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Thermodynamics of Higher Dimensional Black Holes

Revised

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Abstract

We discuss the thermodynamics of higher dimensional black holes with particular emphasis on a new class of spinning black holes which, due to the increased number of Casimir invariants, have additional spin degrees of freedom. In suitable limits, analytic solutions in arbitrary dimensions are presented for their temperature, entropy, and specific heat. In $5+1$ and $9+1$ dimensions, more general forms for these quantities are given. It is shown that the specific heat for a higher dimensional black hole is negative definite if it has only one non-zero spin parameter, regardless of the value of this parameter. We also consider equilibrium configurations with both massless particles and massive string modes.

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1. Introduction

The connection between thermodynamics, event horizons and quantum theory has provided insights on what can be expected from a quantum theory of gravity. Though such a theory is still not at hand, developments in recent years have turned attention to theories in more than four dimensions which may incorporate quantum gravity.[1] Of these, the most promising are superstrings, which are formulated in ten dimensions. In addition to a consistent quantum theory of gravity, superstrings may provide a unified theory of all fundamental forces.

In studying higher dimensional theories, the thermodynamic connection can still provide insights, as has been demonstrated by the recent work on superstrings at high temperatures.[2] Given the central role played by black holes in understanding the relation of thermodynamics to gravity theory in four dimensions, the consequences of higher dimensions on their physics is of prime interest.

For the $N+1$ dimensional Einstein equations, Myers and Perry[3] have shown that a spinning black hole solution is characterized by $[N/2]+1$ parameters ($[N/2]$ is the integer part of $N/2$), $[N/2]$ angular momenta and mass. Analogues of Reissner-Nordstrøm solutions have also been discussed by Myers and Perry. However, the global structure of these solutions is exactly the same as for the four dimensional case and we will not consider them in this paper. What is significant about the spinning solutions is that the angular momenta are not all constrained by cosmic censorship to obey an inequality with respect to the total mass of the black hole — some can take on arbitrarily large values while still assuring the existence of a horizon. The additional observation by Myers and Perry that the laws of black hole thermodynamics are equally applicable in higher dimensions permits an investigation of the thermodynamics of these solutions.

Hawking's discovery[4] that black holes emit particles with a thermal distribution allowed the explicit identification of the thermodynamic quantities, temperature and entropy, and with the geometric quantities, surface gravity κ and horizon area A: $T = \hbar\kappa/(2\pi k_B)$ and $S = k_B A/(4\hbar)$. For black holes in four dimensions, the connection with thermodynamics is summarized in the four laws of black hole thermodynamics.[5] In geometric form, these are, the zeroth law: κ is a constant over the horizon of a stationary black hole. First law: if M is the total mass of the black hole, Ω the angular velocity of the horizon, and J the angular momentum of the black hole, then $dM = \frac{1}{8\pi G_N}\kappa dA + \Omega dJ$. Second law: in any physically allowed process, the total area of a black hole cannot decrease, i.e., $\delta A \geq 0$. Third law: in any physically allowed process, it is impossible to attain $\kappa = 0$. These are the same (with T and S substituted for κ and A) as would be expected for more common thermodynamic systems. However, the statement of the third law is not in agreement with Nernst's since $S \rightarrow$ constant as $T \rightarrow 0$.

Using the fundamental thermodynamic equations for a $3+1$ dimensional black hole, it is straight forward to calculate the various specific heats associated with it.[6] A Kerr black hole in thermal equilibrium with a heat bath at temperature T can reversibly absorb energy from the bath without affecting its angular momentum J. The specific heat at fixed J can be written

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_J = \frac{8MS^3T}{J^2 - 8T^2S^3}. \tag{1.1}$$

When $J = 0$, $C_{J=0} = -M^2$, which is the specific heat of a Schwarzschild black hole. The fact that this is a negative definite quantity indicates that the Schwarzschild black hole cannot be in equilibrium with an infinite heat bath, and will eventually evaporate. This is not unusual for a self-gravitating system — a star contracts and gets hotter as it radiates energy. As the extreme Kerr limit $M^2 = |J|$ for a black hole is approached, $T \to 0$ and $C_J \to 0+$. This sign change is due to a discontinuity in the specific heat at the value $J_c \simeq 0.68M^2$. The discontinuity is indicative of a second order 'phase transition' and for $J > J_c$, a Kerr black hole can, in fact, be in equilibrium with an infinite corotating heat bath. The discontinuity in Eq. (1.1) does not signal a true phase transition since it is not accompanied by any change in the long range order of the black hole 'system'. However it does indicate a change in the stability of the black hole and we will continue to speak of a phase transition in this restricted sense.

In higher dimensions, when only one spin parameter is non-zero, the specific heat of a spinning black hole is negative definite for $N \geq 5$ and exhibits no discontinuity. We demonstrate this explicitly for black holes in $5+1$ and $9+1$ dimensions. Thermodynamically, black holes in higher dimensions with one non-zero spin parameter are unstable in an infinite heat bath regardless of the value of the spin parameter. When more than one spin parameter is non-zero it is possible that the specific heat can become positive. However, it is not clear what the order of the associated phase transition (if any) is. Whether or not the specific heat becomes positive depends in part on the sign of $\partial M/\partial T$ becoming positive. If the specific heat is negative definite, it can be argued in a manner completely analogous to the four dimensional case that stability can hold in a heat bath of finite extent which is corotating with the black hole. We will address this issue for black holes in $5+1$

and $9 + 1$ dimensions.

It has been pointed out that string theory may be relevant to the last stages of black hole evaporation.[7] This is an intriguing suggestion which deserves closer attention. The question of unitary evolution of quantum states in black hole evaporation cannot at present be adequately addressed in the context of string theory. However, it is argued that the existence of a naked singularity as the end product of black hole evaporation can be called into question if at some point in its evolution it becomes entropically favorable for the black hole to fluctuate to a massive string mode. To arrive at this conclusion, a number of assumptions are made. The analysis is carried out in four dimensions, but the density of states $\Omega(E,V)$ used in reference [7] is computed in ten dimensional Minkowski space. Even if we neglect the possible modifications a Schwarzschild background might give, we should keep in mind that we are working with a product space $M^4 \times K$, where K is a compact space. In particular, if the compact space is not simply connected, the mappings $X^\mu(\sigma,\tau)$ from σ, τ space into $M^4 \times K$ results in loops being wrapped about the 'holes' in the compact space. These soliton states could markedly effect the density of states. If we choose to do the analysis in what is effectively a four dimensional background, we must take into account any contribution from compactification for the string (and black hole). Additionally, a particular ordering of scales is required: $M_P > M_c > M_s$, where M_P is the Planck scale, M_c is the compactification scale, and M_s is the string scale. To gain a different perspective, we can consider equilibrium configurations in ten dimensions, thereby removing uncertainties associated with compactification. Though we work in ten uncompactified dimensions, it is possible that 'decompactification' occurs at the end of black hole evaporation (from the point of view of four dimensions) making a ten dimensional analysis relevent.

The paper is organized as follows: in section 2, we review the $N + 1$ dimensional spinning black hole solutions of Myers and Perry and present the relevent thermodynamic quantities generalized to higher dimensions. In section 3 we discuss analytic solutions, in suitable limits, to these equations. In section 4, black hole thermodynamics in $5 + 1$ dimensions is discussed in detail and conditions for equilibrium to hold with massless particles are found. In section 5, we extend our analysis to $9 + 1$ dimensions and consider implications for string thermodynamics. We conclude in section 6 with a discussion of our results.

2. Review of General Results

In this section we review the results of Myers and Perry for spinning black holes in $N + 1$ dimensions (the reader is refered to their paper for details) and write down the equations relevant to the themodynamics of these black holes. We adopt the following conventions: the $N+1$ dimensional flat metric is $\eta_{\mu\nu} = diag(-1, 1, 1, \ldots)$ and greek indices range over all values, $\mu,\ \nu = 0, 1, \ldots, N$, while latin indices range over spatial values, $i, j = 1, \ldots, N$. We set $c = G^{(N+1)} = \hbar = k_B = 1$ ($G^{(N+1)}$ is the gravitational constant in $N + 1$ dimensions).

As mentioned in the introduction, an uncharged, spinning black hole in $N + 1$ dimensions is completely determined by $[N/2] + 1$ parameters (where $[N/2] = \frac{N}{2}$ $(\frac{N-1}{2})$ if N is even (odd)), $[N/2]$ spin parameters and mass. These parameters can be understood in terms of the Casimir invariants of a massive representation of $SO(N) \times R$. There are then $[N/2]$ parameters corresponding to the Cartan subalgebra $\{T^i\}$ which generate commuting rotations in the planes $x^i - y^i$. The angular momentum tensor is described by $[N/2]$ parameters J_i corresponding to rotations in the planes $x^i - y^i$, i.e., the planes in which the black hole is spinning. In the following, we will concentrate on the odd N case, drawing attention to even N solutions where necessary.

