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Multiparticle Superstring Tree Amplitudes

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Abstract

Covariant functional integral methods are used to derive formulas for (Type I, Type II, and heterotic) superstring tree amplitudes having arbitrary numbers of external ground-state bosons, thereby obtaining superstring generalizations of the Koba-Nielsen formula. The cancellation of the infinities of the Type I closed-string diagrams (disk and projective plane)is shown to require a relation between the coupling of the joining-splitting interaction g and the coupling of the exchange interaction κ.

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I. Introduction

In view of the widespread discussion of superstring theories[1] as candidates for finite quantum theories of all known fundamental forces, it is remarkable how few superstring amplitudes have actually been calculated. In contrast to the full-fledged expressions[2] for bosonic string tree and one-loop diagrams, complete expressions exist in the literature[1,3] only for tree and one-loop amplitudes having three or four external particles. In particular, there exists no superstring counterpart of the Koba-Nielsen formula in the literature.[4]

As a consequence, discussions of the finiteness (unitarity) of superstring theories have been based on extrapolating from few-particle to N-particle amplitudes using "pictorial" arguments based on world-sheet diagrams. Clearly, it is important to be able to explicitly calculate superstring amplitudes with any number of external states, both for purposes of checking finiteness, and also for possible phenomenological applications. Ultimately, of course, one would like to be able to calculate arbitrary multiloop amplitudes, and considerable progress[5] has recently been made in developing the tools for doing so.

The reason for the absence of an N-particle formula even for tree amplitudes is that the original calculations were performed using a light-cone gauge operator formalism. Such methods are efficient for the few-particle case, but even in going from four to five particles the labor involved increases enormously.[6] Moreover, the superstring light-cone gauge vertices are strictly valid only for diagrams with 10 or fewer external lines.

In our view, it is more expedient to obtain superstring amplitudes directly from the covariant functional integral,[7] following the example of Polyakov for the bosonic string.[8] In Section 2 we show how functional integral methods may be used in the case of vector emission in open superstrings. We employ a superspace approach throughout, as introduced by Fairlie and Martin [9] (in the context of an "analogue model") for calculating multitachyon emission in the Neveu-Schwarz string. We extend our method to closed strings in Section 3 and to heterotic strings in Section 4. In Section 5 we discuss the cancellation of the infinities[10] of the two tree diagrams with external closed strings (disk and projective plane diagrams) of the Type I superstring: This cancellation turns out to require a relation between two couplings in the theory. Finally, in an Appendix we briefly discuss methods for obtaining the Green's function needed for the amplitude calculations.

II. Multiparticle Amplitudes for Type I Open Superstrings

The functional integral approach for bosonic string amplitudes has been known since the early studies of dual resonance models. With the advent of Polyakov's work,[8] however, several subtle points became more transparent.[11] We begin by summarizing these briefly, taking the example of the open-string multitachyon tree amplitude. We shall see that these points are carried over to the superstring case.

In the open-string case, the world sheet is a two-dimensional manifold M (with coordinates ξ_1, ξ_2) having a boundary; we shall take M to be the unit disk. The amplitude for scattering of N tachyons is

$$A_N^{tach}(k_1, \cdots, k_N) = \left\langle \prod_{I=1}^{N} V^{tach}(k_I) \right\rangle \tag{2.1}$$

where the vertex operator for emission of a tachyon with momentum k_I,

$$V^{tach}(l_I) = \oint_{\partial M} ds_I e^{ik_I \cdot X_I,(z_I, \bar{z}_I)} \tag{2.2}$$

is manifestly reparametrization invariant and has the right conformal dimension. Here the string coordinate X^μ is parametrized using complex world-sheet coordinates $z = \xi_1 + i\xi_2, \bar{z} = \xi_1 - i\xi_2$. We fix the gauge to the conformally flat metric $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^{\varphi(z,\bar{z})}, g_{zz} = g_{\bar{z}\bar{z}} = 0$, so that the line element is given by $ds^2 = g_{z\bar{z}}dzd\bar{z} + g_{\bar{z}z}d\bar{z} = e^{\varphi}d\vartheta^2, with z = e^{i\vartheta}$ on ∂M. The conformal factor φ drops out of the dynamics in 26 spacetime dimensions, the ghost modes decouple entirely in the tree amplitudes, and the averaging in (2.2) is simply with respect to a functional integral $\int [DX]e^{-S}$ over the disk, the gauge-fixed action being (after an integration by parts)

$$S = \frac{-1}{4\pi\alpha'} \int d^2 z X \cdot \triangle X, \tag{2.3}$$

where $\triangle$ is the two-dimensional laplacian, and $d^2z \equiv d\xi_1 d\xi_2$.

The gaussian integration is performed by the usual shift of variables, with an external source defined by $J^\mu(z, \bar{z}) = \sum_{I=1}^{N} k_I^\mu \delta^{(2)}(z - z_I)$; the integration of constant modes, however, requires a separate treatment, due to the invariance of the action under the shift $X \to X + const$. After the standard Faddeev-Popov prescription, it just yields a factor $(2\pi)^{(26)}\delta^{(26)}(\sum_{I=1}^{N} k_I)$ multiplying the standard result due to the integration of the other modes. We shall not make such factors explicit in our amplitude formulas, but we want to emphasize that the momentum conservation condition follows directly from the functional integration procedure, and is not a separate consideration.

