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# Explicit formulae for heavy flavour production

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## Abstract

Explicit formulae are given in QCD for the processes which contribute to the hadroproduction of heavy quarks $Q$. The processes considered are

1. $q + \overline{q} \to Q + \overline{Q}$
2. $g + g \to Q + \overline{Q}$
3. $Q + \overline{Q} \to q + \overline{q} + g$
4. $Q + \overline{Q} \to g + g + g$

All mass effects are included. A comparison of the exact matrix elements with the results of the leading pole and soft gluon approximation is made.

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## I. INTRODUCTION.

In the last year there has been a resurgence of interest in the theory of the production of heavy quark pairs in hadronic reactions. This interest is motivated in part by the promise of reliable experiments on the hadro-production of charm[1], and also by a reported sighting of the top quark in $p\overline{p}$ collisions[2]. The theoretical progress[3,4,5] of the last year can be summarised as follows. It is now believed that the dominant parton reactions leading to the production of a sufficiently heavy quark $Q$ are,

$$\begin{aligned} &(a) \quad q(p_1) + \overline{q}(p_2) \to Q(p_3) + \overline{Q}(p_4) \\ &(b) \quad g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4) \end{aligned} \tag{1}$$

where the four momenta of the partons are given in brackets. The invariant matrix elements squared for processes (a) and (b) have been available in the literature for some time[6,7,8] and are given by,

$$\overline{\sum} \left| M^{q\overline{q}\to Q\overline{Q}}(p_1,p_2,p_3,p_4)\right|^2 = \frac{g^4 V}{2N^2}\left(\frac{\{13\}^2+\{23\}^2}{\{12\}^2}+\frac{m^2}{\{12\}}\right) \tag{2}$$

$$\begin{aligned} &\overline{\sum} \left| M^{gg\to Q\overline{Q}}(p_1,p_2,p_3,p_4)\right|^2 = \\ &\frac{g^4}{2VN}\left(\frac{V}{\{13\}\{23\}}-\frac{2N^2}{\{12\}^2}\right)\left(\{13\}^2+\{23\}^2+2m^2\{12\}-\frac{m^4\{12\}^2}{\{13\}\{23\}}\right) \end{aligned} \tag{3}$$

where the dependence on the $SU(N)$ colour group is shown explicitly, ($V = N^2-1$, $N=3$) and $m$ is the mass of the produced heavy quark $Q$. The matrix elements squared in Eqs.(2,3) have been summed and averaged over initial and final colours and spins, (as indicated by $\overline{\sum}$). For brevity, in this and the following formulae, we have introduced the notation for the dotproduct of two four-momenta.

$$p_i.p_j = \{ij\} \tag{4}$$

The processes of Eq.(1) lead to a cross-section for the production of charmed particles which is predominantly central[5]. In addition, the transverse momentum of the heavy quark or anti-quark is, on the average, of the order of its mass, whilst the transverse momentum of the quark-antiquark pair is small. The theoretical arguments summarised above do not address the issue of whether the charmed quark is sufficiently heavy that the hadroproduction of charmed hadrons in all regions of phase space is described via this mechanism.

In the literature there is another class of processes which contributes significantly at large transverse momentum. As first observed by Kunszt and Pietarinen[9], because of the large cross-section for the production of gluon jets at large transverse momentum the process

$$\begin{array}{rcl} g+g & \rightarrow & g+g \\ & & \hookrightarrow Q+\overline{Q} \end{array} \tag{5}$$

is likely to be the principal source of heavy quarks at large transverse momentum ($q_T >> m$). This mechanism might give a large contribution to heavy particle production because the ratio of the relevant $2 \rightarrow 2$ matrix elements is extremely large. In fact, at 90 degrees in the parton-parton centre of mass, the gluon fusion mechanism is much more likely to produce a gluon than a quark.

$$\frac{\overline{\sum}\left|M^{(gg\rightarrow gg)}\right|^2}{\overline{\sum}\left|M^{(gg\rightarrow q\bar{q})}\right|^2} \approx 200 \tag{6}$$

The validity of this schematic reasoning has been confirmed by analysis of the full $2 \rightarrow 3$ matrix elements including mass effects[9,10].