When N is odd, the metric in Boyer-Lindquist coordinates for an $N+1$ dimensional spinning black hole is (repeated indices are summed):

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\bar{\phi}_i^2) \\ &\quad + \frac{\mu r}{\Pi F}(d\bar{t} + a_i \mu_i^2 d\bar{\phi}_i)^2 + \frac{\Pi F}{\Pi - \mu r} dr^2, \end{aligned} \tag{2.1}$$

where i ranges from 1 to $\frac{N-1}{2}$, r is a radial coordinate, the ϕ_i are angles with period 2π in each plane $x^i - y^i$, and the μ_i are direction cosines with respect to these planes having the range $0 \leq \mu_i \leq 1$. The α coordinate is the result of a rescaling of the one unpaired Kerr-Schild spatial coordinate: $z = r\alpha$ with the range $-1 \leq \alpha \leq 1$ and, the μ_i and α are related via $\mu_i^2 + \alpha^2 = 1$. The spin parameters a_i are defined in terms of the angular momenta by

$$J_i = \frac{2}{N-1} M a_i, \tag{2.2}$$

while the total mass M is written

$$M = \frac{(N-1)A_{N-1}}{16\pi}\mu. \tag{2.3}$$

Here A_{N-1} is the area of the unit $N-1$ sphere:

$$A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{2.4}$$

Finally,

$$F = 1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \tag{2.5}$$

while

$$\Pi = \prod_{i=1}^{(N-1)/2} (r^2 + a_i^2). \tag{2.6}$$

For even N, the metric is similar to (2.1). In particular, there are no unpaired coordinates while the range of i is changed in Eqs. (2.5) and (2.6).

By examining the term $\frac{\mu r}{\Pi F}$ in the metric, one finds that singularities occur at $r = 0$ when any $a_i = 0$ or when all $a_i \neq 0$. These metric singularities correspond to curvature singularities, though for all $a_i \neq 0$, it is possible to extend r to negative values with results similar to those in $3 + 1$ dimensions.

The horizon in this metric occurs when $1/g_{rr} = \frac{\Pi - \mu r}{\Pi F}$ vanishes. Thus, we must consider the roots of

$$\Pi - \mu r = 0. \tag{2.7}$$

A horizon, if it exists, will occur at $r = r_0$ with topology $S^{N-1} \times \mathbf{R}$. To avoid naked singularities μ must be positive and, in turn, for $r > 0$, Eq. (2.7) has only one extremum at $r = \tilde{r}$ instead of a possible $N - 2$ extrema. If all $a_i \neq 0$ then $\Pi - \mu r|_{r=0} > 0$ and there are three possibilities — no horizons, one degenerate horizon or two horizons — corresponding to whether (2.7) at $r = \tilde{r}$ is greater than zero, equal to zero, or less than zero. Without an analytic solution, it is still clear from these remarks that a sufficient condition for the existence of a horizon is that one of the spin parameters vanish, since in this case (2.7) at $r = \tilde{r}$ is negative with one horizon at $r = 0$ and the other at $r = r_+ > 0$. With the existence of a horizon assured, the remaining non-zero spin parameter can take on arbitrarily large values.

The above results can be obtained in a similar fashion when N is even. In this case, (2.7) is replaced by $\Pi - \mu r^2 = 0$. A sufficient condition for the existence of a horizon is that two spin parameters vanish. However, when all $a_i \neq 0$, the solutions can be rather pathological in that μ need not be positive when $r_0 > 0$.

As in $3+1$ dimensions, higher dimensional black holes have a 'static limit' or ergosurface inside of which observers cannot remain stationary; they must move in the direction of rotation of the black hole. The static limit is located where g_{tt} vanishes. This corresponds to solving $F\Pi - \mu r = 0$ for odd N, or $F\Pi - \mu r^2 = 0$ for even N. For all N, the static limit r_s lies outside the horizon at r_+. They coincide when $F = 1$.

We turn now to a brief discussion of the relevant thermodynamic equations. The methods used to derive the laws of black hole thermodynamics in $3+1$ dimensions can be equally applied in higher dimensions and the statement of the four laws is unchanged. In terms of the horizon location at r_+, the surface gravity κ for an $N+1$ dimensional black hole is

$$\kappa = 2\pi T = \begin{cases} \left.\frac{\frac{\partial}{\partial r}\Pi - \mu}{2\mu r}\right|_{r=r_+}, & \text{if } N \text{ is odd;} \\ \left.\frac{\frac{\partial}{\partial r}\Pi - 2\mu r}{2\mu r^2}\right|_{r=r_+}, & \text{if } N \text{ is even.} \end{cases} \tag{2.8}$$

The generalization of Smarr's formula[8] is

$$(N-2)M = (N-1)(\omega_i J_i + \frac{1}{8\pi}\kappa A), \tag{2.9}$$

where $\omega_i = \frac{a_i}{r_+^2 + a_i^2}$. The differential of (2.9), which corresponds to the first law of black hole thermodynamics, becomes

$$dM = \frac{\kappa}{8\pi} dA + \omega_i dJ_i. \tag{2.9}$$

Finally, the area of the horizon A is

$$A = 4S = \frac{A_{N-1}\mu}{2\kappa}\left(N - 2 - \frac{2a_i^2}{r_+^2 + a_i^2}\right). \tag{2.10}$$

The differential form of the first law given in Eq. (2.10) regards S ($\mathcal{A}$) and the J_i as a complete set of global state variables for the black holes we are discussing. Though S and the J_i are generally assumed to be extensive variables, Eq (2.9) is not a homogenous first order function with respect to them. This can be seen by rescaling the variables by their canonical dimension: $J_i \to \lambda^{N-1} J_i$, $\mathcal{A} \to \lambda^{N-1}\mathcal{A}$ while $M \to \lambda^{N-2} M$. Therefore, (2.9) is of degree $\frac{N-2}{N-1}$. In $3+1$ dimensions these scaling relations imply (for a non-spinning black hole) $S \propto M^2$, so the entropy is not an extensive variable. Still, keeping with general practice, we will continue to regard the entropy (and spin) as extensive variables.

Note that the mass associated with the rotation of the black hole is $M - M_{ir}$, where the 'irreducible' mass,

$$M_{ir} = \frac{N-1}{N-2}\frac{\kappa \mathcal{A}}{8\pi}. \tag{2.12}$$

The existence of a static limit defined by the vanishing of the norm ($= g_{tt}$) of the killing vector $n^\mu \partial/\partial x^\mu$ implies that the rotational mass can be reduced by the Penrose process[9] to the minimum value M_{ir}. We will return to this when we discuss thermodynamic processes for black holes.

3. Solutions for $N+1$ dimensions

The task now is to find analytic solutions to the equations of the previous section. Though the solutions we obtain are for black holes of arbitrary dimension, they represent only limiting cases for the spin parameters. More general solutions will be discussed in the next sections.

Knowledge of the solutions to Eq. (2.7) is crucial to understanding Eqs. (2.8)-(2.12). In general there do not exist analytic solutions to Eq. (2.7). As a first step, we will restrict to the case where $[(N-3)/2]$ spin parameters are zero, and keep one, a, non-zero, assuring the existence of a horizon. In the following sections, we will discuss means of loosening these restrictions. For odd N, things simplify greatly — Eq. (2.7) can be written

$$r^{N-2}\left(1+\frac{a^2}{r^2}\right)-\mu=0 \tag{3.1}$$

Similarly, Eq. (2.8) becomes

$$\kappa=2\pi T=\frac{\left(r_+^{N-2}\left[(N-1)+(N-3)a^2/r_+^2\right]-\mu\right)}{2\mu r_+}, \tag{3.2}$$

and

$$\mathcal{A}=4S=\frac{A_{N-1}\mu}{2\kappa}\left(N-2-\frac{2a^2}{r_+^2\left[1+a^2/r_+^2\right]}\right). \tag{3.3}$$

If N is even, Eqs. (3.1) and (3.3) do not change while (3.2) can be written $\kappa = \left(r_+^{N-2}\left[N+(N-2)a^2/r_+^2\right]-\mu\right)/2\mu r_+$, so we we will only write down results for odd N since they differ from even N only by numerical coefficients. Note that in this approximation the static limit exists at the roots r_s of the equation $r^{N-2}(1+(1-\mu_1^2)a^2/r^2)-\mu=0$ with μ_1 the direction cosine with respect to the x^1-y^1 plane.

Analytic solutions can be readily obtained in the following cases: 1) $a^2/r_+^2 \gg 1$, 2) $a^2/r_+^2 \approx 1$, 3) $a^2/r_+^2 \ll 1$. We consider each case in turn.

