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Einstein-Kalb-Ramond Cosmology

J. A. Stein-Schabes and M. Gleiser *

Theoretical Astrophysics Group

Fermi National Accelerator Laboratory

Batavia Illinois, 60510

ABSTRACT

We study possible cosmological solutions to higher dimensional model of gravity with a 3-form taking values in the physical space, and show that it is possible to integrate Einstein's equations exactly for flat physical and internal spaces. We then present a detailed analysis of the possible trajectories in phase plane of the Hubble factors and find the allowed regions for a physically acceptable cosmology. These turn out be rather small.

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* On leave of absence from Department of Mathematics, King's College London. Address after September 1986: Theoretical Astrophysics Group, Fermi National Accelerator Laboratory, Batavia Illinois, 60510

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Unification of gravity with gauge interactions in higher dimensional space-times has been the object of renewed interest for the past 10 years. With the advent of supersymmetry it was hoped that the long standing problem of quantizing the gravitational field would eventually be resolved in a similar way to the $N = 4$ super Yang-Mills model. Nevertheless, once we formulate supergravity theories in higher dimensions, the basic diseases of quantum gravity were still present and only mildly improved. So, the proof that superstring theories [1] were finite to at least one-loop and anomaly free in 10 dimensions [2,3], gave new strength to higher dimensional theories. Gravity at distances of the order the Planck length would have a radically different behaviour, with the gravitational interactions being mediated by massive modes of the string, very much like weak interactions being mediated by massive vector bosons for $E \geq 100GeV$.

With such radically different gravitational behaviour at Planck energies, many changes will take place in our understanding of the initial singularity and evolution of the early Universe [4]. We do not propose to tackle these problems here but only to study the classical evolution of a higher-dimensional model with two dynamical fields, the graviton g_{MN} and the Kalb-Ramond field strength $F_{MNP} = 3\partial_{[M}A_{NP]}$. The interest in studying cosmologies with the above 3-form comes from the fact that it plays a fundamental role in string models and is always present when we take the point-like limit of strings and consider the ten dimensional supergravity action[5,6] as the starting point of a unified field theory.

We will show that, for this simple model, it is possible to integrate Einstein's equations exactly in terms of the Hubble factors for the physical and internal spaces. This procedure has an interest on its own as it provides a powerful tool to extract a wealth of qualitative information about the possible regions of the phase space that allow for viable cosmologies. We shall see that the set of initial conditions that produce a Friedmann-like model is very small(possibly of measure zero).

The action can be written as

$$S = -\int d^{10}z\sqrt{-g^{10}}\left\{\frac{R}{16\pi G} - \frac{1}{6}F_{MNP}F^{MNP}\right\} \tag{1}$$

Setting $16\pi G = 1$ the field equations take the form

$$R_{MN} - \frac{1}{2}g_{MN}R = -T_{MN} \tag{2.1}$$

with

$$T_{MN} = F_{MPQ}F_{M}^{PQ} - \frac{1}{6}F_{PQR}F^{PQR}g_{MN} \tag{2.2}$$

$$\partial_M\left[\sqrt{-g^{10}}F^{MNP}\right] = 0 \tag{3}$$

where capital latin indices run from 0 to 9, greek indices run from 0 to 3 and small latin indices from 4 to 9.

In order to obtain a compactified solution of the field equations, we follow Myung and Kim[7] and write the Kalb-Ramond field components as

$$F_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\Phi^{\sigma} \tag{4.1}$$

$$F_{mnp} = F_{\mu\nu p} = F_{\mu np} = 0 \tag{4.2}$$

where $\Phi^{\sigma} = \partial^{\sigma}\eta(x)$ can be thought of as being the gradient of a space-time dependent scalar field $\eta(x)$. It is easy to check that the Kalb-Ramond field equation (eq. (3)) is trivially satisfied with the above ansatz.

In order to obtain a suitable cosmological description of the above model, we assume, as usual, that the line element is given by the generalized Robertson-Walker metric[8],

$$ds^2 = -dt^2 + R_3(t)^2 \tilde{g}_{ij}(x^l) dx^i dx^j + R_6(t)^2 \tilde{g}_{mn}(y^p) dx^m dx^n \tag{5}$$

