m_τ at low energies, starting from Clebsch-Gordan relations at the unification scale*. Witten [2] has already shown that the standard [3] renormalization group calculation of $\sin^2\theta_W$ can still be used in the context of superstring theories. What we want to know is whether the Clebsch-Gordan relations that determined m_b/m_τ can still hold in superstring theories. Let us say at the outset that there is another interesting possibility that we will not analyze here. This is that the relations between Yukawa couplings that give rise to the m_b/m_τ mass ratio arise purely from properties of the manifold since the Yukawa couplings can be calculated once the

*Of course, as the Higgs sector became more complicated in the course of the development of grand unified models these quantities were viewed less as successes and more as guides to determining which models were phenomenologically viable. The requirement, in such more complicated models, that, for example, m_b/m_τ have the correct value at unification, and that it arise naturally (generally as a consequence of a Clebsch-Gordon relation) put constraints on the structure of the model being considered.

manifold is specified* [4]. The problem that we are considering in this paper however is the one that we have just stated.

How do superstring theories differ from grand unified models in their low energy structure? To answer this let us follow the program of refs. [2, 6] and compactify the heterotic $E_8 \times E_8'$ superstring [7] on non-simply connected Calabi-Yau manifolds [8] of the form K_0/G. Here K_0 is a simply connected Calabi-Yau manifold, G is a discrete group of transformations that acts freely and holomorphically on K_0 and K_0/G is the manifold constructed by, for each $x \in K_0$, identifying the points x and gx for all $g \in G$: points on the manifold K_0/G are the equivalence classes, [x], under this identification. It is easy to see that K_0/G is non-simply connected and that $\Pi_1(K_0/G) \cong G$.

If we now consider <u>single valued</u> (zero mode) particle fields on K_0/G, we find [2, 9] that they can be replaced by particle fields on K_0 that satisfy a boundary condition of the form:

$$\Psi(x_0)=\mathbf{U}_g^{\dagger}\Psi(gx_0) \qquad ...(1)$$

for all $g \in G$. Here, Ψ is a particle field on K_0 and $\mathbf{U}_g$ is an element of the gauge group H (E_6 in the case that we shall deal with mostly). Note that $g \rightarrow \mathbf{U}_g$ is a group homomorphism of G into a <u>discrete</u> subgroup $\overline{G}$ of H. The $\mathbf{U}_g$'s can be shown to be related to the Wilson <u>lines</u> on K_0:

$$\mathbf{U}_g=P\ \exp[\ i\int_{\Gamma_g} A_m(x)\, dx^m] \qquad ...(2)$$

where $A_m(x)$ is a gauge configuration on K_0 (which is the pullback of a gauge

*Such calculations have been performed [5] and generally depend upon a number of (as yet) unknown parameters. These parameters (which fix the manifold) should, in principle, be calculable in the "final" version of string theory.

configuration on K_o/G under the map $x \rightarrow [x]$)with vanishing field strength, x^m denotes the coordinates on K_o (any dependence on the external coordinates has been suppressed) and the path ordered integral is taken along the path Γ_g from x to gx on K_o*.

From eqn (1) we find that if $U_g \neq 1$, then H is broken to the subgroup Σ that commutes with $\overline{G}$ [2].

How are mass relations affected by this formulation? First, note that, given a particle field Ψ in some irreducable rerpresentation of $H \otimes G$**, it will generally be the case that not all of the components of Ψ can satisfy eqn (1). Second, the superpotential on K_o/G is constructed by considering the superpotential on K_o (consistent with all of the symmetries and pseudosymmetries on K_o [2, 6]) and setting all fields that do not satisfy eqn (1) equal to zero. By putting these two statements together we see that in general whatever Clebsch-Gordan relations may have held between elements in a given H multiplet need not hold true now since the particles being related to each other need not even be in the spectrum on K_o/G (i.e., satisfy eqn (1))! In reference [2] this was touted as an advantage over the situation that occurs in traditional grand unified models since, in the traditional case, most mass relations for the first two families that would be obtained in this fashion were not obeyed very well

*For U_g to be path independent, and for the boundary condition as written down here to make sense, K_o must either be flat, or the effects of the spin connection of K_o must be nullified in some way [9] such as through the identification of the spin and gauge connections as considered in ref. [6]. Those particle fields on K_o that do not satisfy eq (1) are <u>not</u> supermassive on K_o/G as is sometimes stated. Rather, they simply are not in the spectrum of viable modes on K_o/G. The boundary condition (1) is valid for both zero modes and higher excitations.

**If G is abelian, this will also be an irreducible representation of H since all of the irreducible representations of G will be one dimensional; however, if G is non-abelian this might not be so. These two cases are discussed at the end of the next section.

if at all (bad for the second family and worse for the first family). It certainly is an advantage for the first two families.

We may now sharpen our question. Given that we will use Wilson lines to break the H symmetry, and thus the integrity of the H multiplets will be destroyed, what constraints are imposed on this breaking if we require that the (good) m_b/m_τ mass relation be a consequence of a Clebsch-Gordan relationship (at the unification scale) between two terms in the superpotential on K_0/G? What further constraints are imposed by requiring that the first two families not be constrained by this same relation?

To answer this question, we will first discuss the G transformation of the zero modes on K_0 (since we need these transformations in order to solve the boundary condition (1))for the case where $H=E_6$ [2] and the cases where H=SO(10) or SU(5)* [10]. We will find that these transformation properties are those given in ref [11] (although our way of arriving at the result is somewhat more transparent for bohemians than that given in ref [11]). Next we will demand that the m_b/m_τ mass ratio is just a Clebsch-Gordan coefficient and see what constraints this puts on the Q=-1/3 and Q=-1 sector fermion mass matrices.

*We will present these two cases for completeness even though manifolds that give rise to SU(5) or SO(10) gauge groups [6] are problematic because of world sheet instantons [19]. However, they may be allowable [20] on orbifolds [21].

(II) -- G transformation properties of the zero modes

We will focus on the E_6 case first. For simplicity we assume that K_0 has the (1, 1) Hodge number $b_{1,1}=1$ so that only one $\overline{\mathbf{27}}$ (corresponding to the Kähler form) appears in the zero mode spectrum on K_0 [6]. We will relax this assumption later and briefly discuss the consequences. Under these circumstances the zero mode content on K_0 is [6] $\{|\chi(K_0)|/2+1\}\,\mathbf{27}+\overline{\mathbf{27}}+\delta\,\mathbf{1} \equiv (n_f+1)\,\mathbf{27}+\overline{\mathbf{27}}+\delta\,\mathbf{1}$, where $\chi(K_0)$ is the Euler characteristic of K_0 and δ is discussed in ref. [10]. What we wish to determine is how the zero modes transform under the action of G on K_0. This is determined uniquely in the present case through consistency with the index theorem <u>if</u> for each possible parameterization of the $\mathbf{U}_g$ matrices there exists a vectorfield configuration A_m (with $F_{mn}=0$) on K_0/G that gives rise to it*. Actually, although this is a sufficient condition, we will see below that only a subset of these parameterizations of the $\mathbf{U}_g$ matrices need have a corresponding vectorfield configutration for there to be a unique determination of the G transformation properties of the modes.

Since G is a symmetry of K_0, the action of $g\in G$ must transform the $\mathbf{27}$'s amongst themselves in such a way as to form a representation of G. We will consider the case in which G is abelian first. Let us assume for the moment that $G=Z_n$. Then, $n_f=pn$ for some integer p. Since G is abelian we can choose a basis in which the $\mathbf{27}$'s do not mix under G (since the irreducible representations of G are all one dimensional). Thus under the action of $g\in G$, $\mathbf{27}_i \to \eta_i\,\mathbf{27}_i$ (no sum) where η_i is an $n^{\underline{th}}$ root of unity constituting a one dimensional irreducible representation of Z_n. Let g_0 be the generator of Z_n and let $\alpha=e^{2\pi i/n}$, then $\mathbf{27}_i \to \alpha^{m_i}\,\mathbf{27}_i$ where the possible values of m_i are 0, 1, 2, ..., n-1, corresponding to the n one-dimensional irreducible representations of Z_n. It is now easy to find out how many $\mathbf{27}$'s transform according

*This question is being investigated by us [9].

to each irreducible representation of G. First choose $\mathbf{U_g} = \mathbf{1}$. According to the character valued index theorem [12] on K_o/G the representation content on K_o/G is $(p+1)\,\mathbf{27}+\overline{\mathbf{27}}$. Thus p+1 **27**'s have $m_i=0$ (i.e., they satisfy $\mathbf{27}_i \rightarrow \mathbf{27}_i$).