The shift of variables is carried out using the Neumann's function N_{disk} satisfying $\triangle N_{disk} = \delta^{(2)}(z - z')$ inside M and $\partial N_{disk}(z - z')/\partial n|_{\partial M} = const$; it is given by

$$N_{disk}(z, z') = \frac{1}{2\pi} ln|z - z'|1 - \bar{z}z'| \,. \tag{2.4}$$

In the Appendix, we discuss methods for constructing Green's functions on the super-Riemann surfaces associated with superstring tree diagrams. The Neumann's function (2.4) can be understood as a truncation to commuting complex variables.

This expression may look unfamiliar, but one can see that, since the source satisfies the condition $\int J(z, \bar{z}) d^2 z = 0$, and since $\bar{z} = 1/z$ on ∂M, (2.4) can effectively be replaced by $ln|z - z'|/\pi$ in the amplitude

$$A_N^{tach} = \frac{(-i)^N}{(vol)} \int \prod_{I=1}^{N} \frac{dz_I}{z_I} e^{\varphi(z_I)/2} exp\left(\pi\alpha' \int d^2z d^2z' J(z) \cdot N_{disk}(z, z') J(z')\right) \;, \tag{2.5}$$

giving

$$A_N^{tach} = \frac{(-i)^N}{(vol)} \int \prod_{I'=1}^{N} \frac{dz'_I}{z'_I} e^{\varphi(z'_I)/2} exp\left(\alpha' \sum_{I,J}^{N} k_I \cdot k_J ln|z_I - z_J|\right) \,. \tag{2.6}$$

Notice the unrestricted sum over I, J, which, together with the short distance behavior $ln|z_I - z_J| \to -ln1/\varepsilon - \varphi(z_I)/2$ as $z_J \to z_I$, accounts for the cancellation of the conformal factor. One can also show that the singularity associated with the $ln1/\varepsilon$ factors is absent due to the momentum conservation condition.

The expression above is invariant under a three-parameter group of $SU(1,1)$ transformations. ((vol) means the group volume). By a gauge-fixing procedure we can make this invariance explicit; then, dividing out the group volume and using

$\bar{z}_I = 1/z_I$, we reach the well known formula

$$A_N^{tach} = (-i)^N \int (d^3F_{abc})^{-1} \prod_{I'=1}^{N} dz'_I \prod_{I<J}^{N} (z_I - z_j)^{2\alpha' k_I \cdot k_J}, \tag{2.7}$$

where $d^3F_{abc} = dz_a dz_b dz_c/[(z_a - z_b)(z_b - z_c)(z_c - z_a)]$, and z_a, z_b, and z_c are three arbitrarily chosen $z'_I s$. The integral includes $\frac{1}{2}(N-3)!$ different orderings of the $z'_I s$ around the boundary of the disk.

The discussion above can now be adapted to the case of Type I open superstring N-particle amplitudes, formulated in terms of covariant functional integrals of a two-dimensional real superfield.

$$X^\mu(z,\bar{z},\theta,\bar{\theta}) = X^\mu(z,\bar{z}) + \frac{1}{\sqrt{2}}\theta\psi^\mu(z,\bar{z}) + \frac{1}{\sqrt{2}}\bar{\psi}^\mu(z,\bar{z})\bar{\theta} + \frac{1}{2}\theta\bar{\theta}F^\mu(z,\bar{z}) \ , \tag{2.8}$$

where ψ is a fermion field and F is a real auxiliary field. The gauge- fixed action is written as

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2z \left[(\partial_\alpha X)^2 - i\bar{\Psi}\sigma\cdot\partial\Psi\right] \\ &= \frac{1}{\pi\alpha'}\int d^2z d\theta d\bar{\theta} \bar{D}X \cdot DX \ , \end{aligned} \tag{2.9}$$

where $\Psi = (\psi, \bar{\psi})^T$ is a two-dimensional Majorana spinor, and the covariant derivatives are $D = -\partial_\theta + i\theta\partial_z$ and $\bar{D} = \partial_{\bar{\theta}} - i\bar{\theta}\partial_{\bar{z}}$. Also, $\psi = \pm i\bar{z}\bar{\psi}$ at the boundary. Note that $D^2 = -i\partial_z, \bar{D}^2 = -i\partial_{\bar{z}}$, and $\{D, \bar{D}\} = 0$. The above definitions of $D, \bar{D}$ are consistent with the supersymmetry transformations $\delta z = i\epsilon\theta$, $\delta\theta = \epsilon, \delta\bar{z} = i\bar{\epsilon}\bar{\theta}, \delta\bar{\theta} = \bar{\epsilon}$.

After an integration by parts, (2.9) becomes an integral over the world sheet involving the super-Laplacian $4D\bar{D}$. Accordingly, we shall need the Green's functions of this operator in order to shift variables in the gaussian integrals. To be precise, the integrations in (2.9) are over a two-dimensional supermanifold having a boundary (a "superdisk"), with the coordinates $(z, \bar{z}, \theta, \bar{\theta})$. which corresponds to the ordinary disk upon truncation to commuting coordinates $z, \bar{z}$. We are, of course, interested in superstring amplitudes (rather than those of the Neveu-Schwarz or Ramond strings); for tree diagrams the truncation to even G-parity is achieved simply by requiring that the vertex operators discussed below be G-parity even.