If we additionally impose on the field $\eta(x)$ the condition of being only time dependent, we obtain the following field equations, for flat 3 and 6 dimensional spaces,

$$3\frac{\ddot{R}_3}{R_3} + 6\frac{\ddot{R}_6}{R_6} = -\frac{1}{2}\dot{\eta}^2 \tag{6.1}$$

$$\frac{d}{dt}\left(\frac{\dot{R}_3}{R_3}\right) + \left(3\frac{\dot{R}_3}{R_3} + 6\frac{\dot{R}_6}{R_6}\right)\frac{\dot{R}_3}{R_3} = \frac{3}{2}\dot{\eta}^2 \tag{6.2}$$

$$\frac{d}{dt}\left(\frac{\dot{R}_6}{R_6}\right) + \left(3\frac{\dot{R}_3}{R_3} + 6\frac{\dot{R}_6}{R_6}\right)\frac{\dot{R}_6}{R_6} = -\frac{1}{2}\dot{\eta}^2 \tag{6.3}$$

The authors of ref. (7) claimed to have found a particular solution to equation (6) where the physical 3-space expand exponentially while the internal space oscillates in time. However, the solutions were obtained under the assumption of having constant expansion rates (H_3 and H_6 constants), which is a strong constraint on the dynamical system. Furthermore, we would like to point out that their solution is inconsistent with the field equations, as these do not accept an oscillating solution for the internal manifold.

Here, we shall integrate eq.(6) in an exact way and later analyse the dynamical behaviour of the system. If we introduce the logarithmic variables $H_3 = \frac{d(lnR_3)}{dt}$ and $H_6 = \frac{d(lnR_6)}{dt}$ we can rewrite eq.(6) as a first order quadratic differential system as follows,

$$3\dot{H}_3 + 3H_3^2 + 6\dot{H}_6 + 6H_6^2 = -\frac{1}{2}\dot{\eta}^2 \tag{7.1}$$

$$\dot{H}_3 + 3H_3^2 + 6H_3H_6 = \frac{3}{2}\dot{\eta}^2 \tag{7.2}$$

$$\dot{H}_6 + 6H_6^2 + 3H_3H_6 = -\frac{1}{2}\dot{\eta}^2 \tag{7.3}$$

substituting (7.1) into (7.2) and (7.3) we get

$$\dot{H}_3 = \frac{3}{2}H_3^2 + 21H_3H_6 + \frac{45}{2}H_6^2 \tag{8.1}$$

$$\dot{H}_6 = -\frac{3}{2}H_3^2 - 12H_3H_6 - \frac{27}{2}H_6^2 \tag{8.2}$$

In order to solve eqs. (8.1) and (8.2) exactly, we write H_3 and H_6 in terms of new variables r and θ as

$$H_3 = rsin\theta \tag{9.1}$$

$$H_6 = rcos\theta \tag{9.2}$$

After some algebra, eq.(8) reduces to

$$\dot{r} = \frac{1}{2}r^2(3sin^3\theta + 39sin^2\theta\cos\theta + 21sin\theta\ cos^2\theta - 27\cos^3\theta) \tag{10.1}$$

$$r\dot{\theta} = \frac{1}{2}r^2(3sin^3\theta + 27sin^2\theta\cos\theta + 69sin\theta\cos^2\theta + 45\cos^3\theta) \tag{10.2}$$

Upon introduction of yet another new variable $x = \tan\theta$, eq.(10) can be writen as a single equation relating r and x as follows

$$\frac{dr}{r} = \left(\frac{x^3 + 13x^2 + 7x - 9}{x^3 + 9x^2 + 23x + 15}\right)\frac{dx}{1 + x^2} \tag{11}$$

By using partial fractions, eq.(11) can be easily integrated to give

$$r = r_0 \left[\frac{(x^2+1)^2(x+5)^3}{(x+3)^6(x+1)} \right] \tag{12}$$

As is well known from the theory of non-linear differential equations, the zeros of eq.(10.2) give the invariant lines in the phase plane $H_3 \times H_6$ [9]. These invariant lines are solutions of eqs. (8.1) and (8.2) where $H_3 = \alpha H_6 (\alpha = constant)$, which in our case will be simply given by

$$H_3 = -H_6 \ ; \ H_3 = -3H_6 \ ; \ H_3 = -5H_6 \tag{13}$$

These lines are shown in fig. 1 with the labels α_1, α_2 and α_3 (see fig. 1) respectively.

Fig. 1

Another important property of our dynamical system is that $H_3 = H_6 = 0$ is a critical point. Using this information together with the invariant lines and the solution (12), we can construct the phase plane diagram $H_3 \times H_6$ as shown in fig.1. Note that the lines α_2 and α_3 were not drawn with the right slope to allow an easier reading of the trajectories.

In what follows we will try to extract all the possible information from the dynamical analysis of the phase plane diagram.

