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PERTURBATION THEORY AND THE SINGLE SEXTUPOLE

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Perturbation theory plays at best an equivocal role in studying the behavior of a nonlinear dynamical system. Even the simplest systems possess complicated orbits, which makes the validity of a perturbative expansion doubtful. From a practical standpoint, however, convergence is seldom the real issue; for example, renormalized perturbative QED is certainly not assured to converge, yet its successes have been overwhelming. Rather, one would like to know whether the *first few* low order terms model the system's behavior "reasonably well" within the phase space region of interest. We shall consider this question for a very simple problem from accelerator theory: the single thin sextupole in one degree of freedom.

The design of a circular accelerator begins with the specification of a *central orbit.* Particles are constrained to remain close to the central orbit, to first order, by inserting quadrupole magnets to act as "lenses" which keep the beam focussed. Hill's equation describes the linearized transverse dynamics.

$$\frac{d^2x}{d\theta^2} + K(\theta)x = 0 \tag{1}$$

Here, x represents the horizontal, let us say, displacement of a particle from the central orbit; θ, the "independent variable," is an angular coordinate which labels points on the central orbit; K is a periodic function related to the transverse gradients of the quadrupoles' magnetic fields. The two independent Floquet solutions of this equation can be written

$$x(\theta) = \sqrt{\beta(\theta)}\, exp(\pm i\psi(\theta)) \tag{2}$$

where the *lattice functions* ψ and β are related by $d\psi = ds/\beta = Rd\theta/\beta$, s being arclength along the central orbit. [1] The function β is periodic, but ψ is not. Instead, it obeys the condition $\psi(\theta + 2\pi) = \psi(\theta) + 2\pi\nu$, where ν is the (horizontal) *tune* of the machine. It counts the number of times a particle oscillates about the central orbit in traversing the accelerator once.

Magnetic fields which vary nonlinearly with x are added to the accelerator either deliberately—to perform resonance extraction or to control certain dynamical effects, such as chromaticity—or accidentally—simply because we cannot build perfect dipoles and quadrupoles. In particular, inserting sextupole fields into the accelerator produces a force quadratic in the displacement variable. Eq. (1) then becomes

$$\frac{d^2x}{d\theta^2} + K(\theta)x + S(\theta)x^2 = 0 \tag{3}$$

where S is a periodic function which characterizes the strength and distribution of the sextupoles.

Now consider the case in which a single thin sextupole is inserted into the ring. "Thin" means that $S(\theta) \propto \delta(\theta)$, which in practice means that x remains unchanged in passing through the sextupole while a suitably defined "momentum", p, undergoes a kick, Δp, given by

$$\begin{aligned} \Delta p &= -\lambda x^2 && (4) \\ \lambda &\equiv -e\beta B''l/2p_3 && (5) \end{aligned}$$

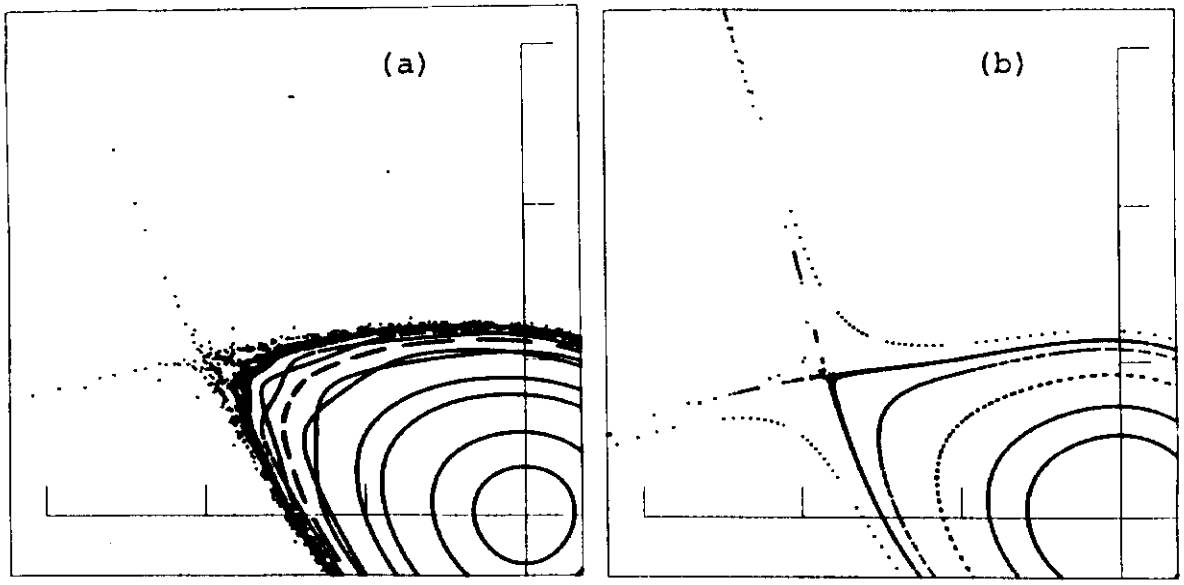

Figure 1: (a) Orbits of the sextupole mapping for $\nu = 0.15$. (b) Second order perturbation theoretic calculation of the stability boundary.

where e is the charge on a proton (the particle), p_3 is its longitudinal momentum, B'' is the (average) second derivative of the sextupole field, l is the length of the sextupole, and β, defined in Eq. (2), is evaluated at the position of the sextupole. The full Poincare map then concatenates this with a phase space rotation through $2\pi\nu$, representing passage through the rest of the accelerator.