1) $a^2/r_+^2 \gg 1$

For this case, Eq (3.1) has the solution $r_+ \approx (\mu/a^2)^{1/(N-4)}$ and we are restricting to $N \geq 5$. For this to be a consistent solution, we require that $a \gg \mu^{1/(N-2)}$. In this approximation, the temperature, from (3.2), is

$$\begin{aligned} T_h &= \frac{\kappa_h}{2\pi} \\ &\approx \frac{(N-4)}{4\pi}\left(\frac{a^2}{\mu}\right)^{1/(N-4)}, \end{aligned} \tag{3.4}$$

with h labeling the high spin solutions, while the entropy, using (3.3) can be written

$$S_h = \frac{1}{4}\mathcal{A}_h$$

$$\approx \frac{1}{4}A_{N-1}a^2\left(\frac{N-4}{4\pi T}\right)^{(N-3)} \tag{3.5a}$$

$$\approx \frac{1}{4}A_{N-1}\mu^{\left(\frac{N-3}{N-4}\right)}a^{-\frac{2}{N-4}}. \tag{3.5b}$$

The specific heat at fixed a is

$$C_a^h = T\left(\frac{\partial S}{\partial T}\right)_a$$

$$\approx -\frac{N-3}{4}A_{N-1}a^2\left(\frac{N-4}{4\pi T}\right)^{(N-3)} \tag{3.6a}$$

$$\approx -\frac{N-3}{4}A_{N-1}a^{-\frac{2}{N-4}}\mu^{\frac{N-3}{N-4}}. \tag{3.6b}$$

For these extreme 'spin-dominated' solutions, $T \sim \mu^{-1/(N-4)}$ at fixed a so the temperature of a higher dimensional black hole decreases as we increase its mass. In $5+1$ dimensions the dependence of T on μ is analogous to the $3+1$ dimensional Schwarzschild solution,

$T \sim \mu^{-1}$, while in $9+1$ dimensions, $T \sim \mu^{-1/5}$. However, for fixed μ, $T \sim a^{2/(N-4)}$, and the temperature diverges with the spin. This is due to the lack of a constraint on the value of the spin parameter. In $3+1$ dimensions the temperature of a black hole *decreases* with increasing spin: as the extreme Kerr limit is approached, $T \to 0$. We will postpone further discussion of this point until the next section when we obtain a more general expression for T in $5+1$ dimensions. There it will be shown that a vestige of the $3+1$ dimensional 'tending to zero' (the Third Law) behavior persists for a less than a certain value. Examination of Eq (3.5b) reveals that the entropy at fixed a is $S \sim \mu^{\frac{N-2}{N-4}}$. In $5+1$ dimensions, $S \sim \mu^2$ so it behaves in a fashion similar to the $3+1$ dimensional Schwarzschild black hole. With μ fixed $S \sim a^{-\frac{2}{N-4}}$ and $S \to 0$ as $a \to \infty$. In $3+1$ dimensions S decreases as a increases so that when a reaches the extreme Kerr limit, $S = \frac{\pi}{2}\mu^2$. Since such a limit does not exist for $N \geq 5$, we see that S simply decreases to zero as a becomes arbitrarily large. Finally, the specific heat for these extreme solutions is negative definite and for fixed μ, $C_a \to 0-$ as $a \to \infty$.

Because something similar to the extreme Kerr limit does not exist for higher dimensional black holes (at least when only one spin parameter is non-zero) one might wonder what effect making the spin arbitrarily large has. Though it is far from clear that any physical differences manifest themselves, in $3+1$ dimensions Davies[6] has suggested that the phase transition indicated by Eq (1.1) signals an instability for the Kerr black hole in which a non-axisymmetric phase is entered with possibly new thermodynamic degrees of freedom appearing. Another indicator of instability, discussed by Smarr[8] in analogy with rotating fluid spheres in Newtonian theory, is the ratio of rotational energy to surface energy $\Omega J/TS$ becoming of order unity, which is the case near the phase transition. This dynamical instability has not been found to occur in $3+1$ dimensions, though it may be present in the higher dimensional solutions we are considering. In higher odd dimensions, the ratio of rotational energy to surface energy is $\omega_i J_i/TS$ which, for the extreme spin dominated solution, with one non-zero spin parameter and N odd, is $\sim \frac{2}{(N-4)}$. By Smarr's criteria, these extreme solutions could be highly unstable for $N=5$, again possibly indicating a transition to a non-axisymmetric phase. However, as $N \to \infty$, this ratio tends to zero. Presumably, performing a perturbative analysis of the spinning solution would shed light on the existence of an instability. However, since the specific heat for higher dimensional solutions does not have a discontinuity, the existence of an instability is, at least thermodynamically, suspect.

In the limit where the number of spatial dimensions becomes arbitrarily large the temperature, entropy, and specific heat for the extreme spin dominated solutions can be written as follows:

$$T_h \sim \frac{N}{4\pi}\left(\frac{a^2}{\mu}\right)^{1/N}, \tag{3.7}$$

$$S_h \sim \frac{1}{4}A_{N-1}a^2\left(\frac{N}{4\pi T}\right)^N \tag{3.8a}$$

$$\sim \frac{1}{4}A_{N-1}a^{-2/N}\mu, \tag{3.8b}$$

$$C_a^h \sim -\frac{N}{4}A_{N-1}a^2\left(\frac{N}{4\pi T}\right)^N \tag{3.9a}$$

$$\sim -\frac{N}{4}A_{N-1}a^{-2/N}\mu. \tag{3.9b}$$

On taking this 'large N limit', we note that the entropy for fixed a, becomes a true 'extensive' quantity in that it is additive — the sum of entropies (areas) of two black holes with masses μ_1 and μ_2 *equals* the entropy (area) of a black hole of mass $\mu_1 + \mu_2$. This is in contrast to the $3+1$ dimensional case where $\mu_1^2 + \mu_2^2 \leq (\mu_1 + \mu_2)^2$. It is easy to see why this occurs since the state variables S, J_i scale in the large N limit like M (the degree of the function, $\frac{N-2}{N-1}$, tends to one) so Eq (2.9) becomes a homogenous first-order function with respect to them.

2) $a^2/r_+^2 = 1 + \delta$

In this limit, $r_+ = (\mu/(2+\delta))^{1/(N-2)}$ so the consistency condition is $a = (1+\delta)^{1/2}(\mu/(2+\delta))^{1/(N-2)}$. The temperature is

$$T = \frac{\theta}{4\pi}\left(\frac{(2+\delta)}{\mu}\right)^{1/(N-2)}, \tag{3.10}$$

where $\theta = (2(N-3) + (N-4)\delta)/(2+\delta)$. The entropy can be written

$$S = \frac{1}{4}A_{N-1}\left(\frac{\mu^{N-1}}{2+\delta}\right)^{1/(N-2)} \tag{3.11a}$$

$$= \frac{2+\delta}{4}A_{N-1}\left(\frac{\theta}{4\pi T}\right)^{N-1}, \tag{3.11b}$$

and finally the specific heat is

$$C_a = -\frac{(2+\delta)(N-1)}{4}A_{N-1}\left(\frac{\theta}{4\pi T}\right)^{N-1} \tag{3.12a}$$

$$= -\frac{(2+\delta)(N-1)}{4}A_{N-1}\left(\frac{\mu}{2+\delta}\right)^{\frac{N-1}{N-2}}. \tag{3.12b}$$

The case when $\delta = 0$ deserves comment. In $3+1$ dimensions, $a = (\mu/2)^{1/(N-2)}$ is of interest because it corresponds to the extreme Kerr limit, and as one would expect, T vanishes there while S is a constant. In higher dimensions, T in this region is non-zero, nor, as we shall see in the next section, does it correspond to the minimum black hole temperature. The temperature and entropy differ from the zero spin case (see below) by factors of $2^{1/(N-2)}$ and $2^{-1/(N-2)}$ respectively. The ratio $\omega J/TS$, assuming one spin parameter, in this limit is $\sim \frac{1}{(N-3)}$.

3) $a^2/r^2 \ll 1$

The solution to Eq (2.13) for this limit is $r_+ \approx \mu^{1/(N-2)}$ and consistency requires that $a \ll \mu^{1/(N-2)}$. This is simply the higher dimensional version of the Schwarzschild solution. As we expect, $\mu = \frac{16\pi}{(N-1)A_{N-1}} M_{ir}$. The temperature becomes

$$T_l \approx \frac{N-2}{4\pi} \mu^{-1/(N-2)}, \tag{3.13}$$

and l labels the low spin solution. The entropy is

$$S_l \approx \frac{1}{4} A_{N-1} \left(\frac{N-2}{4\pi T}\right)^{(N-1)} \tag{3.14a}$$

$$\approx \frac{1}{4} A_{N-1} \mu^{\frac{N-1}{N-2}}, \tag{3.14b}$$

and the specific heat is

$$C_a^l \approx -\frac{(N-1)}{4} A_{N-1} \left(\frac{N-2}{4\pi T}\right)^{(N-1)} \tag{3.15a}$$

$$\approx -\frac{(N-1)}{4} A_{N-1} \mu^{\frac{N-1}{N-2}}. \tag{3.15b}$$

In the large N limit, these become

$$T_l \sim \frac{N}{4\pi} \mu^{-1/N}. \tag{3.16}$$

$$S_l \sim \frac{1}{4} A_{N-1} \left(\frac{N}{4\pi T}\right)^N \tag{3.17a}$$

$$\sim \frac{1}{4} A_{N-1} \mu, \tag{3.17b}$$

$$C_a^l \sim -\frac{N}{4} A_{N-1} \left(\frac{N}{4\pi T}\right)^N \tag{3.18a}$$

$$\sim -\frac{N}{4} A_{N-1} \mu. \tag{3.18b}$$

As in case 1, the entropy for the $N+1$ dimensional Schwarzschild solution becomes an extensive quantity in the large N limit.