In view of the burgeoning theoretical interest in these processes and their potential significance at collider energies, it seemed appropriate to recalculate the results of Kunszt and Pietarinen and attempt to present them in a form which would be accessible to a larger audience. We have therefore recalculated the matrix elements for the processes,

$$(A) \quad Q(-p_1) + \overline{Q}(-p_2) \rightarrow q(p_3) + \overline{q}(p_4) + g(p_5) \tag{7}$$

$$(B) \quad Q(-p_1) + \overline{Q}(-p_2) \rightarrow g(p_3) + g(p_4) + g(p_5) \tag{8}$$

As indicated by the above notation we have calculated the matrix elements squared in the unphysical region in which all momenta are outgoing,

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0 \tag{9}$$

The matrix element for the one gluon four quark process in Eq.(7) is given in terms of the function $A$ as[5],

$$
\begin{aligned}
A(p_1,p_2,p_3,p_4,p_5) \quad = \\
& \left[\frac{\{13\}^2+\{23\}^2+\{14\}^2+\{24\}^2+m^2\left(\{12\}+\{34\}+m^2\right)}{2s\{34\}}\right] \\
& \times\left[\frac{4V^2}{N}\left(\frac{\{13\}}{\{15\}\{35\}}+\frac{\{24\}}{\{25\}\{45\}}\right)\right. \\
& \left.+\frac{4V}{N}\left(\frac{2\{14\}}{\{15\}\{45\}}+\frac{2\{23\}}{\{25\}\{35\}}-\frac{\{13\}}{\{15\}\{35\}}-\frac{\{24\}}{\{25\}\{45\}}-\frac{\{12\}}{\{15\}\{25\}}-\frac{\{34\}}{\{35\}\{45\}}\right)\right] \\
& -\frac{(N^2-4)V}{N}\frac{2m^2}{s\{34\}}\left[\frac{\{13\}-\{14\}}{\{25\}}-\frac{\{23\}-\{24\}}{\{15\}}\right] \\
& +\frac{4V^2}{N}m^2\left[\frac{1}{s^2}\left(\frac{\{35\}^2+\{45\}^2}{\{35\}\{45\}}\right)-\frac{1}{2s}\left(\frac{1}{\{15\}}+\frac{1}{\{25\}}+\frac{1}{\{35\}}+\frac{1}{\{45\}}\right)\right. \\
& \left.-\frac{1}{4\{34\}}\left(\frac{1}{\{15\}}+\frac{1}{\{25\}}+\frac{m^2}{\{15\}^2}+\frac{m^2}{\{25\}^2}+\frac{4}{s}\right)-\frac{\Delta_1^2+\Delta_2^2+\Delta_3^2+\Delta_4^2}{4\{34\}^2}\right] \\
& -\frac{2V}{N}\frac{m^2}{s\{34\}}\left[1+\frac{2\{34\}}{s}+\frac{m^2}{\{15\}}+\frac{m^2}{\{25\}}+\frac{\{35\}^2+\{45\}^2}{\{15\}\{25\}}\right. \\
& \left.+\left(\{13\}-\{14\}\right)\frac{\Delta_1-\Delta_2}{\{34\}}+\left(\{23\}-\{24\}\right)\frac{\Delta_3-\Delta_4}{\{34\}}\right] \qquad (10)
\end{aligned}
$$

where,

$$
\begin{aligned}
& s=(p_1+p_2)^2, \quad \{ij\}=p_i.p_j \quad p_1.p_1=p_2.p_2=m^2 \\
& \Delta_1=\frac{\{13\}}{\{25\}}-\frac{2\{35\}}{s}, \quad \Delta_2=\frac{\{14\}}{\{25\}}-\frac{2\{45\}}{s} \\
& \Delta_3=\frac{\{23\}}{\{15\}}-\frac{2\{35\}}{s}, \quad \Delta_4=\frac{\{24\}}{\{15\}}-\frac{2\{45\}}{s} \qquad (11)
\end{aligned}
$$