To obtain the transformation properties of the remaining (n-1)p **27**'s we can consider a sequence of special cases. Consider the $SU(3)_c \times SU(3)_L \times SU(3)_R$ parameterization of $\mathbf{U}_{g_o}$ that breaks $E_6 \rightarrow SU(3)_c \times SU(3)_L \times SU(2)_R \times U(1)_{R8}$:

$$\mathbf{U}_{g_o} = \mathbf{1}_c \times \mathbf{1}_L \times \begin{bmatrix} \alpha^m & & \\ & \alpha^m & \\ & & \alpha^{-2m} \end{bmatrix}. \qquad \ldots(3)$$

We will consider the sequence of special cases m= 1, 2, ..., n-1 (we have already considered m=0). Under $E_6 \supset SU(3)_c \times SU(3)_L \times SU(2)_R \times U(1)_{R8}$, the **27** decomposes as

$$\mathbf{27} = (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{2}; 1) + (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; -2) + (\mathbf{3}, \mathbf{3}, \mathbf{1}; 0) + (\mathbf{1}, \mathbf{3}, \mathbf{2}; -1) + (\mathbf{1}, \mathbf{3}, \mathbf{1}; 2) \quad \ldots(4)$$

The character valued index theorem says that there must be precisely p copies of the representation $(\mathbf{1}, \mathbf{3}, \mathbf{2}; -1)$ in the spectrum on K_o/Z_n. The boundary condition for $\mathbf{27}_i$ is $\Psi_i(x)=\mathbf{U}_{g_o}^{\dagger}\Psi_i(g_o x)$ or $\Psi_i(x)=\alpha^{m_i}\,\mathbf{U}_{g_o}^{\dagger}\Psi_i(x)$. When applied to the $(\mathbf{1}, \mathbf{3}, \mathbf{2}; -1)$ component of the $\mathbf{27}_i$ this requires that $\alpha^{(m_i - m)} = 1$ if that component is to be in the spectrum on K_o/Z_n. Since we must have exactly p of these, we must have $m_i = m$ for exactly p of the **27**'s. By allowing m to vary from 1 to n-1 we see that exactly p **27**'s must transform according to each irreducible representation of Z_n in this case. We see that only a subset of the parameterizations of the $\mathbf{U_g}$ matrices were needed to uniquely determine of the G transformation properties of the modes. A different set of components of the **27** could have been chosen to do this analysis (in the above case this set only consisted of one component under

$E_6 \supset \Sigma = SU(3)_c \times SU(3)_L \times SU(2)_R \times U(1)_{R8})$. Also, a different parameterization of $\mathbf{U}_{g_o}$ (and thus possibly a different unbroken subgroup Σ) could have been used. If a manifold were found that did not yield this result for the transformation properties of the **27**'s, then we would conclude that a large class of certain conceivable parameterizations of $\mathbf{U}_g$ are not permissible, and that the associated symmetry breaking patterns of E_6 could not occur via the Wilson line mechanism.

An important point to note for use in considering the case in which G is non-abelian is that the character (trace) of each element (except for the identity) of the reducible representation under which all of the n_f **27**'s transform (thus excluding one **27** which is paired with the $\overline{\mathbf{27}}$ and is invariant under G) is zero since we can easily calculate it to be (for the element $g = g_o{}^s$) $p \sum_a \alpha^{as} = p \sum_a e^{(2\pi i\, as)/n} = 0$.

Our result easily generalizes to an arbitrary finite abelian group $G = Z_n \times Z_q \times ... \times Z_r$ if the number of factors is less than or equal to the rank of the gauge group; i.e., 6. Here $n_f = p(nq...r)$ for some integer p. There are exactly p **27**'s trasnsforming as each of the (nq...r) one dimensional irreducible representations. The characters of all of the elements except the identity are again zero. When the number of factors in $G = Z_n \times Z_q \times ... \times Z_r$ is greater than 6 then not all of the representations of G can be embedded in E_6. For example, none of the faithful ones can be embedded. However, the result is still the same as we have just stated and is due to the following argument.

The analysis of the case of $G = Z_n \times Z_q \times ... \times Z_r$ and the case of non-abelian G is relatively easy since the result that we have proven must apply if we restrict ourselves to considering any of its Z_n subgroups. If dim(G) = t, then $n_f = pt$ for some integer p. Since each element $g \in G$ generates a cyclic subgroup of G (a Z_n for some n) it follows from our preceeding discussion of Z_n that the trace of the matrix representing g vanishes (except when g is the identity). The n_f **27**'s will then transform as p copies of the reducible representation **R** of G of dimension t=dim(G) where $\mathbf{R}$ is the unique representation of dimension dim(G) of a finite group that has vanishing characters for all elements except the identity -- the so called "regular representation" [11]. (The remaining 27 will transform trivially as does its $\overline{27}$ partner if $b_{1,1}=1$.) The regular representation is reducible and consists of n_i copies of each irreducible representation of G of dimension n_i. This representation is constructed in the course of proving that for a finite group $\sum_i (n_i)^2 = \dim(G)$. This is consistent with our discussion of the abelian case since there each one dimensional irreducible representation appeared exactly once in $\mathbf{R}$. (For a discussion of how to solve (1) when G is non-abelian, see the appendix.)

In the E_6 case where $b_{1,1}>1$ this result is easily generalized. In this case the zero mode content on K_o is $(n_f + b_{1,1})\,27 + b_{1,1}\overline{27}$. An argument almost identical to that given above gives the conclusion that, as a consequence of the character valued index theorem, n_f ($n_f = p\dim(G)$) of the 27's transform as p copies of the regular representation. The remaining $b_{1,1}27$'s transform conjugate to the way the $b_{1,1}\overline{27}$'s transform thus providing vectorlike pairs of zero modes on K_o/G from the $b_{1,1}(27+\overline{27})$ on K_o. We can say a little bit more. If we denote as $b_{1,1}(K_o/G)$ the value of $b_{1,1}$ on K_o/G, then we know that, by considering the case in which $\mathbf{U}_{g_o}=1$, exactly $b_{1,1}(K_o/G)$ of the $b_{1,1}(27+\overline{27})$ transform trivially under G action on K_o*. A discussion of the G transformation properties of the $\delta\,E_6$ singlets follows along similar lines to this discussion of the $b_{1,1}(27+\overline{27})$.

*In particular cases even more could be said. Consider the (contrived) example where G is non-abelian and has only one non-trivial 1-dimensional irreducible representation in addition to some number of 3 and higher-dimensional irreducible representations. Further, say that $b_{1,1}(K_o)=4$ and $b_{1,1}(K_o/G)=2$, then it follows that two of the 27's from $4(27+\overline{27})$ transform trivially and two of them transform as the non-trivial 1-dimensional irreducible representation (similarly for the $\overline{27}$'s). Of course the numerology of any particular case might not yield a complete determination of the transformation properties of the $b_{1,1}(27+\overline{27})$ under G, but might yield a reasonably small number of possibilities.

The SO(10) and SU(5) cases are treated similarly. For SO(10) the zero-mode content (of chiral superfields) is

$$n_f\,\mathbf{16} + \alpha(\mathbf{16} + \overline{\mathbf{16}}) + \beta\mathbf{10} + \delta\mathbf{1} \qquad \ldots(5)$$

where the numbers α,n_f, β and δ are given by topological invariants [10, 16]. As in the E_6 case, the n_f **16** transform as p (=n_f/dim(G)) copies of the regular representation **R**. A discussion of how the $\alpha(\mathbf{16} + \overline{\mathbf{16}}) + \delta\mathbf{1}$ transform is similar to the discussion of how the $b_{1,1}(27+\overline{27}) + \delta\mathbf{1}$ transforms in the E_6 case. All that is left is to consider the $\beta\mathbf{10}$. Since the representation **10** is real under E_6 (although it is complexifiied since it is a superfield) it yields zero contribution ot the index of the Dirac operator on K_0. Similarly, on K_0/G, choosing $\mathbf{U}_g=1$ for all $g\in G$ we know that the **10**'s that are in the spectrum are those that were invariant under G on K_0: The Dirac index is therefore zero on K_0/G. Since the index does not depend upon the value of $\mathbf{U}_g$ chosen* we can obtain some information on how the **10**'s transform under the action of G from this result. Consider the case where G is abelian (or consider the abelian representations of a non-abelian G) and parameterize $\mathbf{U}_g$ using the $SU(4)_c\times SU(2)_L\times SU(2)_R$ basis of SO(10):

$$\mathbf{U}_g = \begin{pmatrix} \beta & & & \\ & \beta & & \\ & & \beta & \\ & & & \beta^{-3} \end{pmatrix} \times \begin{pmatrix} \gamma & \\ & \gamma^{-1} \end{pmatrix} \times \begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix} \qquad \ldots(6)$$

*The index depends upon the vectorfield only through the field strength. Since the $\mathbf{U}_g$ are computed in the vacuum with F=0, it follows that the index does not depend upon the value of $\mathbf{U}_g$.