4. Thermodynamics of 5 + 1 Dimensional Black Holes

The results obtained so far give us some insight but are of limited use. A great deal of parameter space has been left uncovered. As an example the specific heat appears to be negative definite in the three limits, but it is still possible that a discontinuity exists outside these limits. In addition, when $N \geq 5$ (for N odd) there are additional spin parameters and we would like to be able to say something about them. In the next section we will consider the $N = 9$ case, however, analytically this is a difficult problem so here we consider $N = 5$ which is the simplest case that still exhibits the same qualitative behaviour as $N = 9$.

In $5+1$ dimensions, the horizon(s) exists at a solution r_0 to a fourth order polynomial equation, $\prod_{i=1}^{2}(r^2 + a_i^2) - \mu r = 0$ (Eq (2.7)). The form of the general solution we present in the appendix. In the following discussion we will first consider the less general case in which one of the two spin parameters vanishes, and later we will comment on the situation when both are non-zero.

Setting $a_2 = 0,\ a_1 = a$, there is one horizon at the positive root of the polynomial, r_+, with the other at $r = 0$:

$$r_+ = 2^{-1/3}[(\mu + (\mu^2 + \frac{4}{27}a^6)^{1/2})^{1/3} + (\mu - (\mu^2 + \frac{4}{27}a^6)^{1/2})^{1/3}] \tag{4.1}$$

Note that there are no restrictions on the value of a: it can take on arbitrarily large values. From this equation for r_+ we may proceed to calculate the temperature T. We find, using (2.8), that

$$T = \frac{1}{2\pi}\left(\frac{3}{2r_+} - \frac{a^2}{\mu}\right). \tag{4.2}$$

It is straight forward to obtain the entropy S given T as a function of r_+:

$$S = \frac{\pi\mu}{6T}\left(\frac{27 + 4a^2\xi_5^2}{9 + 4a^2\xi_5^2}\right) \tag{4.3}$$

where $\xi_5 = \frac{3}{2r_+} = 2\pi T + \frac{a^2}{\mu}$.

Finally, the specific heat at fixed a is

$$C_a = -\left(S + \frac{48\pi^2 a^2 \mu \xi_5}{B^2} - \left(\frac{24\pi a^4 \xi_5}{(\mu B^2)} + \frac{TS}{\mu}\right)\frac{\partial\mu}{\partial T}\right), \tag{4.4}$$

where $B = 9 + 4a^2\xi^2$ and

$$\frac{\partial\mu}{\partial T} = 2\pi\left(\frac{a^2}{\mu^2} - \frac{3g(a,\mu)}{2r_+^2}\right)^{-1}.$$

The function $g(a,\mu)$ is given in Eq. (A7) of the appendix. As we will discuss shortly, both T and S are positive definite quntities. In addtition, $\partial\mu/\partial T$ is negative definite. As a result, the specific heat at fixed a for a $5+1$ dimensional spinning black hole is negative definite. With $\mu = 100.0$ in Planck masses, $C_a = -1.22 \times 10^4$ at $a = 0$. The specific heat decreases to $C_a = -1.47 \times 10^4$ at $a = 3.4$ and as a diverges, $C_a \to 0-$. This is contrary to the situation in $3+1$ dimensions mentioned in the introduction — there, the specific heat exhibited a second order phase transition at $a_c \simeq 0.68M$, the specific heat being negative for a less than a_c and positive for a greater than a_c. In the present case, regardless of the value of a, the black hole is unstable; in an infinite heat bath it will radiate and get hotter which, in turn, causes it to radiate more. Note that a spinning black hole preferentially emits counter-rotating particles (with respect to the axis of rotation about the $x^1 - y^1$ plane) since decreasing its spin increases its entropy. The negative definiteness of the specific heat when there is only one non-zero spin parameter (and the absence of infinite discontinuities) appears to be a characteristic of higher dimensional black holes. This is discussed further in the next section. The negative definiteness of the specific heat has also been demonstrated for charged non-rotating black holes in $4+1$ dimensional Kaluza-Klein theory[10]. The temperature, entropy and specific heat, each as a function of the spin parameter a, Eqs. (4.2)-(4.4), with $\mu = 100$, have been plotted in Figures 1-3.

We can consider various thermodynamic processes involving these black holes. When the spin is zero, $T = \frac{3}{4\pi}\mu^{-1/3}$, $S = \frac{1}{4}A_4\mu^{4/3}$ and with minor modification, one can reproduce the familiar results for four dimensional black holes. In particular, because Eq. (2.8) is not homogenous first order in the 'extensive' variables S and J_i, entropy addition is an irreversable process: it is entropically favorable for two black holes in an enclosure to coalesce. We can ask what is the maximum amount of energy that can be extracted when two black holes coalesce. This corresponds to a reversible process in which the initial and final entropies are equal. If μ_f is the final mass and the initial masses are equal, $\mu_i = \mu/2$, then $\mu_f = 2^{-1/4}\mu$. The maximum amount of energy that can be extracted is $(1 - 2^{-1/4})\mu = 0.16\mu$ (in $3+1$ dimensions the maximum energy is $29.29\% \mathrm{of} \mu$). For arbitrary odd N, the fraction of extractable energy is $1 - 2^{-\frac{1}{N-1}}$ which goes to zero for large N, which is what we would expect from the fact that Eq. (2.8) becomes a first-order homogenous equation. When the spin parameter is large, $T = \frac{1}{4\pi}(a^2/\mu)$, $S = \frac{1}{4}A_4 a^{-2}\mu^2$. The entropically favorable process involves coalescence of counter-rotating black holes which simultaneously decreases T and increases S. If μ_i and a_i are the initial mass and spin of an extreme spin dominated black hole then under the assumption that in some process $S_f = S_i$ the maximum available energy is obtained when $a_f = 0$. This implies $\mu_f = (\mu_i/a_i)^{3/2}$ and the available energy is $(1-(\mu_i/a_i^3)^{1/2})\mu_i \approx \mu_i$. Essentially all the mass in the initial black hole can be converted into energy. The conversion cannot be 100% efficient because of the irreducible mass M_{ir} associated with the initial configuration but it can be arbitrarily close since a_i can be arbitrarily large. In general, for odd N, the fraction of extractable energy is $1 - (\mu_i/a_i^{(N-2)})^{\frac{2}{(N-1)(N-4)}}$.

In $5+1$ dimensions with one non-zero spin parameter, if r_+ ever satisfies $r_+ = \frac{3\mu}{2a^2}$, then T will vanish. To establish the validity of the third law, we must check that this does not happen. For a large, $r_+ = \mu/a^2$ so in this limit, as we know, T does not vanish. From $a = 0$ to $a = a_0$, T monotonically decreases with T reaching a minimum at a_0 which is less than $a = (\mu/2)^{1/2}$ (i.e., $a^2 = r^2$, which corresponds to the Kerr limit in $3+1$ dimensions). If $\mu = 100.0$, T decreases from $T = 5.14 \times 10^{-2}T$ at $a = 0$ to $T = 4.08 \times 10^{-2}$ at $a = 5.06$. The minima of T can be obtained using Eq. (4.2) and it is found that r_+ always satisfies $r_+ < \frac{3\mu}{2a^2}$. We conclude from this discussion that T never reaches zero. For $3+1$ dimensional black holes, cosmic censorship prevents T from ever equaling zero, while one might be mislead into thinking that cosmic censorship does not play a role in the present case. It should be pointed out that we have already invoked censorship by setting $a_2 = 0$. That is, we have assured the existence of the horizon by constraining the value of one of the spin parameters. One could then interpret the non-vanishing of T when one a_i is non-zero as a consequence of the constraint imposed by cosmic censorship on the second a_i.

Our discussion so far has been restricted to the case when $a_1 = a$, $a_2 = 0$. We would like to say something about the general case in which both a_1, a_2 are non-zero.