The matrix element for the two quark-three gluon process in Eq.(8) is given in the same notation by the function $B$. In turn $B$ is defined as the sum over the twelve permutations of the function $F^B$ obtained by interchange of the heavy quark momenta ($p_1$ and $p_2$) and the three gluon momenta ($p_3$, $p_4$ and $p_5$).

$$
B(p_1,p_2,p_3,p_4,p_5)=\sum_{permutations} F^B(p_1,p_2,p_3,p_4,p_5) \qquad (12)
$$

and $F^B$ is given by the slightly cumbersome formula,

$$
\begin{aligned}
&F^B(p_1,p_2,p_3,p_4,p_5) = \\
&+\frac{V(N^2+1)}{4N^2}\frac{(\{12\}(\{15\}^2+\{25\}^2+2m^2(\{35\}+\{45\})))}{\{13\}\{14\}\{23\}\{24\}} \\
&-\frac{V(N^2+1)m^2}{N^2}\frac{(s^2+2\{13\}\{14\}+2\{23\}\{24\}+4m^2\{12\})}{8\{13\}\{14\}\{23\}\{24\}} \\
&-V\left[\frac{\{13\}(\{13\}^2+\{23\}^2+2m^2(\{34\}+\{35\}))}{\{34\}\{14\}\{15\}\{25\}}+\frac{(\{15\}^2+\{25\}^2+2m^2(\{35\}+\{45\}))}{2\{34\}\{14\}\{23\}}\right] \\
&+Vm^2\left[\frac{(\{12\}-3m^2)}{\{34\}\{15\}\{25\}}+\frac{(3\{13\}\{23\}+5\{13\}\{25\}-2\{13\}^2)}{\{13\}\{14\}\{23\}\{24\}}\right] \\
&+Vm^4\left[\frac{(\{13\}^2+\{23\}^2+\{15\}^2+\{15\}\{25\}-s\{13\})}{\{34\}\{13\}\{24\}\{15\}\{25\}}+\frac{(\{15\}+\{24\})}{\{13\}\{23\}\{14\}\{25\}}\right] \\
&+2VN^2\left[\frac{\{23\}(\{13\}^2+\{23\}^2+2m^2(\{34\}+\{35\}))}{s\{34\}\{45\}\{25\}}\right] \\
&+VN^2\left[\frac{\{14\}\{24\}(\{14\}^2+\{24\}^2+2m^2(\{34\}+\{45\}))}{s\{34\}\{45\}\{13\}\{25\}}\right] \\
&+2VN^2m^2\left[\frac{(4\{15\}\{25\}+\{13\}\{23\}+\{14\}\{24\}-\frac{1}{8}s(s+2\{34\}))}{s\{34\}\{15\}\{25\}}\right] \\
&+VN^2m^2\left[\frac{(\{34\}-2m^2)}{s\{13\}\{24\}}-\frac{s}{4\{34\}\{15\}\{25\}}\right]-2VN^2m^4\frac{(\frac{1}{4}s^2+\{34\}^2+\{45\}^2)}{s\{34\}\{45\}\{13\}\{25\}} \\
&+4VN^2m^2\left[\frac{(\{34\}^2+\{35\}^2+\{45\}^2)}{s^2\{34\}^2}+\frac{2\{23\}^2}{s\{34\}^2\{15\}}\right] \\
&+V^2m^2\left[\frac{1}{\{34\}^2}\left(\frac{\{13\}}{\{25\}}-\frac{\{23\}}{\{15\}}\right)^2+\frac{2m^2}{\{34\}\{15\}^2}+\frac{m^2(\{34\}+\{45\})}{\{34\}\{45\}\{13\}\{25\}}\right] \\
&+\frac{V^2m^2}{4N^2}\left[\frac{(\{13\}^2+\{23\}^2-(\{13\}+\{23\})\{45\})}{\{13\}\{23\}\{14\}\{25\}}+\frac{2sm^2}{\{13\}\{23\}\{14\}\{25\}}-\frac{4m^4}{\{13\}\{14\}\{25\}^2}\right] \\
&+\frac{V^3m^2}{2N^2}\left[\frac{m^4+4m^2\{24\}-2\{24\}\{25\}}{\{13\}^2\{24\}^2}+\frac{(s\{34\}+2\{15\}\{25\}-\{14\}\{24\})}{\{13\}\{23\}\{14\}\{24\}}\right] \\
&+\frac{Vm^2}{N^2}\left[\frac{m^4}{\{13\}\{23\}\{14\}\{25\}}-\frac{(5s\{13\}+8\{13\}\{23\}+6\{13\}\{25\}-4\{13\}^2)}{4\{13\}\{14\}\{23\}\{24\}}\right]
\end{aligned}
\tag{13}
$$