The decomposition of the **10** under the subgroup $\Sigma_0 = SU(3)_c \times SU(2)_L \times U(1)_{15} \times U(1)_{3R}$, where $U(1)_{15}$ appears in the maximal decomposition $SU(4)_c \supset SU(3)_c \times U(1)_{15}$ and $U(1)_{3R}$ is the $U(1)$ subgroup of $SU(2)_R$, is

$$\mathbf{10} = (\bar{3}, 1; 2, 0) + (3, 1; -2, 0) + (1, 2; 0, 1) + (1, 2; 0, -1). \qquad ...(7)$$

If, under the action of $g \in G$, $\mathbf{10}_i \to \eta_i\, \mathbf{10}_i$ then under the combined action of G and $\overline{G}$ the $(1, 2; 0, 1) + (1, 2; 0, -1)$ transforms into $\eta_i\, \mu^*(1, 2; 0, 1) + \eta_i\, \mu^{*-1}(1, 2; 0, -1)$. In general, at most one of these components will be invariant and thus satisfy the boundary condition (1) (this corresponds to the cases $\mu = \eta_i$ and $\mu^* = \eta_i$). The only case where both components will be invariant is when η_i is the trivial representation of G. For the index theorem to be satisfied when η_i is not trivial there must exist another $\mathbf{10}_j$ that transforms as $\mathbf{10}_j \to \eta_j\, \mathbf{10}_j$ where $\eta_j = (\eta_i)^*$. Thus those of the $\beta\mathbf{10}$'s that do not transform trivially under G break up into two sets that transform as the conjugates of each other under the action of G. It follows that if $\beta = 1$ then the **10** transforms trivially. The situation where G is non-abelian is similar for those representations that can be embedded in SO(10). For those representations that cannot be embedded we do not know how to further analyze the problem (of course, those representations are not important from the point of view of the boundary conditon (1)).

For the SU(5) case the zero-mode content is

$$m\,\overline{\mathbf{5}} + n\,\mathbf{10} + \beta\,(\mathbf{5} + \overline{\mathbf{5}}) + \gamma\,(\mathbf{10} + \overline{\mathbf{10}}) + \delta\,\mathbf{1} \qquad ...(8)$$

Where m, n, β, γ, and δ are topological invariants*. Again, the $m\,\overline{5}$ transforms as m/dim(G) copies of the regular representation R under the action of G, while the $n\,10$ transforms as n/dim(G) copies of R. A discussion of the $\beta\,(5+\overline{5})+\gamma\,(10+\overline{10})+\delta\,1$ is similar to that of how the $b_{1,1}(27+\overline{27})+\ \delta 1$ transforms in the E_6 case.

*Anomaly cancelation in the low energy SU(5) model would seem to require that n=m; however, we do not know of any reason why this should be so for the topological quantities n and m.

(III) -- Fermion mass relations and Wilson lines-- the fate of m_b/m_τ for E_6

In conventional SU(5) grand unification and, in particular, in an almost minimal scheme with some number of $\overline{5}_H$'s of Higgs one expects a mass relation of the form $m_{-1/3}/m_l=3$ to be valid when renormalized down from $m_{-1/3}/m_l=1$ at unification scales* for each family in the absence of mixing. Here $m_{-1/3}$ and m_{l} are the masses of the charged -1/3 quark and the charged lepton respectively. This is a consequence of the fact that the mass terms for each of these particles come from the same Yukawa term $\overline{5}$ $\mathbf{10}$ $\{\sum_i y_i \overline{5}_{Hi}\}$ and thus have a ratio that is just a Clebsch-Gordan coefficient. This relation seems to be obeyed well by the fermions in the third family, not very well by those for the second family, and not at all by those of the first family. Much work has been done in the context of conventional grand unified models to try to reproduce the correct mass relations at tree level (or otherwise) by having a more complicated Higgs sector and perhaps a variety of symmetries to restrict the allowed Yukawa couplings [13]. The success of the relationship $m_b/m_\tau=3$ is one that we wish to understand in the context of the purported low energy effective field theory limit of the superstring.

The character valued index theorem states that on K_0/G the chiral zero mode particle content (we consider the E_6 case as an example) comprises group theoretically the Σ decomposition of n_f $\mathbf{27}$'s where Σ is the subgroup of E_6 that is left unbroken by the $\mathbf{U}_g$'s. However, the various components (irreducible representations under Σ) generally come from different $\mathbf{27}$'s on K_0 for a particular choice of the $\mathbf{U}_g$'s. Given that we know how the $\mathbf{27}$'s transform under G on K_0 we can, by application of the boundary condition $\Psi(x)=\mathbf{U}_g{}^\dagger\Psi(gx)$, determine which components on K_0/G came from which of the $\mathbf{27}$'s on K_0. The superpotential on K_0/G is just that on K_0 (perhaps with an overall proportionality constant) with all of those fields Ψ that

*This assumes that there are not too many colored degrees of freedom contributing to the renormalization group equations between the unification and weak scales.

don't satisfy the boundary condition set to zero. For a conventional E_6 supersymmetric grand unified model with n_f **27**'s there would be many Clebsch-Gordan relations between various Yukawa couplings at the Σ level that come from the same $\mathbf{27}_i\mathbf{27}_j\mathbf{27}_k$ term before conventional breaking of $E_6 \to \Sigma$. In the present context this is generally not so since those terms often come from different couplings in the superpotential as written down on K_0. However, if we believe that the success of the relation $m_b/m_\tau = 3$ is due to a Clebsch-Gordan relation between Yukawa couplings, this would put constraints on the allowed values of the $\mathbf{U}_g$ matrices and hence the possible groups* Σ. Thus, we are interested in knowing when such a relation is possible at tree level when the symmetry is broken by Wilson lines. This translates into determining when the relevant Σ Yukawa couplings come from the same term in the K_0 superpotential. The following is also an example of how Clebsch-Gordan relations can be determined in more general cases using the results of the preceeding section.

We will consider the case where Σ is rank 6 and thus $\overline{G}$ is abelian (G may be abelian or non-abelian). We will generalize to the case where $\overline{G}$ is non-abelian later. We will also neglect intergeneration mixings at first and concentrate on the third family. The smallest rank 6 subgroup (consistent with standard model phenomenology) of E_6 is $\Sigma_0 = SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$. Under Σ_0 the **27** of E_6 decomposes as

$$\begin{aligned}\mathbf{27} = {} & A_1(\mathbf{1},\mathbf{2};-1,1,1) + B_{-1}(\mathbf{1},\mathbf{2};-1,-1,1) + C_{-1}(\mathbf{1},\mathbf{2};-1,0,-2) \\ & + D_2(\mathbf{1},\mathbf{1};2,1,1) + E_0(\mathbf{1},\mathbf{1};2,-1,1) + F_0(\mathbf{1},\mathbf{1};2,0,-2) \\ & + G_{1/3}(\mathbf{3},\mathbf{2};1,0,0) + H_{-2/3}(\mathbf{3},\mathbf{1};-2,0,0) + L_{4/3}(\overline{\mathbf{3}},\mathbf{1};0,0,2) \\ & + J_{2/3}(\overline{\mathbf{3}},\mathbf{1};0,1,-1) + K_{2/3}(\overline{\mathbf{3}},\mathbf{1};0,0,2) \qquad \ldots(9)\end{aligned}$$

*We will also find that it may put constraints on the values of the vacuum expectation values of the singlet fields in the **27** (in the E_6 case) and on the presence or absence of certain Yukawa couplings.

where the subscripts denote the weak hypercharges of the respective components.