As discussed in the appendix, the solution to Eq. (2.7) when both spin parameters are non-zero is rather complicated. We have not attempted to use it in finding the most general form for T, S, or C_{a_1,a_2}. However, we will make a few brief remarks for the case when $a_1 = a_2 = a$. The condition for the existence of a horizon is that $a < 0.69\mu^{1/3}$ which, as mentioned in section 2, corresponds to $\Pi - \mu r|_{r=\tilde{r}} \leq 0$. In this approximation, $S = \frac{A_4\mu}{16\pi T}(\frac{3r_+^2-a^2}{r_+^2+a^2})$, where r_+ is now the full solution to Eq. (2.7) with spin parameters equal. Though it appears that S can vanish if $r_+^2 = \frac{1}{3}a^2$, this does not occur. For example, if $\mu = 100$, then $r_+ = 1.89$ when $a = 3.19$ while $S = 1.24 \times 10^3$. Since r_+ is a monotonicaly decreasing function of a, this is the smallest value S can attain. Similarly, the horizon will vanish before T can go to zero: when $r_+ = 1.89$, $T = 1.54 \times 10^{-3}$. Again, cosmic censorship must be called on to avoid a naked singularity. By doing this we preserve the third law.

When the two spin parameters are equal the specific heat becomes

$$C_{a_1=a_2} = -\left(S - \left[\frac{ST}{\mu} + \frac{8}{6}\pi\frac{\mu r_+ a^2 g'(a,\mu)}{B^2}\right]\frac{\partial\mu}{\partial T}\right). \tag{4.5}$$

We have used the fact that $\partial r_+/\partial T = g'(a,\mu)\partial\mu/\partial T$ where $g'(a,\mu)$ is a function similar to $g(a,\mu)$ in (A7) and set $B = r_+^2 + a^2$. Thus, $C_{a_1=a_2}$ will be a *positive* quantity if the second term of Eq. (4.5) is sufficiently large and negative. In the event that $C_{a_1=a_2}$ changes sign, it will not do so discontinuously and the associated phase transition will not be of second order.

Equilibrium configurations of a $5+1$ dimensional black hole can be found in the same fashion as for $3+1$ dimensions.[11] Examination of Eq (4.3) reveals that the entropy is maximized when $a = 0$, i.e., when a is large ($a \gg \mu^{1/3}$) $S \sim \frac{A_4\mu}{16\pi T}$, and when $a = 0$,

$S \sim \frac{3A_4\mu}{16\pi T}$. This agrees with the results of the last section and implies that if the black hole is spinning, a true equilibrium configuration requires a corotating heat bath.

As an example of equilibrium conditions we consider first the general case of a gas of massless particles in contact with a black hole in a (N dimensional) box of volume V_N and restrict later to $5+1$ dimensions. The discussion will be restricted to zero spin since non-zero a only scales the volume of the box by powers of the dimensionless factor E/a^{N-2} (and a factor depending on the rotational energy of the heat bath). We begin by recalling some results for massless particles in $N+1$ dimensions.

The energy density ρ of massless particles in $N+1$ dimensions is

$$\rho = g_*^{(N)} a_N T^{N+1}, \tag{4.6}$$

where

$$a_N = \frac{N\Gamma\left(\frac{N+1}{2}\right)\varsigma(N+1)}{\pi^{\frac{N+1}{2}}}. \tag{4.7}$$

In (4.7) $g_*^{(N)} = (n_b + (1-1/2^N)n_f)$, and n_b (n_f) are the number of massless bosonic (fermionic) degrees of freedom. Recall that the n_b are the dimension of irreducible representations of the little group for massless particles, SO(N-1) so, for example, vector particles have $n_b = N-1$. For fermions the situation is the same though the representations may be reducible in certain dimensions. With $p = \frac{1}{N}\rho$, the entropy density is

$$s = \frac{N+1}{N} g_*^{(N)} a_N T^N. \tag{4.8}$$

We wish to maximize the total entropy $S = S_H + V_N s$ of the black hole plus massless particles subject to the constraint that the total energy of the system $E = M_H + V_N\rho$ is a constant, which corresponds to an adiathermal box with rigid walls.

In the small spin limit, $a \ll \mu^{1/N-2}$, where

$$S \approx \frac{N+1}{N} g_*^{(N)} a_N V_N T^N + \frac{A_{N-1}}{4} \mu^{\frac{N-1}{N-2}}$$
$$\approx \frac{N+1}{N} g_*^{(N)} a_N V_N \left(\frac{E}{g_*^{(N)} a_N V_N} \right)^{\frac{N}{N+1}} (1-x)^{\frac{N}{N+1}}$$
$$+ \frac{A_{N-1}}{4} \left(\frac{16\pi E}{(N-1)A_{N-1}} \right)^{\frac{N-1}{N-2}} x^{\frac{N-1}{N-2}}. \tag{4.9}$$

Where we have used the energy constraint, $T = \left(\frac{E-M}{g_*^{(N)} a_N V_N}\right)^{1/(N+1)}$, and defined $x = M/E$. Extremising,

$$x^{\frac{2N-1}{N-2}} - x^{\frac{N+1}{N-2}} + \vartheta = 0, \tag{4.10}$$

where

$$\vartheta = \left(4 \frac{N-2}{N-1} \frac{g_*^{(N)} a_N V_N}{A_{N-1}} \left[\frac{E}{g_*^N a_N V_N} \right]^{\frac{N}{N+1}} \left(\frac{(N-1)A_{N-1}}{16\pi E} \right)^{\frac{N-1}{N-2}} \right)^{N+1}. \tag{4.11}$$

Constraints on the existence of roots of Eq. (4.10) can be found by taking $F(x) = x^{\frac{2N-1}{N-2}} - x^{\frac{N+1}{N-2}} + \vartheta$, so that $F'(x) = (\frac{2N-1}{N-2})x^{\frac{N+1}{N-2}} - (\frac{N+1}{N-2})x^{\frac{3}{N-2}}$ which has roots at $x = 0$ and $x = \frac{N+1}{2N-1}$. Since $F(x) \to -\infty$ as $x \to -\infty$, we see that $x = 0$ is a maximum while the second root of $F'(x)$ is a minimum. For $F(x)$ to have roots, we must have $F(\frac{N+1}{2N-1}) \leq 0$. This in turn places a limit in the form of an inequality on the volume of the enclosure (or the total energy contained in that enclosure) in which a higher dimensional black hole can be in equilibrium with massless particles. We can view this equilibrium condition as a condition governing the condensation of a black hole out of the gas of massless particles. If there is a large number of degrees of freedom in the gas of particles their entropy will be large and the ratio of total energy E to a power of the volume ($V^{\frac{N-2}{2N-1}}$) will have to be large in order for black hole condensation to occur. If the black hole is spinning, S_H will be lowered and once again, the ratio of the total energy to the volume has to be larger. The two roots which arise when the inequality is satisfied correspond to low mass and high mass black holes. The two roots coincide when equality holds. It is easy to show that when the inequallity is sufficiently strong the high mass root can correspond to a global maximum for the total entropy.

This analysis implies that in $5+1$ dimensions $\vartheta \leq 0.15$, and the volume of the five dimensional box must satisfy

$$V_5 \leq \frac{0.19\pi^7 E^3}{g_*^{(5)}}. \tag{4.12}$$

A black hole cannot be in thermodynamic equilibrium if the volume (energy) does not satisfy this inequality. If we place such a black hole in a box and the inequality is not satisfied, the black hole will grow and cool slower than the surrounding heat bath thus converting all the energy of the heat bath into black hole mass. When there is sufficient energy, the heat bath is not depleated at a fast enough rate and when the energy in the heat bath is low enough, the two systems will reach a stable equilibrium point described by the second root of (4.9). Note that at the equilibrium point, $E \sim M_H$, and the linear scale of the box, $L \sim M^{3/5}$. Since M has canonical dimension λ^3, the box is not in danger of being crushed by gravity before equilibrium is reached.

5. Thermodynamics of $9+1$ Dimensional Black Holes

In going to $9+1$ dimensions we immediately run into the problem of solving an eighth order polynomial. Again we will simplify matters and restrict to the case with three of the spin parameters set to zero, $a_2 = a_3 = a_4 = 0, a_1 = a$. From the asymptotic forms discussed in section 2, and the analytic solution for T as a function of a, μ, and r_+ in $5+1$ dimensions considered in the previous section, we take the following ansatz for T in $N+1$ dimensions:

$$T = \frac{1}{4\pi}\left(\alpha(N-2)r_+^{-1} + \beta(N-4)\left(\frac{a^2}{\mu}\right)^{1/(N-4)}\right), \tag{5.1}$$

where α and β are determined by fits to the limiting cases. For $a \to 0$, $T \to \frac{N-2}{4\pi} r_+^{-1}$ so that $\alpha = 1$. When a is much greater than $\mu^{1/(N-4)}$, $T \approx \frac{(N-4)}{4\pi}(a^2/\mu)^{1/N-4}$ and $\beta = -\frac{2}{N-4}$. So T has the form

$$T = \frac{1}{2\pi}\left(\frac{N-2}{2}r_+^{-1} - \left(\frac{a^2}{\mu}\right)^{1/(N-4)}\right). \tag{5.2}$$

This reproduces exactly the equation for T for $N = 5$. For $N = 9$, it provides a good fit to the numerical calculation of T, deviating by less than 30% near $a = (\mu/2)^{1/(N-2)}$. This is not suprising since that information was not used in determining the fit. Since Eq (5.2) behaves as we would expect qualitatively, it is adequate for our purposes. If we check T numerically by using the definition of κ, we find that if $\mu = 100.0$ the minimum for T is $T = 0.27$ at $a = 2.01$. If we compare this to the $5+1$ dimensional black hole, we see that this minima is higher and at a smaller value of a. For arbitrarily large N it is possible that T is a monotonically increasing function of a for all values of a.