Expressions for the spin and colour averaged matrix elements for physical processes in terms of the functions $A$ and $B$ are given in Table 1. Fortran routines which generate these functions are available from the authors. We have checked numerically that Eqs.(10,13) agree with the results of ref.(9). Eqs.(10,13) have been obtained with the help of the algebraic manipulation program Schoonschip[11].

## II. THE LEADING POLE AND SOFT GLUON APPROXIMATIONS.

In this section we explore the connection of our exact results with the leading pole approximation and soft gluon approximation. By using the explicit formula Eq.(10), we find that in the limit of small $s_{34} = (p_3 + p_4)^2$, the matrix element for the process

$$g(p_1) + q(p_2) \to Q(p_3) + \overline{Q}(p_4) + q(p_5) \tag{14}$$

tends to,

$$\overline{\sum} \left| M^{gq \to Q\overline{Q}q} \right|^2 \to g^2 \overline{\sum} \left| M^{gq \to gq} \right|^2 \frac{2}{s_{34}} P_{Qg}(z) + O(\frac{m^2}{Q^2 s_{34}}) \tag{15}$$

where,

$$P_{Qg}(z, \frac{m^2}{s_{34}}) = \frac{1}{2} \left[ z^2 + (1-z)^2 + \frac{2m^2}{s_{34}} \right], \quad z = \frac{\{35\}}{\{35\} + \{45\}} \tag{16}$$

and $Q^2$ is a large invariant of the order of $s$, $t$ or $u$. The relevant $2 \to 2$ matrix elements are given in Table 2. This formula illustrates the way in which gluon fragmentation into heavy quark pairs would be inserted into Monte-Carlo programs. The precise definition of the longitudinal variable $z$ may vary between Monte Carlo programs. In a Monte Carlo program, the terms of order $\frac{m^2}{Q^2}$ are dropped. $P_{Qg}$ is the Altarelli-Parisi function for heavy quark production[12]. Note that the term proportional to the mass in $P_{Qg}$ is normally dropped because it is non-leading. It is however necessary in the calculation of the growth of the multiplicity[13].

In a similar way the matrix element for the process,

$$g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4) + g(p_5) \tag{17}$$

in the limit of small $s_{34}$ tends to,

$$\overline{\sum} \left| M^{gg \to Q\overline{Q}g} \right|^2 \to g^2 \overline{\sum} \left| M^{gg \to gg} \right|^2 \frac{2}{s_{34}} P_{Qg}(z, \frac{m^2}{s_{34}}) + O(\frac{m^2}{Q^2 s_{34}}) \tag{18}$$

It is of also of interest to investigate the approximation in which the an emitted gluon is extremely soft. For example in the limit in which $p_5$ tends to zero the matrix element for the process in Eq.(17) becomes

$$
\begin{aligned}
&\overline{\sum}\left|M^{gg\to Q\overline{Q}g}\right|^2 \to \\
&g^2\overline{\sum}\left|M^{gg\to Q\overline{Q}}\right|^2 \left(C_F\left([1,3]+[1,4]+[2,3]+[2,4]-2[1,2]-[3,3]-[4,4]\right)+2N[1,2]\right)\\
&+g^6X(p_1,p_2,p_3,p_4)\left([1,4]+[2,3]-[1,3]-[2,4]\right)\\
&+g^6Y(p_1,p_2,p_3,p_4)\left(2[1,2]+2[3,4]-[1,3]-[2,4]-[1,4]-[2,3]\right)
\end{aligned}
\tag{19}
$$