From the weak hypercharges we can state which components of the superfield, **27**, contain which fermions. Thus, the superfield $G_{1/3}$ is $[u,d]_L$, $L_{4/3}$ is $u_R{}^c$, $H_{-2/3}$ is D_L (the D is sometimes also refered to as g*), D_2 is $e_R{}^c$, and A_1 is $[E, N]_R{}^c$. There is an ambiguity at this stage for $J_{2/3}$ and $K_{2/3}$, and for B_{-1} and C_{-1} depending upon mixings between "light" and "heavy" fields in the mass matrices. Both $J_{2/3}$ and $K_{2/3}$ are candidate $d_R{}^c$ and $D_R{}^c$. Similarly, B_{-1} and C_{-1} are candidate $(\nu,e)_L$ and $(N,E)_L$. E_0 and F_0 are neutral singlets (candidate $\nu_R{}^c$ and $N_R{}^c$). In all of this, u, d, e, and ν are used to denote the fermion type (its quantum numbers); possible family indices are suppressed.

In the SU(5) decomposition of the **27** of E_6 ($\mathbf{27} = (\bar{\mathbf{5}} + \mathbf{10} + \mathbf{1})_{\mathbf{16}} + (\mathbf{5} + \bar{\mathbf{5}})_{\mathbf{10}} + \mathbf{1}$), where the supscripts indicate which SO(10) representation the bracketed quantities come from, we find that $G_{1/3}$ and D_2 and $L_{4/3}$ come from the $\mathbf{10_{16}}$, $J_{2/3}$ and C_{-1} come from $\bar{\mathbf{5}}_{\mathbf{16}}$ and $K_{2/3}$ and B_{-1} come from the $\bar{\mathbf{5}}_{\mathbf{10}}$. $H_{-2/3}$ and A_1 come from the $\mathbf{5_{10}}$. The charged -1/3 quark and the charged -1 lepton mass matrices are two-dimensional (ignoring family indices), and of the four entries in each, two are due to $SU(2)_L$ doublet vacuum values and two are due to $SU(2)_L$ singlet vacuum values. The possibilities for the minimal (necessary) condition that there be a Clebsch-Gordan relation between the mass terms for the charged -1/3 quark and the charged -1 lepton are determined by considering how a doublet entry from the charged -1/3 mass matrix might be equal to one from the charged -1 mass matrix** as a consequence of a Clebsch-Gordan coefficient. There are two possibilities. The first possibility (case I) is that both $G_{1/3}$ and D_2 come from the same **27** on K_0 (another pair of fields -- one of B_{-1} or C_{-1} and one of $J_{2/3}$ or $K_{2/3}$ -- must also come from a common, although different, **27**; we will see that this follows from the condition that

*In ref. [14], for example.

**We then need to determine when these two matrices have an equal eigenvalue.

we have already stated). The second possibility (case II) is that $G_{1/3}$ and B_{-1} and/or C_{-1} come from one ***27*** and D_2 and $J_{2/3}$ and/or $K_{2/3}$ come from another ***27***. To impose these conditions we first parameterize the $\mathbf{U}_g$ matrices as

$$\mathbf{U}_g = \mathbf{1}_c \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \beta\delta & & \\ & \beta\delta^{-1} & \\ & & \beta^{-2} \end{bmatrix} \qquad \text{...(10)}$$

in the $E_6 \supset SU(3)_c \times SU(3)_L \times SU(3)_R$ basis. Then, under $\mathbf{U}_g$, the various components of ***27*** transform as:

$$\begin{array}{lll} A_1 \to \alpha^{-1}\delta\beta\, A_1\,; & E_0 \to \alpha^2\delta^{-1}\beta\, E_0\,; & I_{-4/3} \to \delta^{-1}\beta^{-1}\, I_{-4/3}\,; \\ B_{-1} \to \alpha^{-1}\delta^{-1}\beta\, B_{-1}\,; & F_0 \to \alpha^2\beta^{-2}\, F_0\,; & J_{2/3} \to \delta\beta^{-1}\, J_{2/3}\,; \\ C_{-1} \to \alpha^{-1}\beta^{-2}\, C_{-1}\,; & G_{1/3} \to \alpha\, G_{1/3}\,; & K_{2/3} \to \beta^2\, K_{2/3}\,; \\ D_2 \to \alpha^2\delta\beta\, D_2\,; & H_{-2/3} \to \alpha^{-2}\, H_{-2/3}\,. & \end{array} \qquad \text{...(11)}$$

Thus, in case I, in order for $G_{1/3}$ and D_2 to come from the same ***27*** they must satisfy the boundary condition for a common ***27***. Let this ***27*** transform as $\mathbf{27}_i \to \eta_i\, \mathbf{27}_i$ under G. Then, under the combined action of G and $\overline{G}$ we have $G_{1/3} \to \alpha^* \eta_i\, G_{1/3}$ and $D_2 \to (\alpha^2\delta\beta)^* \eta_i\, D_2$ for this ***27***. For both $G_{1/3}$ and D_2 to satisfy the boundary condition for this $\mathbf{27}_i$ we then need $\alpha^*\eta_i = 1$ and $(\alpha^2\delta\beta)^*\, \eta_i = 1$; hence, we must have $\eta_i = \alpha = \alpha^2\delta\beta$, so $\alpha\beta\delta = 1$; and, by the discussion in the preceeding section we know that there will always be exactly $(1/2)\chi(K_o/G)$ $\mathbf{27}_i$'s that satisfy $\eta_i = \alpha$. This restricts an abelian $\overline{G}$ to be at most $Z_n \times Z_m$. By imposing the condition $\alpha\beta\delta = 1$ we find that the Σ_0 components A_1 and $H_{-2/3}$ transform identically under $\mathbf{U}_g$, B_{-1} transforms identically to $K_{2/3}$, and C_{-1} transforms identically to $J_{2/3}$. Also $L_{4/3}$ transforms identically to both $G_{1/3}$ and D_2.

These were the same groupings as were mentioned above with respect to the SU(5) decomposition of the ***27***. These facts would make it possible for the m_b/m_τ relationship to occur since the components that we want to come from the same SU(5)

$\overline{5}$ as discussed above will indeed come from the same 27 by virtue of the fact that they have the same $\mathbf{U}_g$ transformation properties. In this case the charged 1/3 and charged -1 mass matrices are identical (even when the family structure is put back in). However, this also leads to a fatal problem in that Σ will always contain $SU(5)\times U(1) \times U(1)$ (where the SU(5) is the standard Georgi-Glashow group).

We can see this by counting the minimum number of unbroken E_6 generators given the condition $\alpha\beta\delta = 1$. Under $SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$ the 78 decomposes as :

$$
\begin{aligned}
78 = &(8,1,0,0,0) + (1,3,0,0,0) + 3\,(1,1,0,0,0) + [\,(1,1,0,2,0) \\
&+ (1,2,3,0,0) + (1,1,0,1,3) + (1,1,0,-1,3) + (3,2,-1,-1,-1) \\
&+ (3,2,-1,1,-1) + (3,2,-1,0,2) + (3,1,2,-1,-1) + (3,1,2,1,-1) \\
&+ (3,1,2,0,2) + \text{comp. conj.}] \qquad \ldots(13)
\end{aligned}
$$

Thus, under $\mathbf{U}_g$, this transforms into

$$
\begin{aligned}
&(8,1,0,0,0) + (1,3,0,0,0) + 3\cdot(1,1,0,0,0) + [\,\delta^3(1,1,0,2,0) \\
&+ \alpha^3(1,2,3,0,0) + \delta\beta^3(1,1,0,1,3) + \delta^{-1}\beta^3(1,1,0,-1,3) \\
&+ \alpha^{-1}\delta^{-1}\beta^{-1}(3,2,-1,-1,-1) + \alpha^{-1}\delta\beta^{-1}(3,2,-1,1,-1) \\
&+ \alpha^{-1}\beta^2(3,2,-1,0,2) + \alpha^2\delta^{-1}\beta^{-1}(3,1,2,-1,-1) + \alpha^2\delta\beta^{-1}(3,1,2,1,-1) \\
&+ \alpha^2\beta^2(3,1,2,0,2) + \text{comp. conj.}]. \qquad \ldots(14)
\end{aligned}
$$

For $\alpha\beta\delta = 1$ there are therefore at least 26 unbroken generators, and Σ contains $SU(5)\times U(1)\times U(1)$. This is fatal since symmetry breaking through vacuum values of the zero mode fields that are available cannot break the remaining $SU(5)\times U(1)\times U(1)$ down to $SU(3)_c \times U(1)_{EM}$.