With this approximation for T, the entropy can be written

$$S = \frac{A_8 \mu}{16\pi T}\left(\frac{20a^2\xi_9^2 + 343}{4a^2\xi_9^2 + 49}\right), \tag{5.3}$$

with $\xi_9 = \frac{7}{2}r_+^{-1} = 2\pi T + (a^2/\mu)^{1/5}$. The specific heat for fixed spin is

$$C_a = -\left(S + \frac{448}{15}\pi^4 \frac{a^2\mu\xi_9}{B^2} - \left[\frac{ST}{\mu} + \frac{224}{75}\pi^3 \frac{1}{B^2}\left(\frac{a^{12}}{\mu}\right)^{1/5} \right] \frac{\partial\mu}{\partial T} \right), \tag{5.4}$$

where $B = 49+4a^2\xi_9^2$. These results for the temperature, entropy and specific heat of a 9+1 dimensional black hole are qualitatively similar to those for $5+1$ dimensions. In particular, the specific heat is once again negative definite for negative $\partial\mu/\partial T$. Unfortunately, due to the complexity of the equations, we were not able to perform even a cursory analysis of the general case of all $a_i \neq 0$. However, we expect that the general case will have the same features that were discussed for $5+1$ dimensions.

Applying the results of the previous section, Eqs.(4.10)-(4.11), to $9+1$ dimensions, we find that $\vartheta \leq 0.19$ and the constraint on the volume of the nine dimensional box is

$$V_9 \leq \frac{10.17}{g_*^{(9)}} E^{17/7}. \tag{5.5}$$

The discussion of the previous section is the same for this equation.

In light of string theory, and the suggestion of reference 7 that strings may have significant implications for the last stages of black hole evaporation, it is worth discussing equilibrium conditions in $9+1$ dimensions for black holes with massless and massive string modes.

The entropy for massive string modes has been computed in reference 2. We will confine the discussion to the heterotic string.[12] If E is the energy of a heterotic string configuration, the entropy is $S = -10\ln(E) + \pi(2+\sqrt{2})\sqrt{\alpha'}E$. On dimensional grounds, the inverse string tension α' has the simple relation to the Planck length $\sqrt{\alpha'} = \gamma l_{Pl}$. If the string theory is weakly coupled, $\gamma = M_{Pl}/M_s$ can be large.* Following reference 7, Bowick et al., we assume γ is large (see footnote). If we have a blackhole with mass E, which is less than $\sim \gamma$, in units of Planck mass, Bowick et al. conclude (using the $3+1$ dimensional black hole entropy $S_H^{(4)} = 4\pi E^2$) that $S_H^{(4)}(E) < S_s(E)$. When $E < \gamma$, a black hole can increase its entropy by making a quantum transition to a bunch of massive string excitations. Under these assumptions, this becomes a highly probable transition. Since the specific heat of massive string modes has the form $C = -(\pi(2+\sqrt{2})\sqrt{\alpha'}E - 10)^2/10$, they cannot remain in equilibrium with an infinite heat bath and will evaporate into massless modes. The black hole will have evaporated leaving no remnant.

As pointed out in the introduction, this phenomenon is more clearly, if not more consistently studied in ten non-compact dimensions rather than (four dimensions)$\times$(compact space). It is possible that at the endpoint of black hole evaporation space-time 'decompactifies' making a ten dimensional analysis relevant.

Consider a black hole in $9+1$ dimensions with small or zero spin parameter (maximum entropy). The entropy for the black hole is $S_H^{(10)} = \frac{1}{4}A_8\mu^{8/7}$, compared to the four dimensional entropy $S_H^{(4)} = \frac{1}{4}A_3\mu^2$. The ten dimensional entropy scales much less rapidly with energy than its four dimensional counterpart. If we choose $\gamma = 10$, Bowick et al.

* It is possible that the string theory is weakly coupled allowing a semiclassical approximation, with the non-linear σ model on the world sheet being strongly coupled. However, very general arguments would appear to rule this out and in fact require that the full string theory be strongly coupled.[13] The main result of this is that $M_{Pl} \sim M_s \sim M_{compact}$ implying that not only are 'stringy' effects to be found only at or above the compactification scale, but that the existence of higher dimensional operators imply the inclusion of higher order curvature corrections to the Einstein equations. A consistent approach would require a study of the non-linear σ model on non-trivial backgrounds.[14] This naturally gives the higher order corrections. Black hole solutions for the background field equations have been studied by Callan, Myers and Perry.[15]

show that $S_H^{(4)} < S_s(E)$ is satisfied when $E \leq 8.33$. However, for a ten dimensional black hole $S_H^{(10)}(E) < S_s(E)$ is satisfied for $E \leq 3.19 \times 10^{13}$ i.e. when $E \gg \gamma$. The mass range over which it is entropically favorable for a black hole to fluctuate to massive string modes is enormously larger in ten dimensions than in four dimensions.

We have found the constraint on V_9 for a black hole to be in equilibrium with massless radiation, Eq. (5.5). Bowick and Wijewardhana[2] have discussed the corresponding constraint for equilibrium between the massive and massless string modes. Taking $E = V_9\rho = g_8^{(9)} a_9 V_9 T^{10}$, this constraint is

$$V_9 \leq \frac{(E_r)_{max}}{g_*^{(9)} a_9 T_c^{10}}, \tag{5.6}$$

where $(E_r)_{max} = E + (10T_c/(1 - bT_c)$,

$$T_c = \frac{20bE - 90 \pm \sqrt{8100 + 400bE}}{20b(bE - 10)} \tag{5.7}$$

and $b = \pi(2 + \sqrt{2})\sqrt{\alpha'}$.

Since the massive modes of the string and the black hole both have negative specific heat, the possible equilibrium configurations are 1) black hole and radiation, 2) massive string modes and radiation, and 3) radiation alone. Following Bowick et al., if the total energy is E and the volume V, a V, E phase diagram can be constructed using Eqs. (5.5) and (5.6) with the 'triple point' located where the volume and energy in the equality of (5.5) equal the volume and energy in the equality of (5.6). This triple point is located at $E_{tp} = 4.09 \times 10^{13}$, $V_{tp} = 1.44 \times 10^{30}$ with $n_b = n_f = 4032$ corresponding to the massless modes of the heterotic string. Below the triple point energy, the volume necessary to sustain phase 1 is less than that necessary to sustain phase 2. For energies greater than the triple point, phase 2 does not exist. Our $9+1$ dimensional investigation arrives at the same conclusion as Bowick et al.: starting with a small volume at $E < E_{tp}$, phase 1 exists. As we increase the volume, we enter phase 2 and increasing V further, we find radiation only. It should be stressed that this is an energetics argument only, and not based on any fundamental microscopic analysis. It should be demonstrated for instance that in a curved background, thermal Green's functions (see Gibbons and Perry, reference [11]) are satisfied by string states[16] indicating that in such a background strings have a thermal distribution.

5. Conclusion

The thermodynamics of black holes in $N+1$ dimensions has a rich structure which we have only begun to uncover. As we have shown for spinning black holes, the number of non-zero spin parameters describing the solution has important consequences. When only one such parameter, a, is non-zero, this spin can assume arbitrarily large values while still assuring the existence of a horizon (it is because the other spin parameters are set to zero that the horizon exists for arbitrary a). The thermodynamics of the black hole has unusual characteristics. The temperature decreases as a ranges from zero to a_0 where it has its minimum value (in $3+1$ dimensions this would correspond to the extreme Kerr limit). However since the spin can take on values larger than a_0, we find that the temperature *increases* with the spin as the spin becomes arbitrarily large. Likewise the entropy at a fixed mass decreases with spin as it does in the Kerr solution, but, without a Kerr limit, the entropy tends to zero as the spin diverges. For the $3+1$ dimensional Kerr solution, the specific heat can become positive when the spin is greater than some critical value, where the specific heat goes through an infinite discontinuity. This is not the case in higher dimensions when only one of the spin parameters is non-zero. Then the specific heat is negative definite and the black hole cannot be in equilibrium with an infinite heat bath. We have presented the conditions under which equilibrium with a finite heat bath will hold.