In this equation the eikonal factor for soft gluon emission is represented by a square bracket,

$$
[i,j]=\frac{\{i,j\}}{\{i,5\}\{j,5\}},\quad C_F=\frac{N^2-1}{2N}
\tag{20}
$$

$\overline{\sum}\left|M^{gg\to Q\overline{Q}g}\right|^2$ is given by Eq.(3) and the functions X and Y are given, in the notation of Eq.(4), by

$$
\begin{aligned}
X(p_1,p_2,p_3,p_4) &= \frac{N^2}{4V}\left(\frac{\{12\}+2m^2}{\{23\}}-\frac{\{12\}+2m^2}{\{13\}}+\frac{m^4}{\{13\}^2}-\frac{m^4}{\{23\}^2}+2\frac{\{23\}-\{13\}}{\{12\}}\right)\\
Y(p_1,p_2,p_3,p_4) &= \frac{1}{4VN^2}\left(\{13\}^2+\{23\}^2+2m^2\{12\}-\frac{m^4\{12\}^2}{\{13\}\{23\}}\right)\\
&\left(\frac{1}{\{13\}\{23\}}+\frac{2N^2}{\{12\}^2}\right)
\end{aligned}
\tag{21}
$$

In a similar way the matrix element for the process,

$$
q(p_1)+\overline{q}(p_2)\to Q(p_3)+\overline{Q}(p_4)+g(p_5)
\tag{22}
$$

in the limit of vanishing $p_5$ tends to,

$$
\begin{aligned}
&\overline{\sum}\left|M^{q\overline{q}\to Q\overline{Q}g}\right|^2 \to \\
&g^2\overline{\sum}\left|M^{q\overline{q}\to Q\overline{Q}}\right|^2 \Big(C_F\left(2[1,3]+2[2,4]-[3,3]-[4,4]\right)\\
&+\frac{1}{N}\left(2[1,4]+2[2,3]-[1,3]-[2,4]-[1,2]-[3,4]\right)\Big)
\end{aligned}
\tag{23}
$$

Note that in the leading $N$ approximation the soft radiation is given by an eikonal factor multiplying the lower order hard process and weighted with a colour charge. For general $N$ the soft approximation is no longer necessarily proportional to the lower order cross-section. It may be possible to incorporate the pattern of soft radiation into a Monte Carlo program[14] using an angular ordering constraint[15].

If the leading pole approximation is to be a good approximation then we require that $s_{34} << Q^2$. This constraint will not hold over all phase space or for all choices of $m^2$. It becomes important therefore to understand the limitations of the approximation. For any given choice of $s$ and $m^2$ the generic scale $Q^2$ is maximised when $|t| = |u| = 0.5s$. Thus the best case scattering occurs when the massive quarks recoil against a massless parton in the plane perpendicular to the incident beams. If we confine ourselves to this plane then the kinematics depend only on the energies of the three scattering partons and the relevant cross-sections are most easily represented on Dalitz plots.

In Figure 1 we have generated a contour Dalitz plot of the full matrix element (using Eq.(13)) for the process

$$g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4) + g(p_5) \tag{24}$$

The variables $E_3$, $E_4$, and $E_5$ are the energies of the scattering heavy quark, heavy anti-quark and gluon respectively. The value of $m/\sqrt{s}$ for this plot is 0.02. This value is appropriate for charm or bottom production at the Cern $pp$ collider when the incident partons are at intermediate or large $x$. There are two features to note. First the large peak present for small $E_5$ is simply the soft gluon contribution. We do not expect this to be present in the leading pole approximation. The second feature to note is the ridge which is present for large $E_5$. This ridge arises because $1/s_{34}$ is small in this region of phase space and it is just this feature which we expect the leading pole approximation to reproduce.

