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A Non-Commutative Geometry Model for Closed Bosonic Strings

Siddhartha Sen

School of Mathematics

Trinity College, Dublin 2 Ireland

R. Holman

NASA/Fermilab Astrophysics Center

Fermilab, P.O. Box 500, MS209

Batavia, IL 60510

Abstract

We show how Witten's non-commutative geometry may be extended to describe the closed bosonic string. An *explicit* representation of the $\oint$ and $*$ product for closed strings is provided.

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The possibility that string theories may provide us with a consistent unification of the forces makes it imperative that we understand the geometrical underpinnings of string field theories. Recently, Witten[1] has put forth the idea that string geometry may best be understood by use of the algebraic structure contained in the BRST quantization of string fields [2] to construct a model of a *non-commutative geometry*[3]. Similar notions have also been considered by the Kyoto group[4].

The idea of ref.(1) (henceforth known as I) has great intuitive appeal in the open bosonic string case but seems to need the introduction of the notion of a *non-associative* geometry in the closed string case. The Kyoto group is able to deal with closed strings with their modified geometry and Lykken and Raby[5] have made attempts to amend the axioms in I to deal with this problem (although they must postulate the existence of objects that may,in fact, not exist!).

Our purpose in this Letter is to show how the ideas in I can be extended in a natural way and then to use these extensions to solve some of the problems associated with closed strings in this formalism.

Let us first review the main ideas in I, restricting our discussion to oriented bosonic strings throughout. If Q is the BRST charge operator on string fields, then our object is to construct an associative product $*$ on string fields which will give rise to a graded, associative algebra $\mathcal{B}$ on which Q acts as a derivation:

$$Q(A * B) = Q(A) * B + (-)^A A * Q(B). \tag{1}$$

Here A and B are string fields and $(-)^A$ is the grading of A. One may think of A and B as "differential 1- forms", Q as an "exterior derivative", and $*$ as a "wedge product" (this only works if $Q^2 = 0$, which is true in the critical dimension). Furthermore, in order to construct an action, we will need an integral operator, which we call $\oint$, which is linear and also satisfies

$$\oint A * B \;=\; (-)^{AB} \oint B * A, \tag{2}$$

$$\oint Q(A) \;=\; 0 \tag{3}$$

Eq.(3) is the statement of BRST invariance of the theory.

With these notions in hand, we may define a generalized gauge invariance:

$$\delta A \;=\; Q(\epsilon) + A * \epsilon - \epsilon * A, \tag{4}$$

where ϵ is the gauge parameter. We may also define a field strength F that is gauge covariant under (4):

$$F = Q(A) + A * A. \tag{5}$$

Witten finds that the only non-trivial, gauge invariant actions for A are the Chern-Simons $2p+1$ forms. In particular, his $\oint$ is such that only

$$S(A) = \oint [A * Q(A) + \frac{2}{3} A * A * A] \tag{6}$$

is nonvanishing.

How should the action of $*$ and $\oint$ be chosen so as to give rise to a reasonable theory? First we note that reparametrization invariance can be replaced by BRST invariance so that a particular point (such as $\sigma = \pi/2$, which is the midpoint of the string in Witten's parametrization) can be singled out and left and right halves of the string, S_L, S_R identified. The $*$ operation is then constructed so as to sew S_L to T_R if S_R coincides with T_L (fig 1) (up to corrections for BRST invariance). $\oint S$ is defined to be the operation that sews S_R to S_L (fig 2) (again, with appropriate modifications for BRST invariance).

Having decided on what $*$ and $\oint$ should do, we must find a concrete realization of these operations. Let $X^\mu(\sigma,\tau)$ and $\phi(\sigma,\tau)$ be the string coordinates and the *bosonized* conformal ghosts respectively. The action is given by

$$I = \frac{1}{4\pi}\int d^2\sigma \partial^a X^\mu(\sigma,\tau)\partial_a X_\mu(\sigma,\tau) + \frac{1}{2\pi}\int d^2\sigma[\partial^a\phi(\sigma,\tau)\partial_a\phi(\sigma,\tau) - 3iR\phi(\sigma,\tau)]. \tag{7}$$

where the integrals are over the world-sheet and the $3iR\phi$ accounts for the conformal anomaly (R is the world-sheets' scalar curvature).