The vertex operator[12] for emission of a vector particle having momentum k_I and polarization ζ_I is

$$V^{vect}(k_I,\zeta_I) = \int_{\partial M} dz_I \int d\theta_I \cdot D_I X e^{ik_I \cdot X(z_I,\bar{z}_I\theta_I,\bar{\theta}_I)}, \tag{2.10}$$

where we have made the trivial change from ds_I to dz_I. In contrast to the case of the tachyon vertex discussed above, there is no conformal factor arising from the functional average (since $k_I^2 = 0$), so in (2.10) we just require invariance under analytic reparametrization. It is clear that (2.10) is even under G-parity ($\theta \to -\theta, \psi \to -\psi$).

It is easy to see that the N-particle amplitudes

$$A_N^{vect}(k_1,\ldots,k_N,\zeta_1,\ldots,\zeta_N) = \left\langle \prod_{I=1}^{N} V^{vect}(k_I,\zeta_I) \right\rangle \tag{2.11}$$

can be written

$$A_N^{vect} = \prod_{I=1}^{N} \int_{\partial M} dz_I d\theta_I \zeta_I \cdot \int d\eta_I \left\langle exp\left(i \sum_{J=1}^{N} (k_J - i\eta_J D_J) \cdot X \right) \right\rangle \tag{2.12}$$

by introducing an additional grassmann parameter $\eta_{\mu I}$

Now we perform the superspace path integration, using a source

$$J^\mu(z,\bar{z},\theta,\bar{\theta}) = \sum_{I=1}^{N} (k_I^\mu - i\eta_I^\mu D_I)\delta^{(2)}(z - z_I)(\theta - \theta_I)(\bar{\theta} - \bar{\theta}_I) \tag{2.13}$$

All we need for the functional integration is the appropriate Neumann's function on the boundary of the disk, as calculated in the Appendix,

$$N_{disk}|_{\partial M} = \frac{1}{2\pi} ln|z - z' + i\theta\theta'| \tag{2.14}$$

Now we see that the N-particle amplitude is given by

$$\begin{aligned} A_N^{vect} &= \frac{1}{(vol)} \prod_{I=1}^{N} \int_{\partial M} dz_I d\theta_I \zeta_I \cdot \int d\eta_I \\ &\times \; exp\left(\pi\alpha' \int d^2z d\theta d\bar{\theta}\; d^2z' d\theta' d\bar{\theta}' J \cdot N_{disk}|_{\partial M} J' \right) . \end{aligned} \tag{2.15}$$

Using the fact that on ∂M we have $\bar{z}_I = 1/z_I, \bar{\theta}_I = -i\theta_I/z_I$(cf. the discussion in the Appendix), we obtain

$$\begin{aligned} A_N^{vect} &= \frac{1}{(vol)} \prod_{I'=1}^{N} \int_{\partial M} dz'_I d\theta'_I \varsigma'_I \cdot \int d\eta'_I \\ &\times \; exp\left[\alpha' \sum_{I,J} (k_I - i\eta_I D_I) \cdot (k_J - i\eta_J D_J) ln|z_I - z_J + i\theta_I \theta_J|\right] \end{aligned} \tag{2.16}$$

This formula is reminiscent of the form (2.6) for the multitachyon amplitudes of the bosonic string and its generalizations to fermionic strings.[9] To put this expression into a usable form, we need to carry out the differentiations inside the exponent. Then, as in the bosonic case, we employ a gauge fixing of the $SU(1,1)$ invariance to recast (2.16) as

$$\begin{aligned} A_N^{vect} &= \int (d^3 F_{abc})^{-1} \prod_{I'=1}^{N} dz_{I'} d\theta_{I'} \varsigma_{I'} \cdot \int d\eta_{I'} \prod_{I<J}^{N} (z_I - z_J)^{2\alpha' k_I \cdot k_J} \\ &\times \; exp\left[i\alpha' \sum_{I \neq J}^{N} k_I \cdot k_J \frac{\theta_I \theta_J}{z_I - z_J}\right] exp\left[\alpha' \sum_{I \neq J}^{N} (k_I \cdot \eta_J + k_J \cdot \eta_I) \frac{(\theta_I - \theta_J)}{z_I - z_J}\right] \\ &\times \; exp\left[-i\alpha' \sum_{I \neq J}^{N} \frac{\eta_I \cdot \eta_J}{z_I - z_J}\right] exp\left[-\alpha' \sum_{I \neq J}^{N} \eta_I \cdot \eta_J \frac{\theta_I \theta_J}{(z_I - z_J)^2}\right] . \end{aligned} \tag{2.17}$$

To obtain the final form of a given amplitude, one still has to expand the exponentials in the integrand of (2.17), perform the grassmann integrations, and express $(N-3) z_I$'s in terms of Möbius invariants; these are typically cross ratios

$$x_I = \frac{(z_I - z_a)(z_b - z_c)}{(z_I - z_b)(z_a - z_c)} , \tag{2.18}$$

where $I \neq a, b, c$. We shall conclude this section by demonstrating how to carry out this procedure for the three- and four-particle amplitudes.