In Figure 2 we have shaded that region of the Dalitz plot of Figure 1 where the leading pole approximation and the full matrix element agree to within 20%. We see that there is good agreement for $E_5$ large as expected. For $E_5$ small the leading pole result is much smaller than the full result since the leading pole result does not reproduce the soft gluon peak. In the central portion of the plot the leading pole result over-estimates the full matrix element by a factor of approximately two.

This overestimation is not however very important since the full matrix element is very small in this central region.

The encouraging agreement between the leading pole and the full result which we have just demonstrated represents the best case result. If the mass of the heavy quarks increases or if the value of $\sqrt{s}$ decreases then the leading pole approximation quickly breaks down. For example in Figure 3 we show a plot similar to Figure 2 for the case $m/\sqrt{s} = 0.1$. Note that the 20% band which was present in Figure 2 for $E_5$ large has now disappeared. (There is in fact still a very thin band of agreement right at the large $E_5$ boundary but even this disappears as $m/\sqrt{s}$ increases further.)

These results suggest that the leading pole approximation provides a good description of heavy quark production provided firstly that the produced quarks recoil against a single gluon with large $E_T$, and secondly that the value of $m/\sqrt{s}$ for the production satisfy $m/\sqrt{s} \leq 0.10$.

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# Tables

| Process | $\overline{\sum} \lvert M\rvert^2$ |
|---|---|
| $Q(p_1) + \overline{Q}(p_2) \to q(p_3) + \overline{q}(p_4) + g(p_5)$ | $g^6 A(-p_1, -p_2, p_3, p_4, p_5)/4/N^2$ |
| $q(p_1) + \overline{q}(p_2) \to Q(p_3) + \overline{Q}(p_4) + g(p_5)$ | $g^6 A(p_4, p_3, -p_2, -p_1, p_5)/4/N^2$ |
| $g(p_1) + q(p_2) \to Q(p_3) + \overline{Q}(p_4) + q(p_5)$ | $-g^6 A(p_4, p_3, p_5, -p_2, -p_1)/4/N/V$ |
| $Q(p_1) + \overline{Q}(p_2) \to g(p_3) + g(p_4) + g(p_5)$ | $g^6 B(-p_1, -p_2, p_3, p_4, p_5)/4/N^2$ |
| $g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4) + g(p_5)$ | $g^6 B(p_4, p_3, p_5, -p_1, -p_2)/4/V^2$ |

Table I. Spin and colour averaged matrix elements for physical processes in terms of the functions $A$ and $B$.

| Process | $\overline{\sum} \lvert M\rvert^2$ |
|---|---|
| $q(p_1) + \overline{q}(p_2) \to g(p_3) + g(p_4)$ | $g^4 V/2/N^3(V/t/u - 2N/s^2)(t^2 + u^2)$ |
| $g(p_1) + q(p_2) \to g(p_3) + q(p_4)$ | $g^4/2/N^2(V/s/u - 2N/t^2)(s^2 + u^2)$ |
| $g(p_1) + g(p_2) \to g(p_3) + g(p_4)$ | $4g^4 N^2/V(3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2})$ |

Table II. Spin and colour averaged matrix elements for $2 \to 2$ processes. $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_2 - p_3)^2$.

# Figure Captions

1. Three dimensional Dalitz Plot of the full matrix element squared for the process of Eq. 23 as a function of the energies, $E_3$, $E_4$, and $E_5$ of the three scattering partons for $m/\sqrt{s} = 0.02$. The variable plotted along the $x$-axis is $(E_3 - E_4)/\sqrt{3S}$. The variable plotted along the $y$-axis is $E_5/\sqrt{S}$. The $z$-axis gives the full matrix element squared in units of $S^{-1}$, ( $\sum |M|^2 = \frac{z}{S}$).

2. Comparison of the leading pole and full results for $m/\sqrt{s} = 0.02$. The solid closed curve shown here encloses the kinematically allowed region of phase space. The shaded portion of this plot marks that region of phase space where the leading pole and full results agree to within 20%.