We are now ready to construct an integral for *closed* strings(though the theory that arises from it is not consistent without modifications). The first step is to choose a surface together with a metric on it that is flat near its boundary. For now, we also demand that the boundary components meet *at right angles*. Witten chooses a hemisphere H. We then define

$$\oint \Psi \equiv \int_\Psi DX^\mu(\sigma)D\phi(\sigma)\exp{(-I)} \tag{8}$$

This tells us to evaluate the path integral by only summing over paths on H with boundary conditions (b.c.'s) specified by Ψ. At this point we pause to note that $\oint$ depends crucially on the surface chosen. This freedom of choice and a similar one in the definition of $*$ will be most important for us in the sequel.

Next, we use the definition of $\oint$ for closed strings to define it for open ones. Instead of H, choose a new surface, a quarter sphere , whose boundary consists of two segments S_1 and S_2. On S_1 the b.c.'s are chosen to be those determined by the open string Ψ that we wish to integrate, while those on S_2 correspond to standard open string b.c.'s. The two segments are chosen to meet at right angles and to have zero extrinsic curvature.

How is $*$ defined? As before we pick a surface over which we impose b.c.'s on the paths contributing to the path integral. Following I, we pick a hexagon on whose boundary components we specify open string states which we label A, G_1, B, G_2, F, G_3. The G_i are string states with standard b.c.'s.The dependence of the path integral on F defines a functional of the string coordinates and ghosts that is linear in A and B. We *define* this functional to be $A * B$, and note that this definition also depends on the surface chosen.

Why can't this procedure be carried through for closed strings? The immediate generalization of $*$ to closed strings (i.e. a product that involves the selection of *one* point on the string) gives rise to a commutative but non-associative product. Thus, we should regard closed strings as having *two* points ($\sigma = \frac{-\pi}{2}, \frac{\pi}{2}$, say, if we let σ run from $-\pi$ to $+\pi$) selected and being joined by two segments. The product of two closed strings can then be viewed as the joining of the strings at the two preferred points along matching segments (fig 3). An immediate consequence of this choice is that closed bosonic fields are *bilocal* objects. The action of $\oint$ on closed string fields is taken to be the operation of setting equal the two segments connecting the preferred points and integrating (fig 4).

The fact that two points are selected gives rise to the following problem. In the open string case, the ghost numbers of physical string states are $-\frac{1}{2}$ while $g(A * B)$ was found to be $g(A) + g(B) + \frac{3}{2}$ i.e. $g(*) = \frac{3}{2}$, and a similar calculation shows that $g(\oint) = -\frac{3}{2}$. For closed strings, the physical states have ghost number -1 and the existence of two points changes the 3/2's to 3's, and precludes the use of the Chern-Simons action for closed strings (we shall review these calculations below).

Our modifications of $*$ and $\oint$ will *not* affect the bilocality of closed string fields, but will be able to modify the ghost numbers of $*$ and $\oint$. The crucial point is that these operations depend explicitly on the surfaces we choose for their definitions.