From the rules for grassmann integration ($\int d\theta = 0, \int d\theta\theta = 1$), we see that we need to pick out terms in the expansion of (2.17) that are linear in all the θ_I's and all the η_I's. Let us label the arguments of the successive exponentials in (2.17) using the shorthand notation [0.2], [1.1], [2.0], [2.2], where the first number in brackets indicates the number of η_I's and the second the number of θ_I's. In the three-particle case we have $k_I \cdot k_J = 0$, so the [0.2] terms can be ignored, and it turns out that the only contribution to the amplitude in this case comes from terms of the form [1.1][2.2], giving

$$-(2\alpha')^2\left(-\frac{k_2\cdot\eta_3\theta_3}{z_2-z_3}+\frac{k_1\cdot\eta_3\theta_3}{z_3-z_1}\right)\eta_1\cdot\eta_2\frac{\theta_1\theta_2}{(z_1-z_2)^2}+cyclic\quad perm\,. \tag{2.19}$$

The result is

$$A_3^{vect}=-(2\alpha')^2[(\varsigma_1\cdot\varsigma_2)(\varsigma_3\cdot k_1)+(\varsigma_2\cdot\varsigma_3)(\varsigma_1\cdot k_2)+(\varsigma_3\cdot\varsigma_1)(\varsigma_2\cdot k_3)]\ . \tag{2.20}$$

Terms of the form$[1.1]^3$ are absent due to the anticommuting property of the grassmann variables; such terms get multiplied by a factor $(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)=0$. As a result, there are no terms in the amplitude having the tensor structure $(\varsigma\cdot k)(\varsigma\cdot k)(\varsigma\cdot k)$(but which are known to be present in the corresponding bosonic-string amplitudes).

In the four-particle case one gets terms in the amplitude having tensor structure $(\varsigma\cdot\varsigma)(\varsigma\cdot\varsigma)$ from terms of the form $[2.2]^2,[2.0][0.2][2.2]$, and $[2.0]^2[0.2]^2$, as well as terms with tensor structure $(\varsigma\cdot\varsigma)(\varsigma\cdot k)(\varsigma\cdot k)$ from terms $[2.0][0.2][1.1]^2,[2.2][1.1]^2$, and $[1.1]^4$. The final expression is

$$\begin{aligned} A_4^{vect} &= (2\alpha')^2\left[\frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)}+(st\to tu)+(st\to us)\right] \\ &\times\ K(\varsigma_1,k_1;\varsigma_2,k_2;\varsigma_3,k_3;\varsigma_4,k_4) \end{aligned} \tag{2.21}$$

where the kinematic factor K is given in Ref. 1.

Final comments are in order. In arriving (2.17), we used $SU(1.1)$ invariance to eliminate three commuting variables. Actually, (2.16) is invariant under the graded extension of $SU(1,1)$ group and it is possible to use its odd elements to eliminate two anti-commuting variables say θ_1 and θ_2. The final answer does not change except for a numerical coefficient. We will see in section 5 that the gauge fixing of odd elements plays an important role when the amplitude is divergent.

III. Multiparticle Amplitudes for Closed Superstrings

It is straightforward to generalize the formulas obtained for Type I open superstrings to the case of the sphere diagram with external (Type I or Type II) closed superstrings. The action takes the same form as (2.9) except that the integration is now over the entire complex plane. The vertex operator for massless bosonic closed-string states (for brevity we call it simply V^{grav}) is given by

$$V^{grav}(k_I, \varsigma_I^{\mu\nu}) = \int d^2 z_I \int d\theta_I d\bar{\theta}_I \varsigma_I^{\mu\nu} D_I X_\mu \bar{D}_I X_\nu e^{ik_I \cdot X}, \tag{3.1}$$

where the polarization tensor $\varsigma_I^{\mu\nu}$ is traceless-symmetric for the graviton, antisymmetric for the antisymmetric tensor field, and equal to the metric tensor for the dilaton.

We can proceed to the formula for the N-particle amplitudes by analogy with the open string case; here the trick is to introduce two grassmann parameters $\eta_{\mu I}$ and $\bar{\eta}_{\mu I}$

$$\begin{aligned} A_N^{grav} \quad & (k_1, \ldots, k_N, \varsigma_1^{\mu\nu}, \ldots, \varsigma_N^{\mu\nu}) = \left\langle \prod_{I=1}^{N} V^{grav}(k_I, \varsigma_I^{\mu\nu}) \right\rangle \\ = \quad & \prod_{I=1}^{N} \int d^2 z_I d\theta_I d\bar{\theta}_I \int d\eta_{\mu I} d\bar{\eta}_{\nu I} \left\langle exp\left(\sum_{J=1}^{N} (k_J - i\eta_J D_J - i\bar{\eta}_J \bar{D}_J) \cdot X \right) \right\rangle \end{aligned} \tag{3.2}$$

The source field is now

$$J(z, \bar{z}, \theta, \bar{\theta}) = \sum_{I=1}^{N} (k_I - i\eta_I D_I - i\bar{\eta}_I \bar{D}_I) \delta^{(2)}(z - z_I)(\theta - \theta_I)(\bar{\theta} - \bar{\theta}_I) \tag{3.3}$$

The Green's function on the sphere (cf.Appendix) is:

$$G_{sphere} = \frac{1}{2\pi} ln|z - z' + i\theta\theta'| \ . \tag{3.4}$$

We readily obtain the formula for N-particle closed-string amplitudes:

$$\begin{aligned} A_N^{grav} \quad = \quad & \frac{1}{(vol)} \prod_{I'=1}^{N} \int d^2 z_{I'} \int d\theta_{I'} d\bar{\theta}_{I'} \varsigma_{I'}^{\mu\nu} \int d\eta_{\mu I'} d\bar{\eta}_{\nu I'} \\ \times \quad & exp\left[\frac{\alpha'}{2} \sum_{I,J}^{N} (k_I - i\eta_I D_I - i\bar{\eta}_I \bar{D}_I) \cdot (k_J - i\eta_J D_J - i\bar{\eta}_J \bar{D}_J) \right. \\ \times \quad & \left. ln|z_I - z_J + i\,\theta_I \theta_J| \right] \ . \end{aligned} \tag{3.5}$$