In case II the results are different. The various possibilities for the necessary condition and their consequences are as follows:

1) (a) $G_{1/3}$ and B_{-1} come from the same **27**; then $\alpha=\alpha^{-1}\delta^{-1}\beta \Rightarrow$ $\alpha^2\delta\beta^{-1}=1$.

(b) $G_{1/3}$ and C_{-1} come from the same **27**; then $\alpha=\alpha^{-1}\beta^{-2} \Rightarrow$ $\alpha^2\beta^2=1$.

2) (a) D_2 and $K_{2/3}$ come from the same **27**; then $\alpha^2\delta\beta=\beta^2 \Rightarrow$ $\alpha^2\delta\beta^{-1}=1$.

(b) D_2 and $J_{2/3}$ come from the same **27**; then $\alpha^2\delta\beta=\delta\beta^{-1} \Rightarrow$ $\alpha^2\beta^2=1$.

Condition (1a) is compatible with condition (2a), and condition (1b) is compatible with condition (2b). We refer to these two situations as case IIA and case IIB respectively. Case IIA and IIB cannot both be valid since examination of the transformation properties of the **78** reveals that there would be (at least) 26 unbroken generators, and thus Σ would again contain an $SU(5)\times U(1)\times U(1)$ subgroup (this is not the Georgi-Glashow SU(5)), leading to the problems discussed above. In case IIA an examination of the transfromation properties of the **78** reveals that, when the condition $\alpha^2\delta\beta^{-1}=1$ is imposed, there are (at least) 20 unbroken generators. Thus Σ contains an $SU(4)\times SU(2)_L\times U(1)\times U(1)$ subgroup (where the SU(4) is the Pati-Salam extension of the standard color $SU(3)_c$) which is phenomenologically acceptable insofar as vacuum values of **27** and $\overline{\mathbf{27}}$ can break this group down to $SU(3)_c\times U(1)_{EM}$. Similarly, in case IIB Σ contains an $SU(4)\times SU(2)_L\times U(1)\times U(1)$ subgroup.

For both of these cases, in order to examine the charged -1/3 and charged -1 mass matrices in greater detail, we first construct the various (fermion bilinear) mass terms that are group theoretically possible in order to see what Higgs representations would be needed for their existence.

-1/3 quark

$$G_{1/3}J_{2/3} \sim (\mathbf{3},\mathbf{2};1,0,0)(\bar{\mathbf{3}},\mathbf{1};0,1,-1) \sim (\mathbf{1},\mathbf{2};1,1,-1)$$

$$G_{1/3}K_{2/3} \sim (\mathbf{3},\mathbf{2};1,0,0)(\bar{\mathbf{3}},\mathbf{1};0,0,2) \sim (\mathbf{1},\mathbf{2};1,0,2)$$

$$H_{-2/3}J_{2/3} \sim (\mathbf{3},\mathbf{1};-2,0,0)(\bar{\mathbf{3}},\mathbf{1};0,1,-1) \sim (\mathbf{1},\mathbf{1};-2,1,-1)$$

$$H_{-2/3}K_{2/3} \sim (\mathbf{3},\mathbf{1};-2,0,0)(\bar{\mathbf{3}},\mathbf{1};0,0,2) \sim (\mathbf{1},\mathbf{1};-2,0,2)$$

-1 lepton

$$C_{-1}D_2 \sim (\mathbf{1},\mathbf{2};-1,0,-2)(\mathbf{1},\mathbf{1};2,1,1) \sim (\mathbf{1},\mathbf{2};1,1,-1)$$

$$B_{-1}D_2 \sim (\mathbf{1},\mathbf{2};-1,-1,1)(\mathbf{1},\mathbf{1};2,1,1) \sim (\mathbf{1},\mathbf{2};1,0,2)$$

$$C_{-1}A_1 \sim (\mathbf{1},\mathbf{2};-1,0,-2)(\mathbf{1},\mathbf{2};-1,1,1) \sim (\mathbf{1},\mathbf{1};-2,1,-1)$$

$$B_{-1}A_1 \sim (\mathbf{1},\mathbf{2};-1,-1,1)(\mathbf{1},\mathbf{2};-1,1,1) \sim (\mathbf{1},\mathbf{1};-2,0,2)$$

Let us now assume that there are Higgs transforming as the conjugate of each of the $SU(2)_L$ doublet representations $(\mathbf{1},\mathbf{2};1,1,-1)$ and $(\mathbf{1},\mathbf{2};1,0,2)$ and Higgs transforming as the conjugate of each of the singlet representations $(\mathbf{1},\mathbf{1};-2,1,-1)$ and $(\mathbf{1},\mathbf{1};-2,0,2)$. The charged -1/3 mass matrix is then of the generic form

$$\begin{array}{c} \\ G_{1/3} \\ H_{-2/3} \end{array} \begin{array}{c} \begin{array}{cc} J_{2/3} & K_{2/3} \end{array} \\ \begin{pmatrix} m_1 & m_2 \\ M_1 & M_2 \end{pmatrix} \end{array} = \mathbf{M}_{-1/3} \qquad \ldots(14)$$

In case IIA the charged -1 mass matrix has the form (as a consequence of the unbroken SU(4) subgroup of SU(5))*

$$\begin{array}{ccc} & C_{-1} & B_{-1} \\ \begin{array}{c} D_2 \\ A_1 \end{array} & \multicolumn{2}{c}{\begin{pmatrix} \tilde{m}_1 & m_2 \\ \tilde{M}_1 & \tilde{M}_2 \end{pmatrix}} \end{array} = \mathbf{M}_{-1} \qquad \ldots(15)$$

at tree level, where we are considering only the contributions of the renormalizable terms in the superpotential. Only one of the doublet breaking terms (m_2) in the charged -1/3 matrix is the same as that in the charged -1 matrix, and the singlet breaking terms in the charged -1/3 matrix are different from those in the charged -1 matrix.

In case IIB the charged -1 mass matrix is

*In standard SU(5), the 5 of Higgs only breaks SU(5) to SU(4) where $\{d^c, e\}_L$ forms a 4-plet and $\{d, e^c\}_L$ forms a $\overline{4}$-plet. It is because of this that Clebsch-Gordan relations arise since the mass term from $\overline{5}\cdot\mathbf{10}$ is of the form $\overline{4}\cdot 4$ with respect to this SU(4) (which is not Pati-Salam). In our case the situation is slightly different. In the SU(5) decomposition of the product of two 27's, $\mathbf{27}_1\mathbf{27}_2$, the relevant mass terms come from $((\overline{5}_1)_{\mathbf{10}}\mathbf{10}_2 + (\overline{5}_1)_{\mathbf{16}}\mathbf{10}_2 + (\overline{5}_2)_{\mathbf{10}}\mathbf{10}_1 + (\overline{5}_2)_{\mathbf{16}}\mathbf{10}_1)$ with a common Yukawa coupling. If this were a traditional grand unified model with the doublet breaking comming from 5_H's of Higgs then, by our preceeding statements, this would give rise to four mass terms all with equal values, two for charge -1/3 and two for charge -1. In the E_6 superstring cases that were considering here two of these terms (one of charge -1/3 and one of charge -1) contain fields that are not in the spectrum on K_o/G. The surviving terms come entirely from the $(\overline{5}_1)_{\mathbf{16}}\mathbf{10}_2$ term in case I, whereas in cases IIA one term comes from $(\overline{5}_1)_{\mathbf{10}}\mathbf{10}_2$ and the other comes from $(\overline{5}_2)_{\mathbf{10}}\mathbf{10}_1$ and in cases IIB one term comes from $(\overline{5}_1)_{\mathbf{16}}\mathbf{10}_2$ and the other comes from $(\overline{5}_2)_{\mathbf{16}}\mathbf{10}_1$.

$$\mathbf{M}_{-1} = \begin{pmatrix} m_1 & \widetilde{m}_2 \\ \widetilde{M}_1 & \widetilde{M}_2 \end{pmatrix} \qquad ...(16)$$

Again the singlet breaking terms in the charged -1/3 matrix are different from those in the charged -1 matrix, and only one of the doublet breaking terms (m_1 in this case) in the charged -1/3 matrix is the same as that in the charged -1 matrix.