Let us first show how $g(\oint)$ depends on the integrated curvature of the defining surfaces. Consider the anomalous conservation law for the ghost current $K^a \equiv \frac{1}{\pi}\partial^a\phi$

$$\partial_a K^a + \frac{3iR}{2\pi} = 0 \tag{9}$$

Let S be an arbitrary surface and perform the following path integral:

$$\int_\Psi DX^\mu(\sigma)D\phi(\sigma)exp(-I)\int_S[\partial_a K^a + \frac{3iR}{2\pi}] = 0 \tag{10}$$

After an integration by parts, the first term results in the component of K^a normal to ∂S integrated over ∂S. This then changes the b.c.'s of the path integral to those specified by $-ig(\Psi)\Psi$. The last term changes the b.c.'s to those given by $(\frac{3i}{2\pi}\int_S R)\Psi$. Finally, linearity of $\oint$ tells us:

$$-i(g(\Psi) - \frac{3}{2\pi}\int_S R)\oint\Psi \ = \ 0 \tag{11}$$

which implies that $\oint\Psi$ vanishes *unless* $g(\Psi) = \frac{3}{2\pi}\int_S R$. For a quarter sphere the integral of R is π and $\oint$ carries ghost number -3/2. This can be modified in a variety of ways. Let us consider two possibilities. First, we can add h *handles* to the surfaces discussed above. This will change $g(\oint)$ by $3h$ $(\frac{3h}{2})$ in the case of a hemisphere (quarter-sphere). We may also add b *boundaries* (i.e. holes) to the hemisphere which changes $g(\oint)$ by $\frac{3b}{2}$.

We may also relax the condition that the surface S be embedded with zero extrinsic curvature if we modify the conformal anomaly appropriately. This changes the ghost number of $\oint$ to

$$-\frac{3}{2\pi}[\int_S R + 2\int_{\partial S}\kappa], \tag{12}$$

where κ is the *extrinsic* curvature and the relative normalization is determined by computing ghost zero modes on a surface with boundary, such as a disk[6].

Does this notion of integration correspond to what we want? To see that it does, let's replace the modified hemisphere by a flat surface having two points at which the curvature is singular (fig 5). In the limit that the size δ of the flaps approaches zero, we are left with the intuitive picture of closed string integration discussed earlier. Furthermore, we are now able to compute what the ghost insertions should be in order to preserve BRST invariance. If we take the curvature singularity points to be $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ then we must insert:

$$\exp i\lambda_1\phi(-\frac{\pi}{2})\ \exp i\lambda_2\phi(\frac{\pi}{2}) \tag{13}$$

with $\lambda_1+\lambda_2=g(\oint)$. We will now show that these insertions suffice in order for the theory to be BRST invariant for appropriate choices of $g(\oint)$.

To show that $\oint Q(\Psi)=0$ for any closed string state Ψ, we write out $\oint\Psi$ as a path integral:

$$\oint\Psi\equiv\int DX^\mu(\sigma)D\phi(\sigma)[e^{i\lambda_1\phi(\frac{\pi}{2})}\prod_{\sigma=0}^{\frac{\pi}{2}}\delta(X^\mu(\sigma)-X^\mu(\pi-\sigma))\delta(\phi(\sigma)-\phi(\pi-\sigma))]\times$$
$$[e^{i\lambda_2\phi(-\frac{\pi}{2})}\prod_{\sigma=0}^{-\frac{\pi}{2}}\delta(X^\mu(-\sigma)-X^\mu(\sigma-\pi))\delta(\phi(-\sigma)-\phi(\sigma-\pi))]\Psi(X^\mu,\phi). \tag{14}$$

We may now simplify our task considerably if we note the following facts. From eq.(14), it can be seen that we are treating the closed string as if it were composed of two open strings, one parameterized from 0 to π and the other from $-\pi$ to 0. These open strings are then joined at 0 and $\pi(\equiv-\pi\ mod\ 2\pi)$. The former open string has a ghost insertion $e^{i\lambda_1\phi(\frac{\pi}{2})}$ at $\sigma=\frac{\pi}{2}$ while the latter one has an insertion $e^{i\lambda_2\phi(-\frac{\pi}{2})}$ at $\sigma=-\frac{\pi}{2}$ (the midpoints of the two open strings). Thus, to show that $\oint Q(\Psi)=0$ we can reproduce the proof in I using the new ghost insertion (with λ_i replacing $-\frac{3}{2}$). Eqs.(32, 33) of I then imply that $\oint Q(\Psi)=0$ only if $\lambda_i+\frac{1}{2}$ is an odd integer. Since $\lambda_1+\lambda_2=g(\oint)$, this implies that $g(\oint)$ must also be odd. These insertions also insure that $\oint Q_L,Q_R(\Psi)$ vanish.