This expression factorizes into analytic and antianalytic ($i.e.$, $z-$ and $\bar{z}-$ dependent) parts, as a result of which the remainder of the calculation mainly goes through as in the open string case. The sphere amplitudes are invariant under a six-parameter group of Möbius transformations, leading to a factor $(d^6 F_{abc})^{-1}$, where $d^6 F_{abc} = d^2 z_a d^2 z_b d^2 z_c / [|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2]$, and eventually eliminating the group volume denoted by (vol).

Equation (3.5) correctly reproduces the known three-and four-particle amplitudes:

$$A_3^{grav} = \varsigma_1^{\mu_1\nu_1}\varsigma_2^{\mu_2\nu_2}\varsigma_3^{\mu_3\nu_3} t_{\mu_1\mu_2\mu_3}(k_1/2, k_2/2, k_3/2) t_{\nu_1\nu_2\nu_3}(k_1/2, k_2/2, k_3/2)$$

$$A_4^{grav} = \varsigma_1^{\mu_1\nu_1}\varsigma_2^{\mu_2\nu_2}\varsigma_3^{\mu_3\nu_3}\varsigma_4^{\mu_4\nu_4} \pi K_{\mu_1\mu_2\mu_3\mu_4} K_{\nu_1\nu_2\nu_3\nu_4}$$
$$\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)} \tag{3.6}$$

IV. Multiparticle Amplitudes for Heterotic Strings

A few different formulations of the heterotic string are now known. For the purpose of deriving N-particle amplitudes, we adopt the fermionic formulation in which the rank-16 local gauge symmetry is realized as a current algebra defined on the world sheet.

The action is

$$\begin{aligned} S_{het} &= \frac{1}{4\pi\alpha'}\int d^2 z[(\partial_\alpha X)^2 - i\bar{\Psi}\sigma_- \cdot \partial_- \Psi - i\bar{\Psi}_a \sigma_+ \cdot \partial_+ \Psi_\alpha] \\ &= \frac{1}{\pi\alpha'}\int d^2 z d\theta d\bar{\theta}[-i\bar{\theta}\partial_{\bar{z}} X \cdot DX] - \frac{i}{2\pi\alpha'}\int d^2 z \bar{\psi}_a \partial_z \bar{\psi}_a\,, \end{aligned} \tag{4.1}$$

where we have introduced a fermion field $\bar{\psi}_a$, transforming as the vector representation of the gauge group; the antianalytic part of the supercovariant derivative has been suitably truncated. The first term can further be written as

$$-\frac{i}{\pi\alpha'}\int d^2 z d\theta \partial_{\bar{z}} X(z,\bar{z},\theta,\bar{\theta}=0) DX(z,\bar{z},\theta,\bar{\theta}=0). \tag{4.2}$$

The vertex operator for gravitons (as well as antisymmetric tensors and dilatons) is given by

$$V_{het}^{grav}(k_I, \varsigma_I^{\mu\nu}) = \int d^2 z_I d\theta_I \varsigma_I^{\mu\nu} D_I X_\mu i\partial_{\bar{z}} X_\nu e^{ik_I \cdot X(z_I,\bar{z}_I,\theta_I,\bar{\theta}_I)} \,. \tag{4.3}$$

The Green's function associated with the operator $-4iD\partial_{\bar{z}}$ is

$$G_{het} = \frac{1}{4\pi} ln(z - z' + i\theta\theta') + \frac{1}{4\pi} ln(\bar{z} - \bar{z}') \tag{4.4}$$

and we immediately arrive at the heterotic counterpart of (3.5)

$$\begin{aligned} A_N^{grav} &= \frac{1}{(vol)} \int \prod_{I=1}^{N} d^2 z_I d\theta_I \varsigma_I^{\mu\nu} \int d\eta_{\mu I} \frac{\partial}{\partial \bar{\eta}_{\nu I}} exp \left[\frac{\alpha'}{4} \sum_{I,J}^{N} (k_I - i\eta_I D_I + \bar{\eta}_I \partial_{\bar{z}_I}) \right. \\ &\times \left. \left. \cdot (k_J - i\eta_J D_J + \bar{\eta}_J \partial \bar{z}_J)[ln(z_I - z_J + i\theta_I\theta_J) + ln(\bar{z}_I - \bar{z}_J)] \right] \right|_{\bar{\eta}_I = 0} \end{aligned} \tag{4.5}$$

Unlike the closed-string amplitudes discussed in Section 3, the heterotic three-graviton amplitude (which has not previously been calculated) contains order-α' corrections to the local field-theory limit:

$$A_3^{grav} = \left(\frac{\alpha'}{2}\right)^4 \varsigma_1^{\mu_1\nu_1} \varsigma_2^{\mu_2\nu_2} \varsigma_3^{\mu_3\nu_3} t_{\mu_1\mu_2\mu_3} \left(t_{\nu_1\nu_2\nu_3} + \frac{\alpha'}{2} k_{2\nu_1} k_{3\nu_1} k_{1\nu_3} \right) . \tag{4.6}$$

In general, the N-graviton amplitudes of the heterotic string contain more tensor structures, associated with order-α' corrections, than the (Type I or Type II) closed-string amplitudes.