3. Comparison of the leading pole and full results for $m/\sqrt{s} = 0.10$. Notation is as for Figure 2.

Figure 1

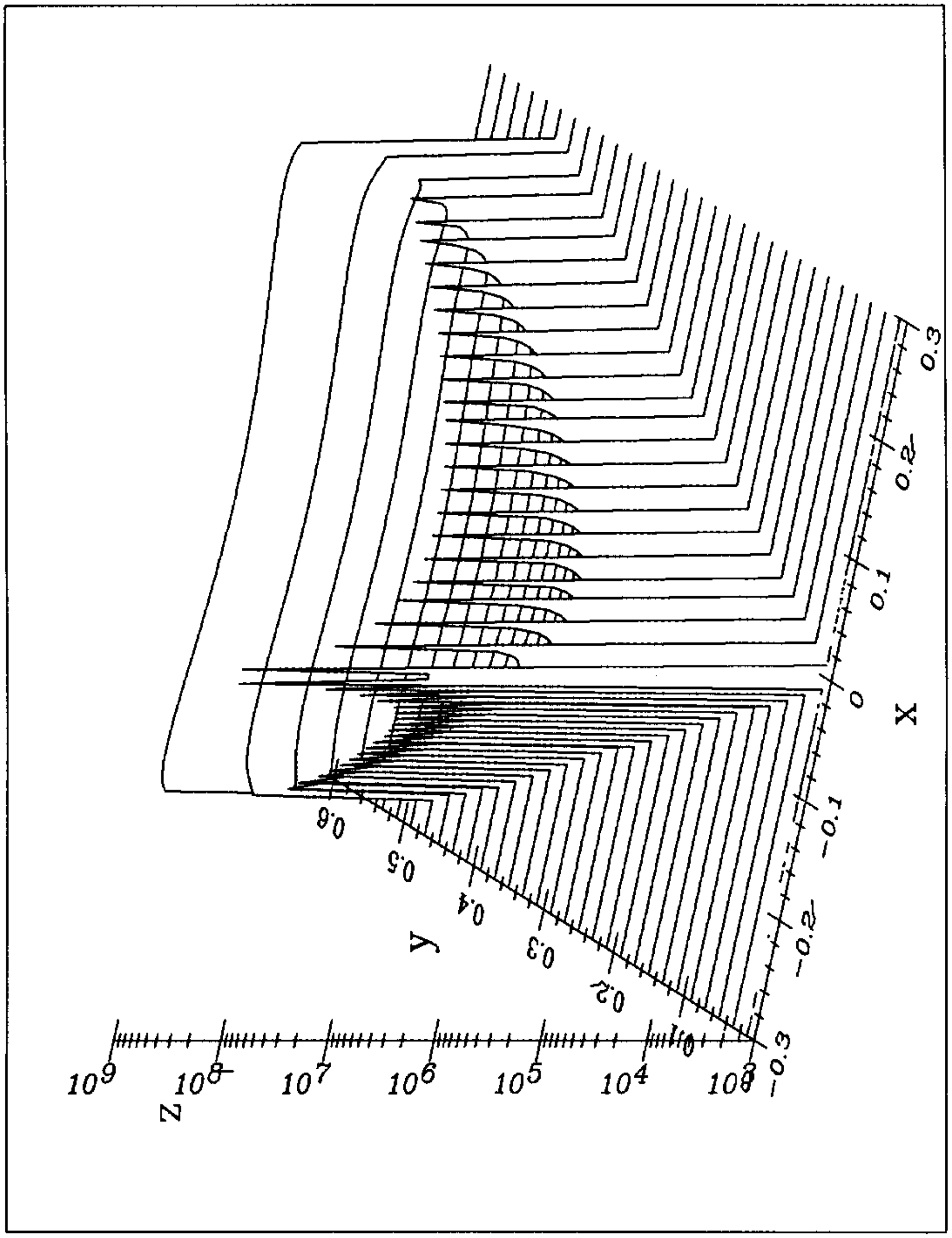

Figure 2

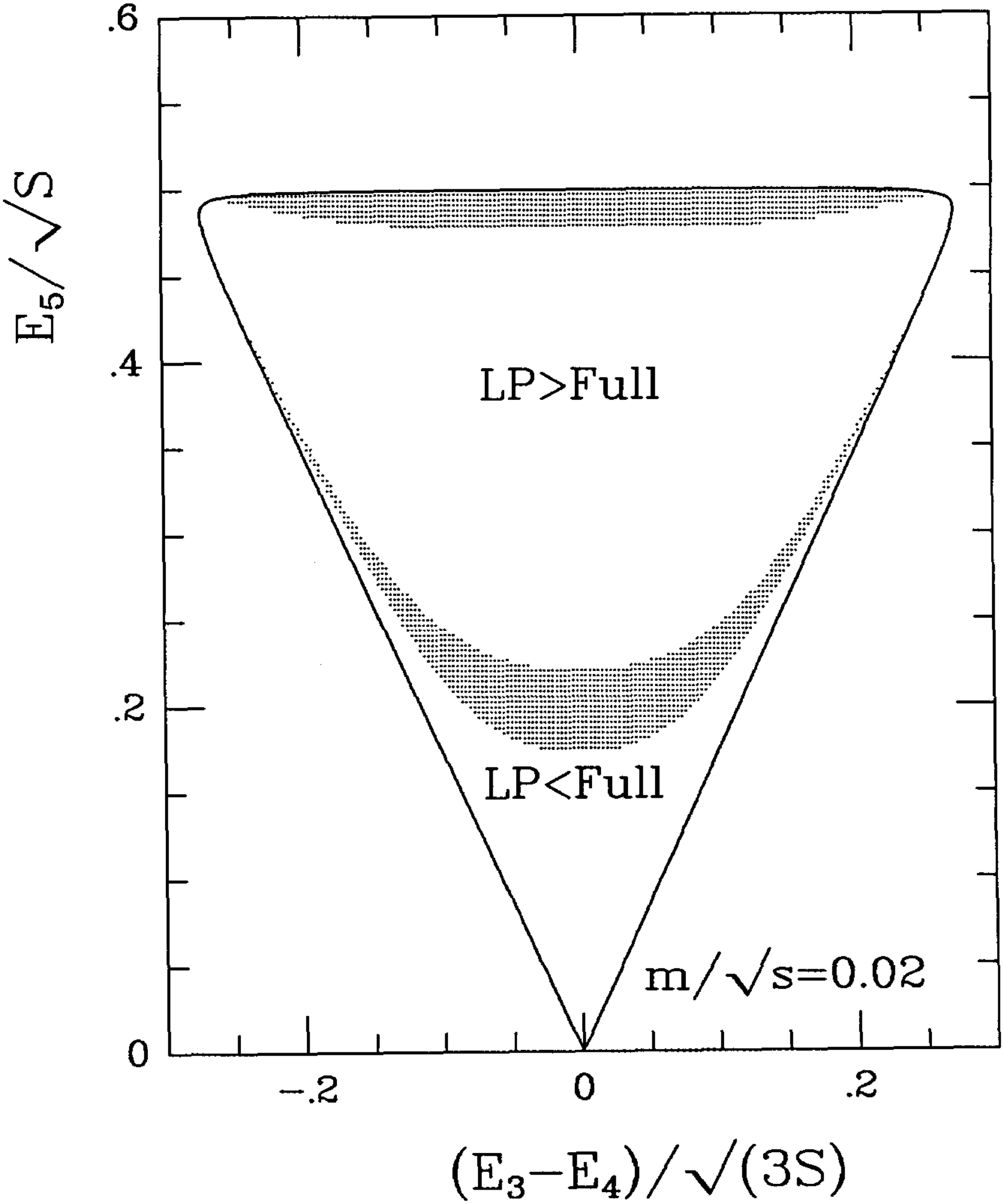

Figure 3