To diagonalize these matrices (consider $\mathbf{M}_{1/3}$; $\mathbf{M}_{-1}$ of course is treated identically) and determine their eigenvalues we need to consider the diagonalization of $\mathbf{M}\,\mathbf{M}^\dagger$ and $\mathbf{M}^\dagger\,\mathbf{M}$. The former yields $\mathbf{V}$ and the latter yields $\mathbf{W}$ where $\mathbf{V}^\dagger\mathbf{M}\mathbf{W}=\mathbf{M}_D$, $\mathbf{M}_D$ is the diagonalized matrix, and $\mathbf{V}$ and $\mathbf{W}$ are unitary matrices. $\mathbf{V}$ gives the $G_{-1/3}$, $H_{2/3}$ (and D_2, A_1) mixing, and $\mathbf{W}$ gives the $J_{2/3}$, $K_{2/3}$ (and C_{-1}, B_{-1}) mixing. For $M_1, M_2 > m_1, m_2$ we find that the mass (squared) eigenvalues are (in the simplified case where the elements of $\mathbf{M}$ are taken to be real)

$$M_1{}^2+M_2{}^2+(m_1M_1+m_2M_2)^2/\{M_1{}^2+M_2{}^2\} \qquad ...(17)$$

and

$$(m_1M_2-m_2M_1)^2\,/\,\{M_1{}^2+M_2{}^2\}. \qquad ...(18)$$

For $\mathbf{M}\,\mathbf{M}^\dagger$ the eigenvector coresponding to the small eigenvalue is

$$[1,\ (m_1M_1+m_2M_2)/\{M_1{}^2+M_2{}^2\}]. \qquad ...(19)$$

Thus, in order to preserve GIM naturalness we need $(m_1M_1+m_2M_2)/\{M_1{}^2+M_2{}^2\} \ll 1$. For $\mathbf{M}^\dagger\,\mathbf{M}$ the small eigenvalue has the eigenvector

$$[-M_2/\{M_1{}^2+M_2{}^2\}^{1/2},\ M_1/\{M_1{}^2+M_2{}^2\}^{1/2}]. \qquad ...(20)$$

Thus $J_{2/3}$, $K_{2/3}$ (and C_{-1}, B_{-1}) mixing is determined by the ratio, M_1/M_2 ($\widetilde{M}_1/\widetilde{M}_2$), of the singlet entries in the mass matrices.

Lets now return to case IIA and ask what else is needed in order to have an m_b/m_τ mass relation that is a consequence of a Clebsch. We want the small eigenvalues in $\mathbf{M}_{-1}$ and $\mathbf{M}_{1/3}$ to both be equal to $m_2{}^2$:

$$(\widetilde{m}_1\widetilde{M}_2-m_2\widetilde{M}_1)^2 \;/\; \{\widetilde{M}_1{}^2+\widetilde{M}_2{}^2\} = (m_1M_2-m_2M_1)^2 \;/\; \{M_1{}^2+M_2{}^2\} = m_2{}^2 \quad ...(21)$$

The only way that this can be satisfied independently of m_1, $\widetilde{m}_1$, and m_2 is to have $M_1 \gg M_2$ and $\widetilde{M}_1 \gg \widetilde{M}_2$. In this event, the mass eigenstates for the charge conjugate of the right handed charged 1/3 quark is primarily $K_{2/3}$ and the left handed charged -1 lepton is primarily B_{-1}. In case IIB the results are that we need $M_2 \gg M_1$ and $\widetilde{M}_2 \gg \widetilde{M}_1$. Here the mass eigenstates for the charge conjugate of the right handed charged 1/3 quark is primarily $J_{2/3}$ and the left handed charged -1 lepton is primarily C_{-1}.

The analysis that we have presented here for the mass matrix of just one family (the third family) is clearly valid when the full family structure is put back in where the entries in the above matrices are now matrices in family space. For the moment let us ignore mixings between families. Then, each family will have a charge -1/3 mass matrix of the form (14) and a charge -1 mass matrix of the form (15) (we will consider case IIA for this discussion; case IIB is entirely analogous). Without any further assumptions we would guess that if $M_1 \gg M_2$ and $\widetilde{M}_1 \gg \widetilde{M}_2$ for the third family, then this would be true for the first two families. If this were so then the first two families would have the same $m_{-1/3}/m_{-1}$ mass relation as the third family. However, this need not be so since the quantities M_1, M_2, $\widetilde{M}_1$, and $\widetilde{M}_2$, are different linear combinations of products of Yukawa couplings and singlet vacuum expectation values for each family. It is conceivable that $M_1 \gg M_2$ and $\widetilde{M}_1 \gg \widetilde{M}_2$ for the third family and that $M_2 \gg M_1$ and $\widetilde{M}_2 \gg \widetilde{M}_1$ for the first two families. In this case the first two families would not have a $m_{1/3}/m_{-1}$ mass relation due to a Clebsch-Gordan coefficient, rather, it would be a consequence of particular values of vacuum values and Yukawa couplings. Another way that this can happen (that relies less on serendipitous values of couplings and vacuum values) is for $M_1 \approx M_2$ and $\widetilde{M}_1 \approx \widetilde{M}_2$ (or even $M_2 \gg M_1$ and $\widetilde{M}_2 \gg \widetilde{M}_1$) for the first two families, and for M_2 and $\widetilde{M}_2$ to vanish identically for the third family as a consequence of the relevant Yukawa couplings vanishing for discrete symmetry [2] or topological reasons [4]. A different way this idea can be implemented is to have $M_1 \gg M_2$ and $\widetilde{M}_1 \gg \widetilde{M}_2$ for the third family and to have one or both of M_1 and $\widetilde{M}_1$ vanish identically for each of the first two families. A third possibility is to have $M_1 \gg M_2$ and $\widetilde{M}_1 \gg \widetilde{M}_2$ for the third family and to have m_2 vanish identically for the first two families. There are many other possibilities along these lines. In particular some linear combination of the preceeding examples could possibly work. An analogous discussion holds when the family mixing is put back in (even in the case of "non-standard" [17] mass matrices, although the anlaysis could become problematic in weird cases). We thus arive at the conclusion that we can have the mass relation $m_b/m_\tau=1$ as a consequence of a Clebsch-Gordan coefficient at unification without this being a necessary relation for the first two families even if all masses are non-zero at tree level.

We note that some phenomenological considerations can further constrain our results. Thus, for example under some circumstances the vacuum values of fields with the quantum numbers of the neutral field in $\overline{5}_{16}$ might be required to vanish [18]. These are the neutral fields that gives rise to the $G_{1/3}K_{2/3}$ and D_2B_{-1} entries in the mass matrices (14), (15) and (16). If these entries vanish then $m_2=\widetilde{m}_2=0$. Hence, under these circumstances case IIA would not yield he m_b/m_τ mass ratio as a consequence of a Clebsch-Gordan relation.

What happens if some of the fields that contribute to the matrices (14), (15) and (16) come from the **27**'s in $b_{1,1}(\mathbf{27} + \overline{\mathbf{27}})$? If these fields do not mix significantly with those in the "regular" families, then the analysis that we have presented here suffers no change. More generally the matrices (14), (15) and (16) are matrices each entry of which is itself a matrix with family indices. In this case then some of the entries in the matrices (14 - 16) will will have a higher matrix dimension than others. The problem is then similar to that of intergenerational mixing. It could conceivably become complicated in situations where mixings are large. When they are small we do not see any problem arising. It would be interesting to anlayze "non-standard" cases in this context.

We can now consider the case in which G and $\overline{G}$ are non-abelian. This is only slightly more complicated than the preceeding case, but leads to the conclusion that it is inconsistent with having the m_b/m_τ mass ratio being a consequence of a Clebsch-Gordan relation. To see this we first note that the only parameterizations of $\mathbf{U}_g$ for non-abelian $\overline{G}$ that are consistent with unbroken electric charge are of the form

$$\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \mu^{-2} & \\ & \mathbf{V}_g \end{bmatrix} \qquad \text{...(22)}$$

in the $SU(3)_c \times SU(3)_L \times SU(3)_R$ basis where $\mathbf{V}_g$ is a 2×2 matrix representation* of G and $\det(\mathbf{V}_g) = \mu^2$. The only non-abelian irreducible representations of $\overline{G}$ that are relevant for phenomenololgy are 2-dimensional ones. Next we need the decomposition of the **27** under

*Note that we have changed the notation here slightly from that in ref. [15] by having a μ^{-2} (rather than μ) in $\mathbf{U}_g$ and thus having $\det(\mathbf{V}_g) = \mu^2$.