In the case of vector emission, the vertex operator invariant under analytic reparametrization and (1,0) supersymmetry and having the right conformal weight is

$$\begin{aligned} V_N^{vect}(k_I, \varsigma_I^m, T_I) &= \int d^2 z_I \int d\theta_I \varsigma_I^\mu D_I X_\mu \psi_a (T_I)_{ab} \psi_b \\ & e^{ik_I \cdot X(z_I, \bar{z}_I, \theta_I, \bar{\theta}_I)} \end{aligned} \tag{4.7}$$

In the present treatment, $(T_I)_{ab}$ must be a generator of $SO(16)\times S0(16)$ subgroup of $E_8 \times E_8$. The N particle amlitude is

$$\begin{aligned} V_N^{vect} &= \left\langle \prod_{I=1}^{N} V_{het}^{vect}(k_I, \varsigma_I^\mu, T_I) \right\rangle \\ &= \prod_{I=1}^{N} \int d^2 z_I d\theta_I \varsigma_I^\mu \int d\eta_{\mu I} (T_I)_{ab} \int d\bar{\chi}_{Ia} d\bar{\chi}_{Ib} \end{aligned}$$

$$\times \left\langle exp\left[\sum_{J=1}^{N}[i(k_J - i\eta_J D_J)\cdot X + \bar{\chi}_{Ja}\bar{\psi}_a]\right]\right\rangle \quad , \tag{4.8}$$

where the $\bar{\chi}_{Ia}$ are new Grassmann and the superfield is to be evaluated at $\bar{\theta}_J = 0$.

The functional average with respect to the $\bar{\psi}_a$ field is carried out using the fact that

$$\frac{1}{\pi}\frac{\partial}{\partial z}\left(\frac{1}{\bar{z}-\bar{z}'}\right) = \delta^{(2)}(z-z') \tag{4.9}$$

We obtain

$$\begin{aligned} A_N^{vect} = & \int \prod_{I=1}^{N} d^2 z_I d\theta_I \varsigma_I^{\mu} \int d\eta_{\mu I}(T_I)_{ab} \int d\bar{\chi}_{Ia} d\bar{\chi}_{Ib} \\ \times & \; exp\left[\frac{\alpha'}{4}\sum_{I,J}^{N}(k_I - i\eta_I D_I)\cdot (k_J - i\eta_J D_J)(ln(z_I - z_J\right. \\ + & \left. i\theta_I\theta_J) + ln(\bar{z}_I - \bar{z}_J)) - \frac{i\alpha'}{2}\sum_{I,J}^{N}\frac{\bar{\chi}_I\cdot\bar{\chi}_J}{\bar{z}_I - \bar{z}_J}\right] \quad . \end{aligned} \tag{4.10}$$

Let us finally check the four-particle vector emission amplitude. (The three-particle case agrees with (2.20)). Expanding the integrand and performing various integrations of grassmann variables and cross ratios, we obtain the known result.[3]

$$\begin{aligned} A_4^{vect} = & -\pi\alpha'^6\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(\alpha' s/4)\Gamma(\alpha' t/4)\Gamma(\alpha' u/4)}K(\varsigma_1,k_1;\varsigma_2,k_2;\varsigma_3,k_3;\varsigma_4,k_4) \\ \times & \left[\frac{tr(T_1T_2)tr(T_3T_4)}{(\alpha' s/4)(1+\alpha' s/4)} + \frac{tr(T_1T_4)tr(T_2T_3)}{(\alpha' t/4)(1+\alpha' t/4)} + \frac{tr(T_1T_3)tr(T_2T_4)}{(\alpha' u/4)(1+\alpha' u/4)}\right. \\ & \left. + \frac{tr(T_1T_2T_3T_4)}{(\alpha' s/4)(\alpha' t/4)} + \frac{tr(T_1T_3T_4T_2)}{(\alpha' s/4)(\alpha' u/4)} + \frac{tr(T_1T_4T_2T_3)}{(\alpha' t/4)(\alpha' u/4)}\right] \end{aligned} \tag{4.11}$$

The reader who has tried the calculation using operator methods will appreciate the relative simplicity of our approach.

V. Disk and Projective Plane Amplitudes for Type I Closed Strings

Since Type I superstrings are nonoriented, the sum over surfaces in the functional integral approach must include non-orientable as well as orientable surfaces. Thus, at the one-loop level, there are annulus and Mõbius strip diagrams with external open string states attached at the boundaries. They are separately divergent but the infinities have been shown to cancel in the case of gauge group $SO(32)$.

As pointed out in the original infinity cancellation paper by Green and Schwarz,[10] even at the tree level, there are divergent Type I diagrams involving closed-string external states. (To be more precise, these diagrams are $0(\kappa^{N-1})$ in N graviton scattering. They lie between the sphere diagram $(0(\kappa^{N-2}))$ and the torus diagram $(0(\kappa^{N}))$. See discussions below). In one of these diagrams, the world sheet has the topology of a disk, with closed-string states attached to its interior. (This diagram may also be visualized as a sphere with a disk removed. (Figure 1.)) In the other, the world sheet is a non-orientable surface without boundary, called the projective plane (RP^2), which can be pictured as a disk with opposite points on the boundary identified (Figure 2.)) In appendix, we derive Neumann functions for upper Riemann surface analog to these diagrams. (Apparently, there is a sign ambiguity for each of these expressions. $(cf.(A.7),(A.8))$ We will see below that the sign ambiguity disappears in the expression for the amplitudes).