When all (or some) spin parameters are non-zero, the situation can more closely resemble that for $3 + 1$ dimensions. We have considered a $5 + 1$ dimensional black hole with $a_1 = a_2 = a$ and shown that for a horizon to exist, $a < 0.69\mu^{1/3}$. The temperature monotonically decreases with a approaching zero as a approaches this limit. The entropy has a non-zero value at this limit. It is possible that the specific heat can become positive (this has not been checked), however, if it does it will not be via a second order phase transition.

We have considered the thermodynamics of a $9 + 1$ dimensional black hole and the massive and massless modes of the heterotic superstring. Our results are similar to those of Bowick, Smolin and Wijewardhana. In particular, in $9 + 1$ dimensions the mass range over which it is entropically favorable for a black hole to fluctuate to massive string modes is much larger than in $3 + 1$ dimensions.

We have not considered in our paper two crucial elements, whether our results survive through compactification and microscopic arguments for the assumption of a thermal string distribution in curved space. These are key issues which will require more study.

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Appendix

In $N+1$ dimensions, with N odd, horizons occur for solutions of Equation (2.7):

$$\prod_{i=1}^{(N-1)/2} (r^2 + a^2) - \mu r = 0. \tag{A1}$$

For $N = 5$ we have the fourth-order polynomial

$$r^4 + (a_1^2 + a_2^2) r^2 + a_1^2 a_2^2 - \mu r = 0, \tag{A2}$$

which has two spin parameters a_1 and a_2. In section 4 we considered the case when $a_2 = 0$, $a_1 = a$. The general solution to $(A2)$ can be obtained in the following manner. If the polynomial has the form $\xi^4 + p\xi^2 + q\xi + r = 0$, its cubic resolvent is $t^3 - pt^2 - 4rt + (4pr - q^2) = 0$. In our case, $p \equiv a_1^2 + a_2^2$, $q \equiv -\mu$, and $r \equiv a_1^2 a_2^2$. By making the transformation $t_0 = y_1 + (a_1^2 + a_2^2)/3$ the real root of the transformed cubic resolvent equation is:

$$\begin{aligned} y_1 = &\left(\frac{2A(a_1,a_2) + 27\mu^2 + \sqrt{729\mu^4 + 108\mu^2 A(a_1,a_2) - 432 B(a_1,a_2)}}{54} \right)^{\frac{1}{3}} \\ &+ \left(\frac{2A(a_1,a_2) + 27\mu^2 - \sqrt{729\mu^4 + 108\mu^2 A(a_1,a_2) - 432 B(a_1,a_2)}}{54} \right)^{\frac{1}{3}}, \end{aligned} \tag{A3}$$

with

$$A(a_1, a_2) \equiv a_1^6 + a_2^6 - 33(a_1^4 a_2^2 + a_1^2 a_2^4),$$

$$B(a_1, a_2) \equiv a_1^2 a_2^2 (a_1^8 + a_2^8 - 4(a_1^6 a_2^2 + a_1^2 a_2^6) + 6 a_1^4 a_2^4).$$

So, finally, the four roots of the original quartic equation are given by the roots of the two quadratic equations.

$$\xi^2 \pm \sqrt{t_0 - p}\,\xi \pm \sqrt{t_0^2 - 4r} + \frac{t_0}{2} = 0. \tag{A4}$$

We are interested in solution(s) which are positive definite. If we take the particular case of $a_2 = 0$ (or $r = 0$) it is easy to see that $\xi > 0$ only for the $(-)$ combination of signs. Thus, the solutions are

$$2\xi = \sqrt{t_0 - p} \pm \sqrt{-p - t_0 + 4\sqrt{t_0^2 - 4r}}. \tag{A5}$$

Choosing the particular case $a_2 = 0$, $a_1 = a$,

$$y_1 = \frac{1}{\sqrt[3]{4}}\left[\left(\mu + \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{\frac{2}{3}} + \left(\mu - \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{\frac{2}{3}}\right].$$

The root of the resolvent is $t_0 = y_1 + a^2/3$ and equation (A5) has solutions (setting $\xi_1 = r_1 = r_+$ and $\xi_2 = r_2$)

$$r_+ = \frac{1}{\sqrt[3]{2}}\left[\left(\mu + \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{\frac{1}{3}} + \left(\mu - \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{\frac{1}{3}}\right], \tag{A6}$$

and $r_2 = 0$. To find the specific heat we need in addition,

$$\frac{\partial r_+}{\partial T} = g(a,\mu)\frac{\partial \mu}{\partial T},$$

where

$$g(a,\mu) = \frac{1}{3\sqrt[3]{2}}\left[\left(\mu + \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{-\frac{2}{3}}\left(1 + \mu\left(\mu^2 + \frac{4}{27}a^6\right)^{-\frac{1}{2}}\right)\right.$$
$$\left. + \left(\mu - \sqrt{\mu^2 + \frac{4}{27}a^6}\right)^{-\frac{2}{3}}\left(1 - \mu\left(\mu^2 + \frac{4}{27}a^6\right)^{-\frac{1}{2}}\right)\right]\frac{\partial \mu}{\partial T}. \tag{A7}$$

These are the solutions we will use in section 4.

The solutions of (A5) when both spin parameters are non-zero have a much more complicated form then (A6). However, when $a_1 = a_2 = a$ it is possible to find a constraint on a in order for a horizon to exist. We require that the term in the radical of Eq.(A3) not go negative. If we impose this condition then it is easy to see that $a < 0.69\mu^{1/3}$. When both spin parameters are equal, their value cannot be arbitrarily large which is a very different situation from Eq. (A6) which has no such constraint.

Figure Captions

Figure 1. The temperature for a $5+1$ dimensional spinning black hole, at fixed μ, as a function of the single spin parameter a. Scales are in Planck units.

Figure 2. The entropy for a $5+1$ dimensional spinning black hole, at fixed μ, as a function of the single spin parameter a. Scales are in Planck Units.

Figure 3. The specific heat for a $5+1$ dimensional spinning black hole, at fixed μ, as a function of the single spin parameter a. Scales are in Planck Units.

6. References

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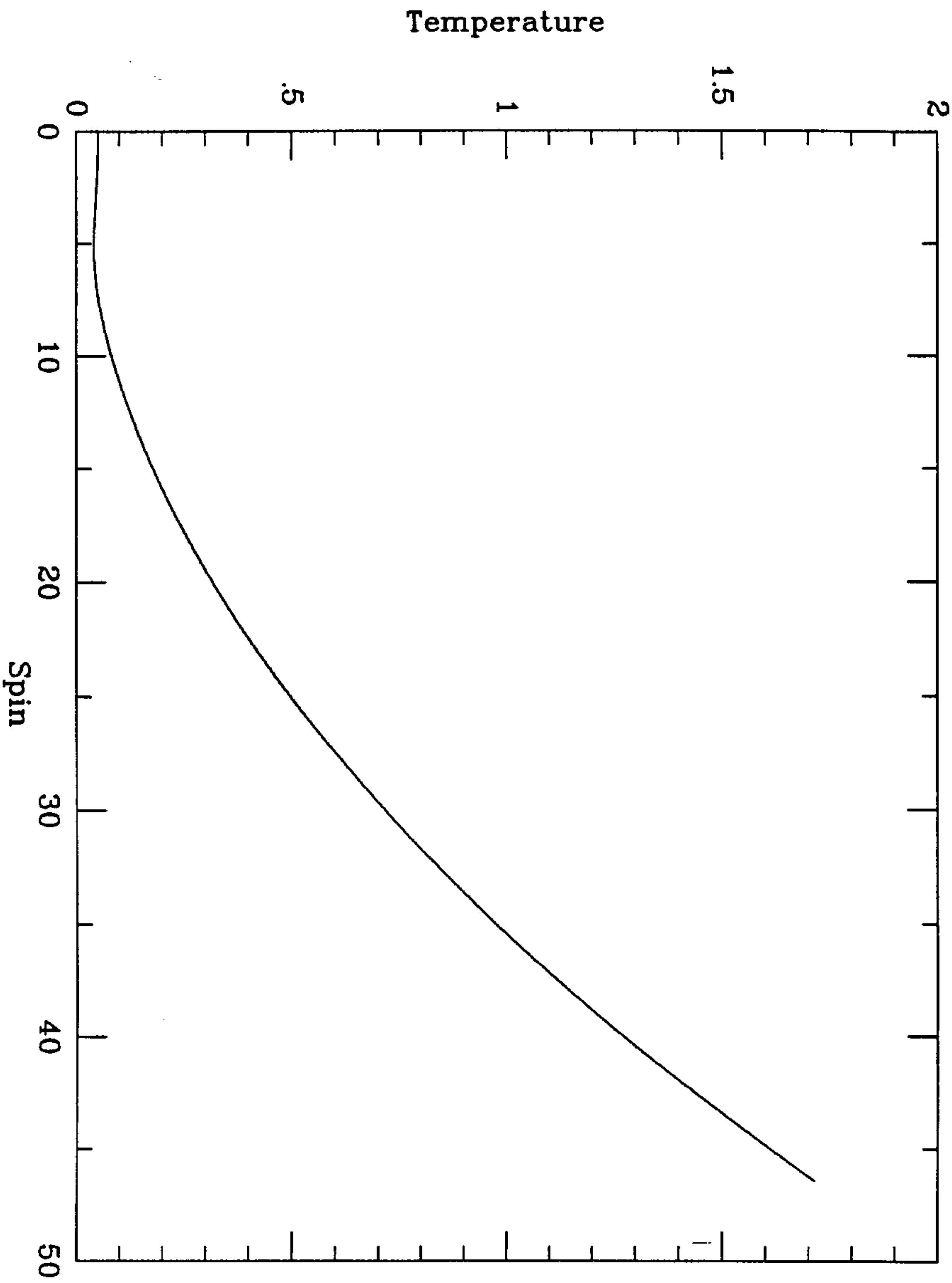

Figure 1.