$$\begin{pmatrix} x \\ p \end{pmatrix} \Leftarrow \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \begin{pmatrix} x \\ p - \lambda x^2 \end{pmatrix} \tag{6}$$

We can set $\lambda \equiv 1$ without loss of generality by rescaling, $x \to x/\lambda$ and $p \to p/\lambda$. This is in keeping with Hénon's observation that any area preserving quadratic map can be put into a one-parameter form. [2]

We have studied this mapping in the tune range $0 < \nu < \frac{1}{2}$; Figures 1a and 2a illustrate a few orbits at the tunes $\nu = 0.15$, $\nu = 0.29$ respectively. The tic marks on the axes are separated by 0.5. The general features in these drawings are not surprising: (i) near the origin there are smooth (on the scale of the observations) KAM tori; (ii) as one gets farther in phase space a structure of islands and sub-islands develops; (iii) which finally breaks into a chaotic sea, nonetheless contains stable islands of its own.

It is hopeless to expect perturbation theory to say much about the rich fine-scale structure—which the figures exhibit rather poorly—of this mapping; it is, after all, the existence of this structure which makes us uneasy about the meaning of a perturbative expansion. However, the principal feature of interest is the *stability boundary*, and perturbation theory does enable us to calculate its position and shape surprisingly well. Figures 1b and 2b illustrate calculations done by applying Deprit's algorithm to the Hamiltonian associated with Eq.(6). [3] The dynamics in Figure 1 is dominated by a first order integer resonance, which must be put explicitly into the new Hamiltonian. With the appropriate distortion, also given by the perturbation expansion, *the separatrix of the resonance then can be associated with the stability boundary of the exact mapping.* By making this identification, we can compute the location of the latter to better than 10%.

Figure 2 is a remarkable case. Its most dramatic feature is the very large 2/7 resonance

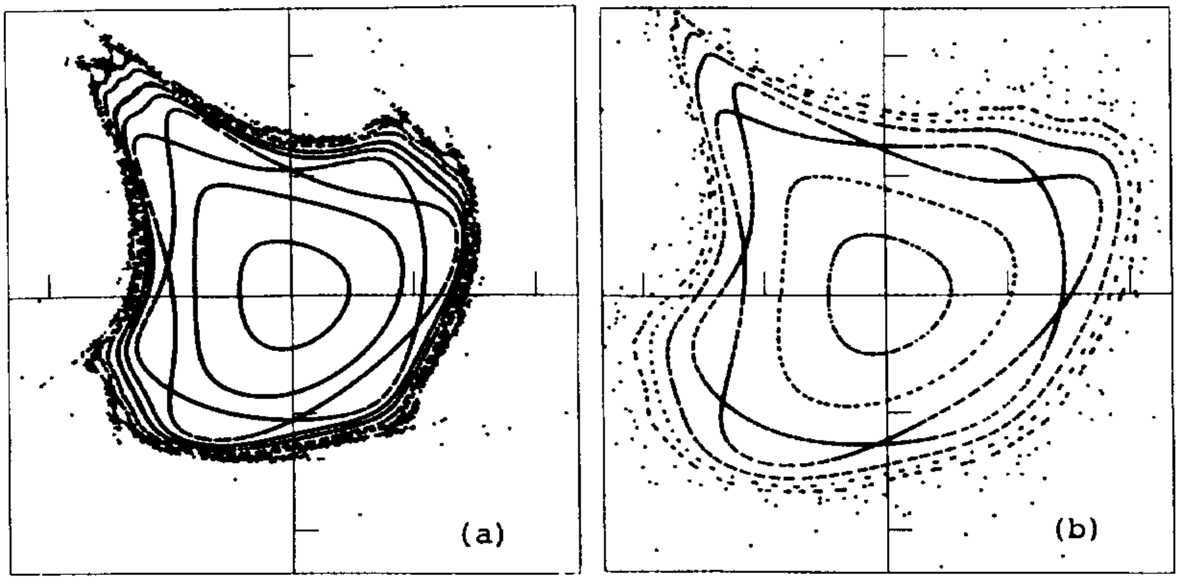

Figure 2: Same as Figure 1, but with $\nu = 0.29$.

which produces a system of seven islands. Seventh "order" resonances (i.e., resonances with winding number seven) should not appear until fifth order in the perturbation expansion, while the island chain is certainly more than a fifth order effect. In fact it is due to an *interference* between the 1/3 resonance, which appears at first order in the perturbation expansion, and the 1/4 resonance, which appears at second order. This is confirmed in Figure 2b which shows the perturbation theoretic prediction when those two resonances are explicitly taken into account.

Carrying out similar comparisons at other values of the tune we have found that second order perturbation calculations can usually predict the stability boundary within 5-15% accuracy when the dominant resonances are put into the new Hamiltonian.

Of course, the real situation is far more complicated. At the minimum we must include both transverse directions in any realistic analysis of sextupole effects. This would change the horizontal force to something proportional to $x_1^2 - x_2^2$, where x_1 and x_2 represent the horizontal and vertical displacements from the central orbit, while introducing a vertical force proportional to $x_1 x_2$. The dynamics are in fact derivable from a Hamiltonian with a potential term of the form $g(\theta)(x_1^3 - 3x_1x_2^2)$. If g were constant we would recapture the Hénon-Heiles potential. In addition, more than sextupoles must be taken into account: octupoles produce cubic forces, decapoles produce quartic forces, and so forth. The "general" Hamiltonian representing transverse dynamics of a storage ring will possess harmonic polynomials in the transverse variables multiplying periodic functions of θ. The analysis of such Hamiltonian systems is a major challenge for accelerator theorists.

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