$E_6 \supset SU(3)_c \times SU(2)_L \times SU(2)_N \times U(1)_{8L} \times U(1)_{8N}$:

$$\begin{aligned} \mathbf{27} = {} & A_{-1}(1, 2, 2; -1, 1) + B_1(1, 2, 1; -1, -2) + C_0(1, 1, 2; 2, 1) \\ & + D_2(1, 1, 1; 2, -2) + E_{1/3}(3, 2, 1; 1, 0) + F_{-2/3}(3, 1, 1; -2, 0) \\ & + G_{2/3}(\overline{3}, 1, 2; 0, -1) + H_{-4/3}(\overline{3}, 1, 1; 0, 2) \qquad \ldots(23) \end{aligned}$$

where $SU(2)_N$ is the SU(2) contained in $SU(3)_R$ that sits in the lower right 2x2 bolck. The particle content of each of the components of the **27** under this decomposition as well as their $SU(2)_N$ and $SU(2)_L$ transformation properties is shown in Table I. As before there are two possible cases (see the appendix for a discussion of the boundary condition (1) in the non-abelian case):

Case I: $e_R{}^c$ and d_L come from the same multiplet of **27**'s <u>and</u> e_L and/or E_L and $d_R{}^c$ and/or $D_R{}^c$ come from the same multiplet of **27**'s

Case II: $e_R{}^c$ and $d_R{}^c$ and/or $D_R{}^c$ come frome the same multiplet of **27**'s <u>and</u> d_L and e_L and/or E_L come from the same multiplet of **27**'s.

It is clear that case II cannot work since, for example, $e_R{}^c$ (D_2) is a **1** under $SU(2)_N$ while (e_L, E_L) (A_{-1}) is a **2** under $SU(2)_N$: both of these cannot come from the same multiplet of **27**'s since it would have to transform as both a **1** and **2** under G in order to satisfy the boundary condition. In case I this problem does not arise since both D_2 and $E_{1/3}$ are **1**'s under $SU(2)_N$ and both A_{-1} and $G_{2/3}$ are both **2**'s under $SU(2)_N$. However, another problem prevents success in this case. D_2 and $E_{1/3}$ can indeed come from the same **27** (this is no different than what arose in the abelian case since they are both **1**'s under $SU(2)_N$ and hence $\overline{G}$) if the constraint $\alpha^2\mu^{-2} = \alpha \Rightarrow \alpha = \mu^2$ is satisfied. For A_1 and $G_{2/3}$ to each give rise to a linear combination that satisfies the boundary condition and which comes from a single multiplet of 27's (a particular 2-dimensional irreducible represenation under G) they must transform under <u>equivalent</u> 2-dimensional irreducible representations under G. A_{-1} transforms as part of a 3 under $SU(3)_R$ and therefore transforms as $\alpha^{-1}V_g$ under $\overline{G}$. In contrast, $G_{2/3}$ transforms as a $\overline{3}$ under $SU(3)_R$ and hence transforms as $V_g{}^\dagger = V_{g^{-1}}$. We can easily prove that these two representations are equivalent if and only if the V_g commute for all $g \in G$. If $\alpha^{-1}V_g$ is equivalent to $V_g{}^\dagger$ then there is a unitary matrix S (independent of g) such that $\alpha^{-1}V_g = SV_g{}^\dagger S^\dagger$ for all $g \in G$. Now let $g = g_1 g_2$ and $\alpha = \alpha_1\alpha_2$ then $\alpha^{-1}V_g = \alpha_1{}^{-1}\alpha_2{}^{-1}V_{g1}V_{g2}$ and $SV_g{}^\dagger S^\dagger = S(V_{g1}V_{g2})^\dagger S^\dagger = SV_{g2}{}^\dagger V_{g1}{}^\dagger S^\dagger = SV_{g2}{}^\dagger S^\dagger SV_{g1}{}^\dagger S^\dagger = \alpha_1{}^{-1}\alpha_2{}^{-1}V_{g2}V_{g1}$. Hence $V_{g1}V_{g2} = V_{g2}V_{g1}$. Thus $\overline{G}$ cannot be non-abelian consistent with having the m_b/m_τ mass ratio a consequence of a Clebsch-Gordan relation.

(IV) -- The m_b/m_τ mass relation in the SO(10) and SU(5) cases

The solution to the problem of when the m_b/m_τ mass relation can be a consequence of a Clebsch-Gordan relation in the SO(10) and SU(5) cases can be easily obtained from the analysis that we have just done of the E_6 case. To do this we need only to explicitly break E_6 to SO(10) (or SU(5)) while noting two differences between these cases and the E_6 case. The first difference is in how to specify what explicit breaking is needed. For our purposes this explicit breaking just appears in the superpotential as an increase in the number of Yukawa couplings. Thus, for example, in the SO(10) decomposition of E_6 the superpotential term $\mathbf{27}_1\ \mathbf{27}_2\ \mathbf{27}_3$ will contain terms such as $\mathbf{16}_1\ \mathbf{10}_2\ \mathbf{16}_3$ and $\mathbf{16}_1\ \mathbf{16}_2\ \mathbf{10}_3$. In the E_6 case all of these terms would have Clebsch-Gordan relations between their coefficients. When we explictily break E_6 to SO(10) all of these coefficients become independent Yukawa couplings. The second distinction was discussed at the end of section II where we considered the G transformation properties of the zero-modes in the SO(10) and SU(5) cases. There we noted that we do not have much information (in the SO(10) case, say) on the G transformation properties of the zero-modes that transform as the **10** representation. However, in the SO(10) decomposition of the E_6 case, we do know how the SO(10) **10**'s that come from the **27**'s that transform as copies of the regular representation R transform under G: They simply transform as the same number of copies of R. In the SO(10) case this information is lost. The best that we can do is to consider the consequences of assuming how many **10**'s there are and how they transform under G.

Let us consider the SO(10) case (and consider the SU(5) case as a further truncation of the SO(10) case) and determine whether the truncation (via explicit breaking) as we have just discussed still allows for the m_b/m_τ mass relation to be a consequence of a Clebsch-Gordan relation. Then we need to determine under what conditions this relation is not necessarily implied for the first two families. Thus we need to truncate case I and cases IIA and IIB from our analysis of E_6. We can make quick work of case I. Since in the E_6 case Wilson line symmetry breaking left an unbroken SU(5) that could not be reduced to the standard model through the vacuum values of zero-mode fields, the same situation obtains here. It again occurs in the SU(5) case. To consider the truncation of cases IIA and IIB we must remind ourselves which components of the ***27*** come from which SO(10) representations. Refering to eqn (9) and the discussion in the paragraph just preceeding eqn (10) we see that $G_{1/3}$, D_2, $L_{4/3}$, $J_{2/3}$ and C_{-1} come from **16** and $K_{2/3}$, B_{-1}, $H_{-2/3}$ and A_1 come from the **10**.

In case IIA we had $G_{1/3}$ and B_{-1} come from the same ***27*** ($\mathbf{27}_1$) and we had D_2 and $K_{2/3}$ come from the same (another) ***27*** ($\mathbf{27}_2$). In the product $\mathbf{27}_1\mathbf{27}_2$ the relevant terms are (under SO(10)) $\mathbf{16}_1\,\mathbf{10}_2$ (from which the product $G_{1/3}K_{2/3}$ comes) and $\mathbf{16}_2\mathbf{10}_1$ (from which the term D_2B_{-1} comes). In the E_6 case these terms had the same (common) Yukawa coupling as a coefficient. In the present case they do not, thus the m_b/m_τ mass ratio cannot be a consequence of a Clebsch-Gordan relation in the truncation of case IIA. It also follows that in the SU(5) case there can be no Clebsch-Gordan relation in the m_b/m_τ mass ratio in the truncation of case IIA.