We now turn to the modification of $*$, treating the open string case first. In this case, we use the same hexagon as before but now attach handles and boundaries to its interior, modifying its ghost number as for $g(\oint)$. We may also allow the boundary components to have non-zero extrinsic curvature. The ghost number $g(*)$ then becomes an arbitrary parameter.

For closed strings, we again consider them to be made up of two open strings with parameterization $[0, \pi]$ for one and $[-\pi, 0]$ for the other. As in I, we replace the open string hexagon by a rectangle with a wedge cut into it (fig. 17 in I). We then construct the mirror image of this surface and set the two side by side, with the open end of the wedges facing each other. Finally, we identify the relevant points on both rectangles and wedges so that each side represents a closed string. Since the open string surface could be modified so as to make $g(*)$ arbitrary, the same is true for closed strings. From the above picture, we see that we have ghost insertions at $\sigma = \pm\frac{\pi}{2}$ in the definition of $*$ for closed strings. We write this as $e^{i\mu_1\phi(+\frac{\pi}{2})}e^{i\mu_2\phi(-\frac{\pi}{2})}$ with $\mu_1 + \mu_2 = g(*)$. Eqs. (1,2) can be shown to hold as in I, as can the associativity of $*$.

We have shown by explicit construction that $g(\oint)$ and $g(*)$ can be chosen arbitrarily for both open and closed strings. We now use this in order to construct an interacting closed bosonic string action.

Let us first enumerate the constraints on such actions. First they must be gauge invariant under the transformations of eq.(4) (or some suitable modification thereof; see below). Second, we have only two free parameters, $g(*)$ and $g(\oint)$ to use. Finally, we must still have that Q has ghost number one, while physical closed string states have ghost number -1. If we are to include an explicit kinetic term for the string fields (this may not be necessary or desirable; see ref. (7)), then the action can only be some modification of the Chern-Simons three form in eq.(6). If we wish to keep eq.(6) as the <u>closed</u> string action, we must choose surfaces such that $g(*) = +2$, $g(\oint) = -1$ (note that this value of $g(*)$ is compatible with the one required by the gauge transformation law of eq.(4). Also note that $g(\oint)$ is odd as required by BRST invariance). If, however, we choose to consider a modified kinetic term (as in ref. (5)) such as

$$\oint A * [\bar{c}, Q] A, \tag{15}$$

where $\bar{c}$ is the fermionic antighost, and modify the gauge transformation law appropriately[5], we find that $g(\oint) = -3$, $g(*) = +3$ is required so that the cubic interaction term have non-zero integral. Again we note that $g(\oint)$ is odd in this case. In this case we need only consider surfaces with <u>no</u> extrinsic curvature (in fact, the hemisphere described earlier can be used to construct $\oint$), while this was not the case for the previous case above.

For completeness, we write out the cubic interaction term out explicitly:

$$\oint A * A * A \equiv \int DX_i^\mu(\sigma) D\phi_i(\sigma) e^{(\lambda_1+2\mu_1)\phi(+\frac{\pi}{2})} e^{(\lambda_2+2\mu_2)\phi(-\frac{\pi}{2})} \; x$$

$$\prod_{i,\mu} \prod_{\lambda=0}^{\frac{\pi}{2}} \delta(X_i^\mu(\lambda) - X_{i+1}^\mu(\pi - \lambda)) \delta(\phi_i(\lambda) - \phi_{i+1}(\pi - \lambda))$$

$$\prod_{\lambda=0}^{-\frac{\pi}{2}} \delta(X_i^\mu(-\lambda) - X_{i+1}^\mu(\lambda - \pi)) \delta(\phi(-\lambda) - \phi(\lambda - \pi)) \qquad (16)$$

where X_i^μ, ϕ_i $i = 1, 2, 3$ are the coordinates and ghosts for the three strings. Note that due to the δ functions, it is not necessary to specify which ϕ is present in the ghost insertions.