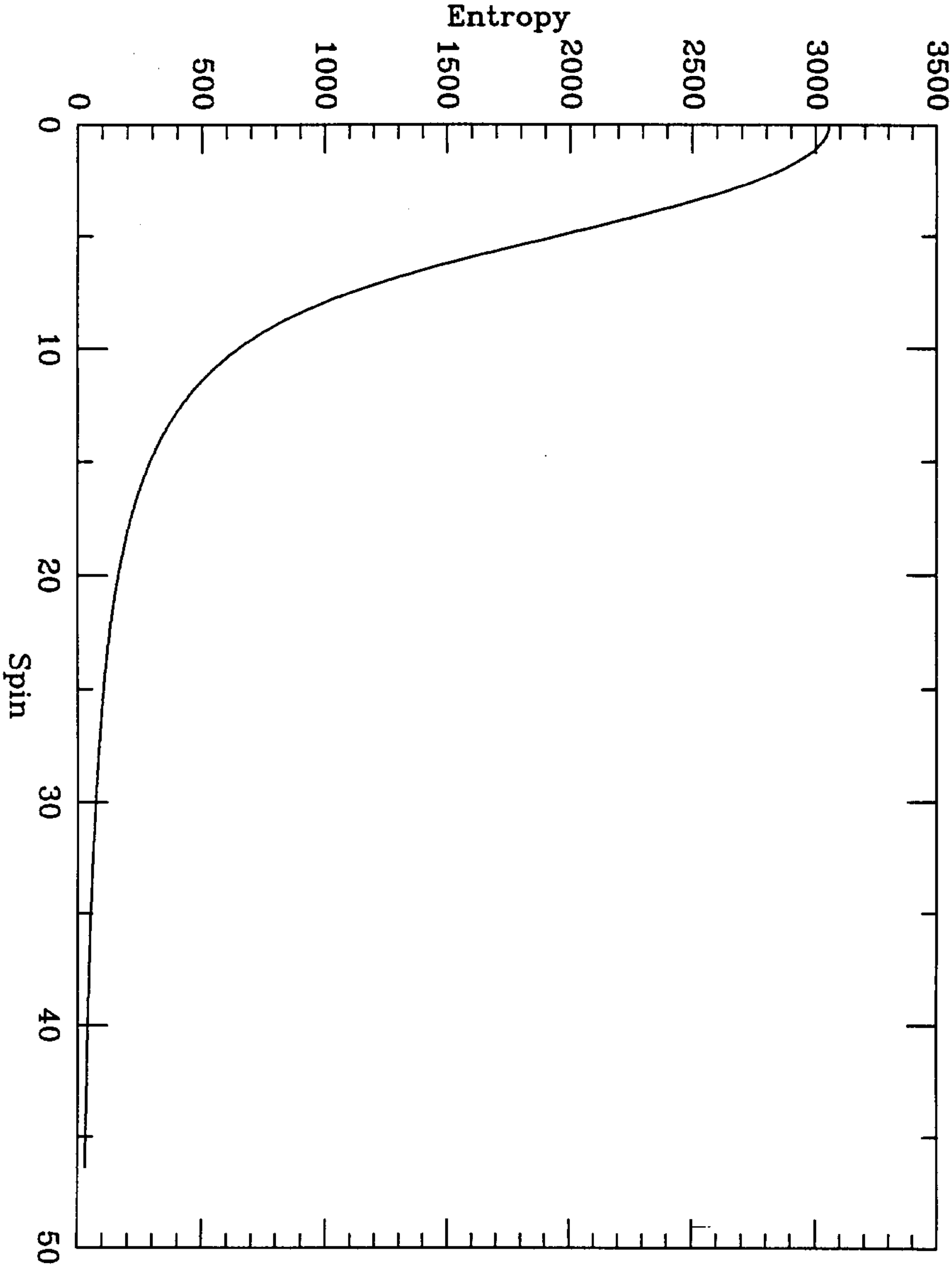
Entropy
0
500
1000
1500
2000
2500
3000
3500
Spin
0
10
20
30
40
50

Figure 2.

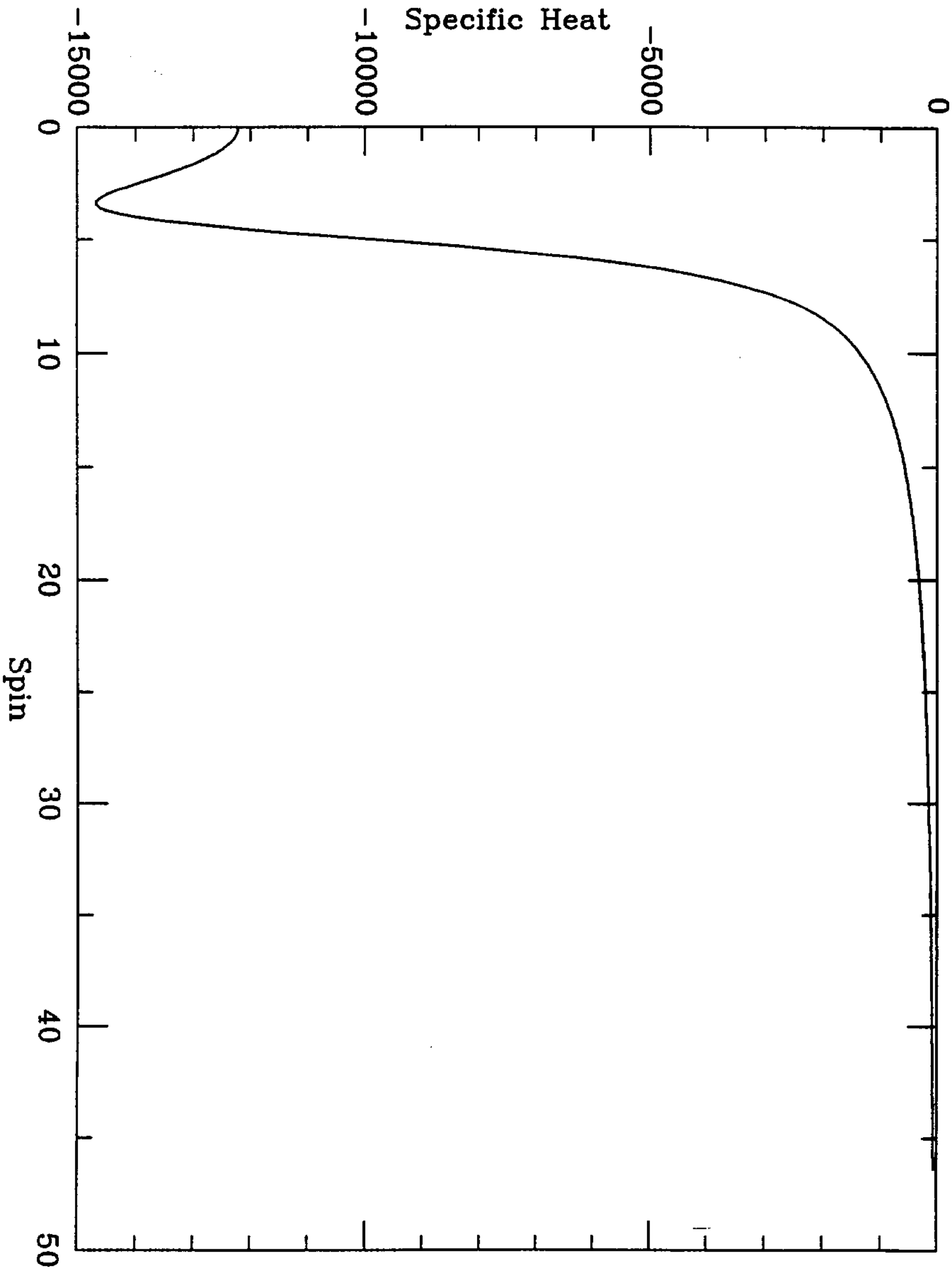

Figure 3.