Green and Schwarz argued that these diagrams in the bosonic string theory are divergent and conjectured that the infinities would cancel in the superstring case at least for gauge group $SO(32)$. Here, we shall show how to perform such a calculation in the superstring case, and demonstrate that the infinity cancellation requires a relation between two couplings associated with two basic types of string[13] interactions, namely the coupling of a joining - splitting interaction - g and the coupling of an exchange interacton - κ. (Note that the dilaton vacuum expectation value dynamically generated will fix the dimensionless combination g^4/κ^3 only). We will see that the fixing of grassman variables using the odd elements of graded $SU(1,1)$ is crucial to the cancellation of infinities.

The essential ingredients to write down the amplitudes for these diagrams are in the previous section and appendix. Let us first derive the amplitudes for the disk. The N particle amplitudes of massless bosonic closed string states are written as (3.1) and (3.2) if we perform the functional averge over the disk.

The relevant Neumann function $(cf.(A.7))$ is

$$N_{disk} = \frac{1}{2\pi} ln|z - z' + i\theta\theta'||1 - \bar{z}z' \pm \bar{\theta}\theta'| \quad , \tag{5.1}$$

where the second factor is due the image charge and involves the sign ambiguity mentioned before. The formula for the N particle amplitudes then reads

$$
\begin{aligned}
A^{grav}_{N,disk} &= \frac{1}{(vol)} \prod_{I'=1}^{N} \int d^2 z'_I \int d\theta_{I'} d\bar{\theta}_{I'} \varsigma^{\mu\nu}_{I'} d\eta_{\mu I'} d\bar{\eta}_{\nu I'} \\
& exp \left[\frac{\alpha'}{2} \sum_{I,J}^{N} (k_I - i\eta_I D_I - i\bar{\eta}_I \bar{D}_I) \cdot (k_J - i\eta_J D_J - i\bar{\eta}_J \bar{D}_J) \right. \\
& \left. ln|z_I - z_J + i\theta_I\theta_J||1 - \bar{z}_I z_J \pm \bar{\theta}_I \theta_J| \right] . \qquad (5.2)
\end{aligned}
$$

It is straightforward to perform the differentiations in the exponent.

While the integration involves both the $z'_I s$ and $\bar{z}'_I s$ as well as the $\theta'_I s$ and $\bar{\theta}'_I s$ in case of the closed string external states, the invariance of the expression is dictated by the fact that the world sheet topology is a disk. In fact, a careful examination tells us that (5.2) is invariant under

$$
\begin{aligned}
z_I &\rightarrow \frac{az_I + b + i\theta_I \alpha}{cz_I + d + i\theta_I \beta} \\
\theta_I &\rightarrow -\alpha + \beta \frac{az_I + b}{cz_I + d} + \theta_I \frac{1 + \frac{i}{2}\alpha\beta}{cz_I + d} , \qquad (5.3)
\end{aligned}
$$

where a, b, c and d are complex numbers satisfying $|a|^2 - |c|^2 = 1$ and $ad - bc = 1$, and $\alpha = \pm i\bar{\beta}$ is a grassman number. (the sign conforms to (5.1).)

A set of all matrices$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$forms the group $SU(1,1)$ as mentioned in section two, and the entire transformations (5.3) forms its graded extension. The (vol) therefore denotes the volume of graded $SU(1,1)$. Let us, for a while, restrict our attention to the transformations generated by the even elements. A salient feature of the expression (5.2) is that it does not possess full $SL(2,C)$ invariance. In consequence, one cannot eliminate three complex variables by fixing the gauge of the $SL(2,C)$ transformations as we did for the type II superstring. We shall see shortly that this is precisely the origin of the infinities.

Let us make the following change of variables which corresponds to an element of $SU(1,1)$ and therefore eliminates three real parameters in (5.2):

$$z_1 = \bar{x} ,$$

$$
\begin{aligned}
z_2 &= \frac{r+\bar{x}y}{xr+y}\ , \\
z_I &= \frac{z'_I+\bar{x}y}{xz'_I+y} \quad (3\le I\le N) \quad \text{and} \\
\theta_I &= \frac{\theta'_I}{Cz'_I+\bar{a}}
\end{aligned}
\tag{5.4}
$$

where $y=\bar{a}/a, x=c/a, r=real$ and we fixed z_1 and the phase of z_2 to be zero, Rescaling z'_I's by $r=\sqrt{\lambda}$ and $z'_I=\sqrt{\lambda}w_I(3\le I\le N)$, we reach