First we note that along the invariant lines we can obtain the exact solution for the scale factors as functions of time;

i) $\alpha_1 = -1$, we find that

$$H_3(t) = -H_6(t) = \frac{H_0}{1-3H_0 t} \ ; R_3 \sim t^{-\frac{1}{3}} \ ; \ R_6 \sim t^{\frac{1}{6}} \tag{14.1}$$

ii) $\alpha_2 = -3$, we find that

$$H_3(t) = -3H_6(t) = \frac{H_0}{1+3H_0 t} \ ; \ R_3 \sim t^{\frac{1}{3}} \ ; \ R_6 \sim t^{-\frac{1}{9}} \tag{14.2}$$

iii) $\alpha_3 = -5$, we find that

$$H_3(t) = -5H_6(t) = \frac{H_0}{1 + \frac{9}{5}H_0 t} \; ; \; R_3 \sim t^{\frac{5}{9}} \; ; \; R_6 \sim t^{-\frac{1}{9}} \tag{14.3}$$

where H_0 is the initial values for H_3 (H_6). The solution described by (14.1 and 14.3) are special cases of those found by Chodos and Detweiler in ref. (8). They represent Kasner solutions asymptotically.

The arrows for the critical lines are obtained by analysing the sign changes in $H_3(t)$ and $H_6(t)$ for each solution (or that of $\dot{r}$ in eq.(10. 1). The reader can easily verify that only solutions (14.1) and (14.3) are compatible with the field equations for $\dot{\eta} = 0$. It is not sufficient to find the power law solutions from the critical lines since the consistency condition, eq. (6.1) has to be satisfied together with eqs. (6.2) and (6.3). Alternatively, we could see that the whole region *between* the α_1 and α_3 lines (regions II, III, V and VI) is excluded (shadow region in fig.1) since it would produce a negative or imaginary radius (trajectories in these parts of the phase plane are of no physical significance).

As a second step we have to follow the allowed trajectories for all possible initial conditions for H_3 and H_6. We can do this by studying each of the allowed regions, namely, regions I, IV and the two invariant lines α_1 and α_3. We will concentrate on the region $H_3 \geq 0$ since we consider that only an initially expanding 3-space is of physical relevance. Nevertheless, once the behaviour for region I is understood, then the behaviour for region IV follows from symmetry arguments.

If the initial conditions are such that both H_3 and H_6 are positive (quadrant I), the physical radius R_3 expands (H_3 increases) until a maximum expansion rate is reached after which a deceleration phase takes over, eventually the model gets to a point where there is no expansion or contraction ($H_3 = 0$). The internal radius (R_6), on the other hand, expands with an increasing acceleration, the model approaches the invariant line α_1 asymptotically. This is not a satisfactory solution since the model describes a contracting

three space and an expanding internal space. However, we should be careful with the timescales with which this solutions approach the invariant line. With this in mind we could say that the subregion of I than lies in the second quadrant, could be acceptable as it describes an expanding three space and a contracting internal space, but only for a finite time.

Turning now to region IV we obtain a universe where the three radius expands, then contracts and decelerates to eventually reach a static point ($H_3 \rightarrow 0$). The internal space on the other hand contracts for some time, then expands and also tends to a static point ($H_6 \rightarrow 0$). The model in this region takes an infinte time to reach the critical point. Unfortunately, the critical point is a saddle point so its stability depends strongly on the type of perturbations. If we assume the Universe starts in that position then its easy to see that if perturbations displace the solution towards the upper half of the phase space then the solution is unstable, while the contrary occurs if the perturbation, acts the other way.

The same can be said about the invariant lines concerning their stability as solutions of the model. The only acceptable solution is that describe by the α_3 line (eq. 14.3). This case is not bad since it provides, at least qualitatively, the right type of behaviour, an expanding physical universe and a contracting internal one. This solution is stable as long as perturbations are such that trajectories do not enter the forbiden regions (shaded region).

Going back through all the analysis, we find a surprisingly small region of the whole phase plane to be of possible physical relevance. If we assume a strictly conservative point of view, only the α_3 line (and possibly a small neigbourhood around the line) in the IInd quadrant provides an expanding universe, with a power law behaviour very similar to the radiation dominated scenarios of conventional 4-dimensional cosmologies, and a shrinking internal space, as desired from observations. We expect that when the internal

radius approaches the singularity, quantum effects will play a balancing role, stopping the collapse.

Although this is a highly simplified model, it is reasuring to know that this behaviour is not completly ruled out a priori.

In any case, it does not seem possible to obtain an exponential inflation in this model. If we start with an unaturally large internal space, assume adiabatic expansion for the whole 10-dimensional space-time and wait long enough (since a power law expansion is relatively slow), the shrinking of the internal space may generate the correct amount of entropy [10].