To conclude, we have shown how the ideas of I could be extended to generate an integral and an associative product relevant for closed bosonic strings. We have also extended the definitions of $\oint$ and $*$ for open strings, which may be important when actions without kinetic terms are considered(7). Using the freedom of choice in $g(*)$ and $g(\oint)$, we have shown that the Chern-Simons 3-form or modification of it is the only possible gauge invariant action for closed bosonic strings.

A possible flaw in our formulation of closed strings is that the on-shell-constraint that the number of right movers equal the number of left movers is not obviously satisfied. One way to see whether this is true or not is to construct the Fock space of our theory following Gross and Jevicki(8). This work is now in progress(9), as is a calculation of Virasoro-Shapiro amplitudes following Giddings(10)

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Figure Captions

Fig. 1: The $*$ operation for open strings. The point $\sigma = +\frac{\pi}{2}$ is selected and $S * T$ sews the left half of S to the right half of T if S_R coincides with T_L.

Fig. 2: The $\oint$ operation for open strings. $\oint$ S sews S_L to S_R.

Fig. 3: The $*$ product for closed strings. The points $\sigma = \pm\frac{\pi}{2}$ are selected and if the right half of S coincides with the left half of T we sew S_L to T_R.

Fig. 4: The $\oint$ operation for closed strings. This also sews S_L to S_R.

Fig. 5: Representation of the closed string $\oint$ via a modified hemisphere that is flat except for two singular points at $\sigma = \pm\frac{\pi}{2}$. When δ tends to zero, we are left with Fig. 4, but with computable ghost insertions at $\sigma = \pm\frac{\pi}{2}$.

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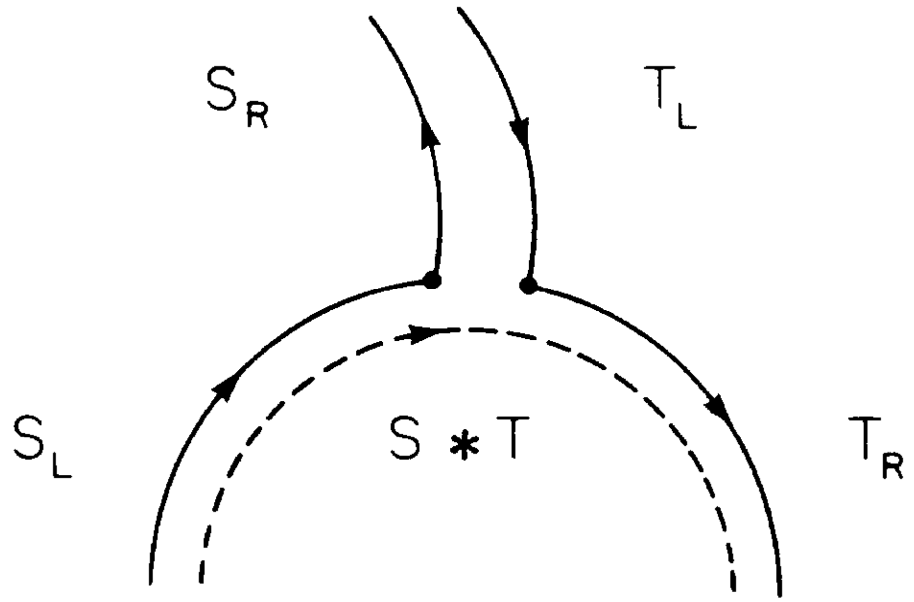