The situation in case IIB is different. There $G_{1/3}$ and C_{-1} came from the same ***27*** ($\mathbf{27}_1$) and D_2 and $J_{2/3}$ came from the same ***27*** ($\mathbf{27}_2$). In the product $\mathbf{27}_1\mathbf{27}_2$ the relevant terms in this case are (under SO(10)) $\mathbf{16}_1\,\mathbf{16}_2$ from which <u>both</u> the terms $G_{1/3}J_{2/3}$ and D_2C_{-1} come. They are thus related via a Clebsch-Gordan coefficient. (In the SU(5) truncation of this we find that (under $\mathbf{16}=\overline{\mathbf{5}}+\mathbf{10}+\mathbf{1}$) the term $G_{1/3}J_{2/3}$ comes from $\mathbf{10}_1\overline{\mathbf{5}}_2$ and the term D_2C_{-1} comes from $\mathbf{10}_2\overline{\mathbf{5}}_1$. These two terms now have distinct Yukawa couplings and thus a Clebsch-Gordan relation is precluded.) We complete the analysis of this case by considering a spectrum of possibilities for how the **10**'s transform. One extreme example is where there are no SO(10) **10** zero-modes on K_0 transforming under G so as to have fields transforming as B_{-1} and $K_{2/3}$ in the spectrum on K_0/G. In this case we will have the same relation, $m_{-1/3}/m_{-1}=1$, for all families at the unification scale since the mass matrices are one dimensional (neglecting family indices) as compared to the E_6 case in which they were two dimensional. An intermediate set of cases is to have **10**'s that transform under G so as to have some copies of the fields B_{-1} and $K_{2/3}$ (by our discussion from section II on the G transformation properties of the **10**'s we see that an equal number of copies of A_1 and $H_{-2/3}$ will also be in the spectrum); but, fewer than the number of families. If, for the sake of illustration we neglect family mixing, then those families that have one dimensional mass matrices will necessarily have the relation $m_{-1/3}/m_{-1}=1$, while those that have two dimensional mass matrices need not have this relation as we discussed in the preceeding section. However, there is one subtlety here in that it is necessary for there to also be at least one SO(10) **1** in the spectrum on K_0/G, and that it obtain a vacuum expectation value so that the terms M_1, M_2, M_1, and M_2 do not all vanish in eqns (14) and (16). Thus, from this point of view, we would want to have at least two copies of the fields B_{-1} and $K_{2/3}$, so that the mass relation $m_{-1/3}/m_{-1}=1$ need not be necessary for the first two families. In the case of three copies the situation would be the same as that encountered in the E_6 case. If there were more than three copies, the situation would be similar to the E_6 case where some of the relevant fields were comming from the $b_{1,1}(27+\overline{27})$.

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APPENDIX -- The boundary condition in the non-abelian case

The boundary condition (1) is obtained by requiring that allowable modes on K_0/G are single valued. (1) can be derived from this demand as follows*. If $\psi(x)$ is a field on K_0 (possibly with E_6 indices) then as Ψ is parallel transported from x to x+dx, ψ will change by a gauge transformation: $\psi(x) \rightarrow \exp[\, iA_m dx^m]\psi(x)$. As we move from the point x to the point gx on K_0, we pick up the full factor $\mathbf{U}_g$; however, for $\psi(x)$ to be single valued as a mode on K_0/G, the result of parallel transporting from x to gx must equal the value of the field ψ at gx, $\psi(gx)$. Thus, we must have

$$\mathbf{U}_g\psi(x)=\psi(gx). \qquad \text{(A-1)}$$

This assumption assumes that any potentially path dependent factors that might have been picked up in the course of the parallel transport have canceled amongst themselves [9]. Let us assume that the $\mathbf{U}_g$ constitute an n dimensional irreducible representation of $\overline{G}$, where n>1. Then for ψ to satisfy (A-1) it must come from a representation Ψ that transforms as the same irreducible representation of G, $\mathbf{V}_g$, where $\mathbf{U}_g$ is <u>numerically</u> equal to $\mathbf{V}_g$, but they act on different spaces. That is to say, ψ is some linear combination of the components of $\mathbf{\Psi}$. We wish to determine this linear combination. The field $\mathbf{\Psi}$ is an nxn matrix (in the E_6 case we refered to this as comming from a multiplet of **27**'s). The first index of $\mathbf{\Psi}$ is a gauge group index and the second index is a representation index under G (thus $\overline{G}$ acts from the left and G acts from the right). Using this we can write down the boundary condition more accurately. ψ is a linear combination of the components of $\mathbf{\Psi}$ which we write as

$$\psi = \mathrm{Tr}(\mathbf{A}\mathbf{\Psi}) \qquad \text{.(A-2)}$$

*This was discussed in ref [15]; however, the argument was a bit backwards even though it lead to the correct conclusion as a result of the combination of two incorrect statements. It is corrected here, and is discussed in much greater detail in [9].

where $\mathbf{A}$ is a numerical nxn matrix to be determined. The boundary condition then reads

$$\mathrm{Tr}(\mathbf{A}\mathbf{U}_g\boldsymbol{\Psi}(x)) = \mathrm{Tr}(\mathbf{A}\boldsymbol{\Psi}(gx)) \qquad \text{(A-3)}$$

We thus see how $\mathbf{U}_g$ appears in the boundary condition. We now have to determine how $\mathbf{V}_g$ appears; i.e., how do we express $\boldsymbol{\Psi}(gx)$ in terms of $\boldsymbol{\Psi}(x)$ and $\mathbf{V}_g$? A naive guess would be $\boldsymbol{\Psi}(gx) = \boldsymbol{\Psi}(x)\mathbf{V}_g^T$; however, this is not right. The correct answer follows if we carefully examine what the notation means. The quantity $\boldsymbol{\Psi}(x)$ denotes the field $\boldsymbol{\Psi}$ evaluated at the point x. Thus it is true that under the action of g the field $\boldsymbol{\Psi}$ transforms as $g\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{V}_g^T$, and this is true evaluated at each point. The transpose is necessary since, if $g=g_1g_2$, then $g\boldsymbol{\Psi}=g_1(g_2\boldsymbol{\Psi})=g_1(\boldsymbol{\Psi}\mathbf{V}_2^T)=\boldsymbol{\Psi}\mathbf{V}_2^T\mathbf{V}_1^T$, where we recognized that $g_1(\boldsymbol{\Psi}\mathbf{V}_1^T)$ means g_1 acting on the field $(\boldsymbol{\Psi}\mathbf{V}_1^T)$. If we compare this to $g\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{V}_g^T$ we see that $\mathbf{V}_g^T = \mathbf{V}_2^T\mathbf{V}_1^T$ and thus $\mathbf{V}_g = \mathbf{V}_1\mathbf{V}_2$ as is necessary for the mapping $g\rightarrow\mathbf{V}_g$ to be a group homomorphism. The relationship between $(g\boldsymbol{\Psi})$ evaluated at the point x and $\boldsymbol{\Psi}$ evaluated at the point gx is $(g\boldsymbol{\Psi})(x)=\boldsymbol{\Psi}(gx)$. Thus, $(g_1g_2\boldsymbol{\Psi})(x) =(g_1(g_2\boldsymbol{\Psi}))(x) = (g_2\boldsymbol{\Psi})(g_1x) =\boldsymbol{\Psi}(g_2g_1x)$. As a consequence of this and $g\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{V}_g^T$, it follows that

$$\boldsymbol{\Psi}(gx)=\boldsymbol{\Psi}(x)\mathbf{V}_g. \qquad \text{(A-4)}$$

Thus, the boundary condition (A-3) reads

$$\mathrm{Tr}(\mathbf{A}\mathbf{U}_g\boldsymbol{\Psi}(x)) = \mathrm{Tr}(\mathbf{A}\boldsymbol{\Psi}(x)\mathbf{V}_g). \qquad \text{(A-5)}$$

Hence, $\mathrm{Tr}\{(\mathbf{A}\mathbf{U}_g - \mathbf{V}_g\mathbf{A})\Psi(x)\}=0$. Since this is true for all values of $\Psi(x)$, it follows that $(\mathbf{A}\mathbf{U}_g - \mathbf{U}_g\mathbf{A})=0$ for all $g\in G$, where we have now replaced $\mathbf{V}_g$ by $\mathbf{U}_g$ since this is just a matrix equation. From Schur's lemma it now folows that $\mathbf{A}$ is proportional to the identity matrix. Thus, finally, we see that there is exactly one linear combination of the components of Ψ that satisfies the boundary condition. From (A-2) this is (since $\mathbf{A}\propto\mathbf{1}$)

$$\psi = \mathrm{Tr}(\Psi) \tag{A-6}$$

within a normalization constant.

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TABLE I

		$SU(2)_N$	$SU(2)_L$
$G_{2/3}$	$= \begin{pmatrix} d_R{}^c \\ D_R{}^c \end{pmatrix}$	2	1
$F_{-2/3}$	$= D_L$	1	1
$E_{1/3}$	$= \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	1	2
A_{-1}	$\supset \begin{pmatrix} e_L \\ E_L \end{pmatrix}$	2	2
D_2	$= e_R{}^c$	1	1
B_1	$\supset E_R{}^c$	1	2