It must be said that quantum effects coming from one loop corrections on the matter fields, or the Casimir forces, will probably play an important role at these energies, and were neglected for simplicity. We intend to gradually add more structure to the field equations to understand precisely the function of each of the extra terms. For example, with the inclusion of the Yang-Mills field plus a gaugino condensate as an effective cosmological constant[11], the internal radius will acquire a potential that may lead to an inflationary phase in the physical space-time. If this is correct, superstrings will be as viable cosmologically as they are as candidates for a finite theory of gravity. And, surely, these two aspects should be manifestly unified in a coherent theory of physical interactions.

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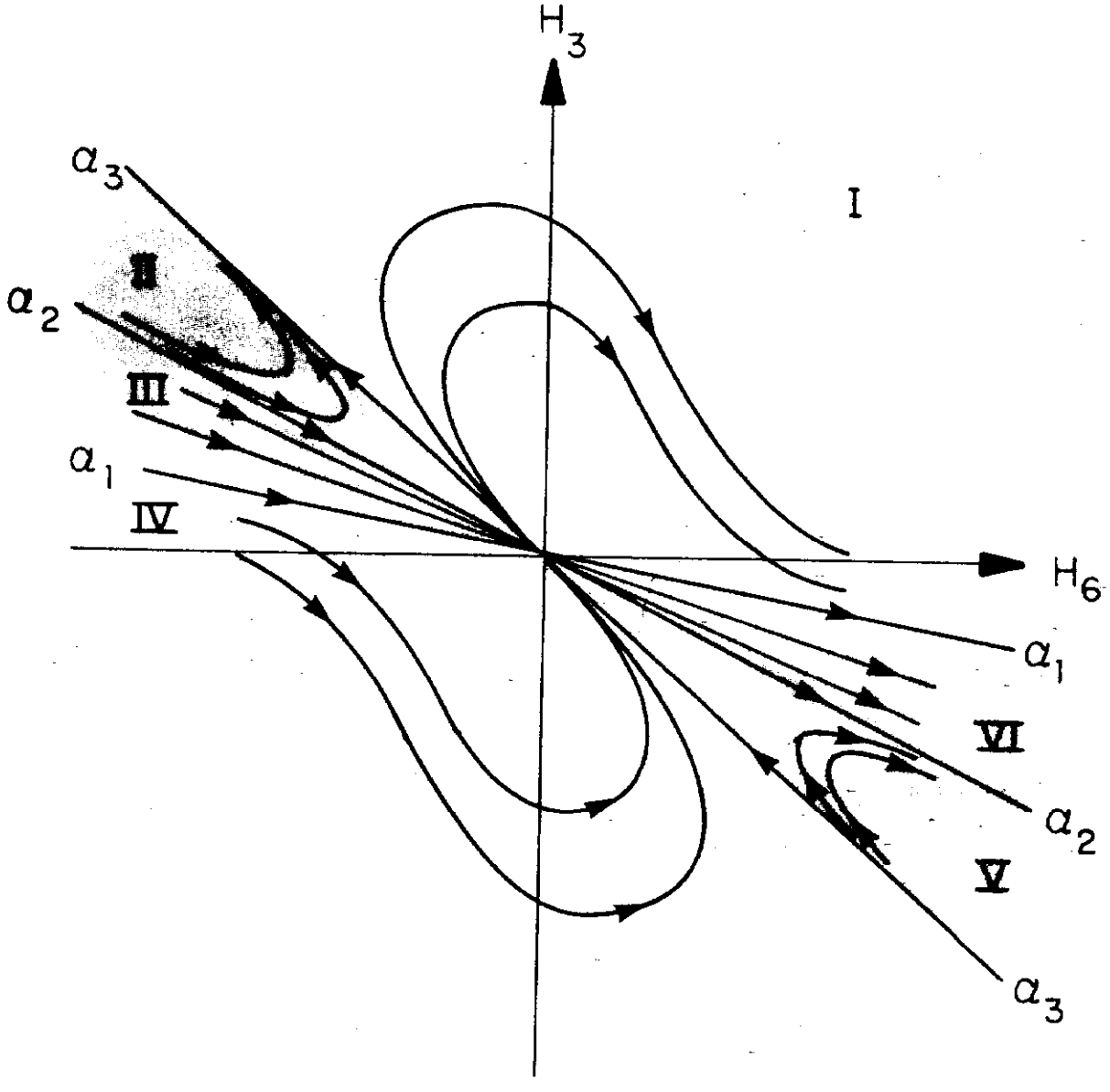
Figure Caption

Fig. 1 The Phase Plane trajectories are shown for a variety of possible initial values for H_3 and H_6. Note that the shadow region is excluded from analytical considerations.


H_3
H_6
α_3
α_2
α_1
I
II
III
IV
V
VI
α_1
α_2
α_3