FIGURE 1

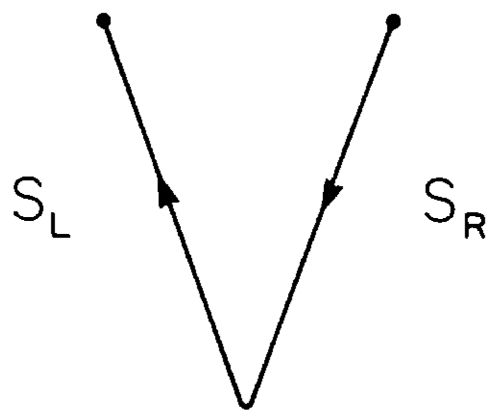

FIGURE 2

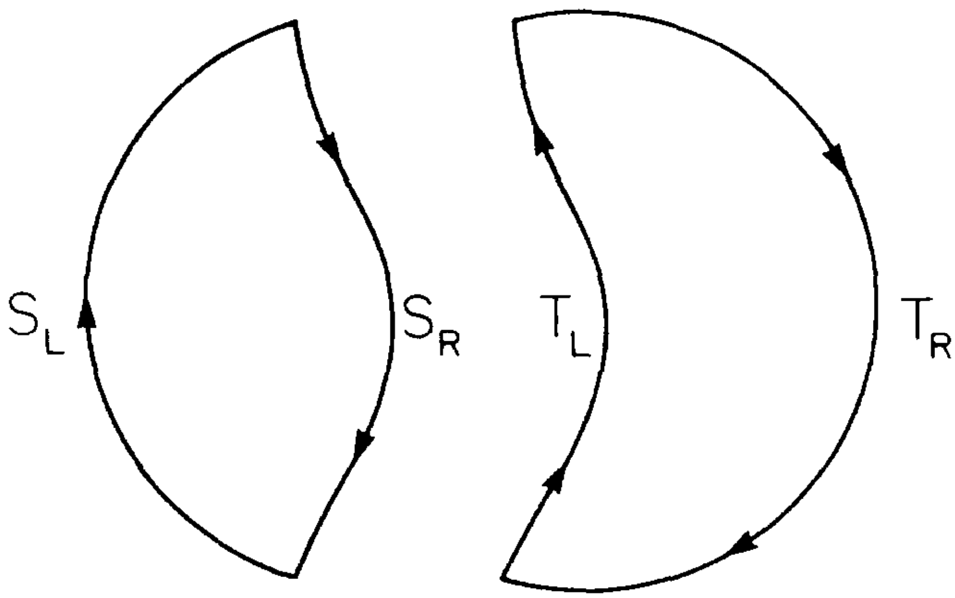

FIGURE 3

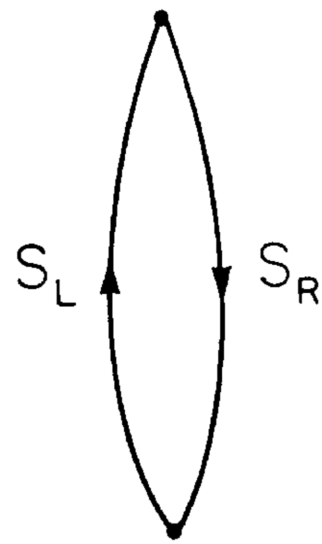

FIGURE 4

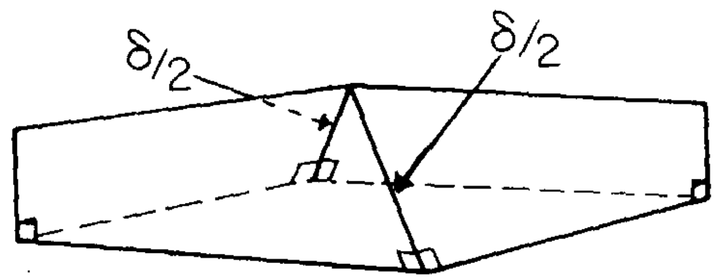

FIGURE 5