$$
\begin{aligned}
A^{grav.}_{N,disk} = \; & \frac{1}{(vol)'}\int_0^1 d\lambda\lambda^{N-2}\int_{\lambda|w_{I'}|^2\le 1}\prod_{I'=3}^{N} d^2w_{I'}\prod_{J'=1}^{N}\varsigma^{\mu\nu}_{J'}\int d\theta_{J'} \\
& d\bar{\theta}_{J'}d\eta_{\mu J'}d\bar{\eta}_{\nu J'}\prod_{I<J}|w_I-w_J|^{\alpha' k_I\cdot k_J}\prod_{I<J}|1-\lambda w_I\bar{w}_J|^{\alpha' k_I\cdot k_J} \\
exp\; & \left(i\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}^{N}k_I\cdot k_J\frac{\theta_I\theta_J}{(w_I-w_J)}+\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}^{N}(k_I\cdot\eta_J+k_J\cdot\eta_I)\frac{(\theta_I-\theta_J)}{(w_I-w_J)}\right. \\
& \left.-\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}^{N}\frac{\eta_I\cdot\eta_J}{(w_I-w_J)}-\frac{\alpha'}{4\lambda}\sum_{I,J}^{N}\frac{\eta_I\cdot\eta_J\theta_I\theta_J}{(w_I-w_J)^2}\right) \\
exp\; & \left(i\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}k_I\cdot k_J\frac{\bar{\theta}_I\bar{\theta}_J}{(\bar{w}_I-\bar{w}_J)}+\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}^{N}(k_I\cdot\bar{\eta}_J+k_J\cdot\bar{\eta}_I)\frac{(\bar{\theta}_I-\bar{\theta}_J)}{(\bar{w}_I-\bar{w}_J)}\right. \\
& \left.-\frac{\alpha'}{4\sqrt{\lambda}}\sum_{I,J}^{N}\frac{\bar{\eta}_I\cdot\bar{\eta}_J}{(\bar{w}_I-\bar{w}_J)}-\frac{\alpha'}{4\lambda}\sum_{I,J}^{N}\frac{\bar{\eta}_I\cdot\bar{\eta}_J\bar{\theta}_I\bar{\theta}_J}{(\bar{w}_I-\bar{w}_J)^2}\right) \\
exp\; & \left(-\frac{\alpha'\sqrt{\lambda}}{2}\sum_{I,J}^{N}\frac{\eta_I\cdot k_J\theta_I\bar{w}_J}{1-\lambda w_I\bar{w}_J}+\frac{\alpha'\sqrt{\lambda}}{2}\sum_{I,J}^{N}\frac{\bar{\eta}_J\cdot k_I\bar{\theta}_J w_I}{1-\lambda w_I\bar{w}_J}\right. \\
& \left.-\frac{\alpha'}{2}\sum_{I,J}\frac{\eta_I\cdot\bar{\eta}_J\theta_I\bar{\theta}_J}{(1-\lambda w_I\bar{w}_J)^2}\right) \\
exp\; & \left(\mp\frac{\alpha'}{2}\sum_{I,J}^{N}\frac{k_I\cdot k_J\theta_I\bar{\theta}_J}{1-\lambda w_I\bar{w}_J}\mp\frac{\alpha'}{2}i\sum_{I,J}^{N}\frac{\eta_I\cdot k_J\bar{\theta}_J}{1-\lambda w_I\bar{w}_J}\right. \\
& \left.\mp\frac{\alpha'}{2}i\sum_{I,J}^{N}\frac{\bar{\eta}_J\cdot k_I\theta_I}{1-\lambda w_I\bar{w}_J}\mp\frac{\alpha'}{2}\sum_{I,J}^{N}\frac{\eta_I\cdot\bar{\eta}_J}{1-\lambda w_I\bar{w}_J}\right)
\end{aligned}
\tag{5.5}
$$

Here w_1 and w_2 are set to 0 and 1 respectively.

The arguments of the successive exponentials in (5.5) can be classified into two categories. The first class of terms, which contain $\frac{1}{\sqrt{\lambda(w_I - w_J)}}$ or $\frac{1}{\sqrt{\lambda(\bar{w}_I - \bar{w}_J)}}$, is the same as we saw in the sphere diagram of the type II superstring. We denote it by T_1. These terms are singular at $\lambda = 0$. The second class of terms, which contain $\frac{1}{\lambda - w_I \bar{w}_J}$, is, on the other hand, unique to the disk diagram. These terms originate from the second factor of the Neumann's function (5.1). We call it T_2. These terms are regular at $\lambda = 0$. Note that the sign ambiguity which exists in some terms in T_2 disappears since they have to appear even times to saturate the grassman variable integration.

It is now clear that the leading divergences of the expression (5.5) arise from products of terms all of which are in T_1. A simple power counting tells us that they diverge like $\frac{d\lambda}{\lambda^2} \sim \frac{1}{\lambda} as \lambda \to 0$. We also observe that the infinities arise from the region of integration where all the z_I's (the original variables) approach the origin. (Equivalently, one can imagine letting the radius of the disk go to infinity, keeping the z_I's fixed). These singularities cannot be interpreted as due to the propagation of physical particles. One could have eliminated them if there were $SL(2, C)$ invariance. The lack of $SL(2, C)$ invariance is thus the origin of the infinities.

So far, the qualitative discussions on the structure of infinities are more or less the same as what has been observed in closed string tachyon amplitudes in bosonic string.[10] One has $\int \frac{d\lambda}{\lambda^2}$ infinities and cannot hope to obtain a cancellation through for instance, principal value prescription. In the superstring amplitude, however, (5.5) may not be considered as a final expression. One can use the invariance due to the odd elements of the graded $SU(1,1)$ group to eliminate one complex θ integration. In the previous section, doing this did not create anything new. But here, it does improve the degrees of divergences. It offers more effective parametrization of the integrand.

The residual transformations under which (5.5) is invariant are

$$\begin{aligned} z'_I &\rightarrow z'_I(1 - i\theta'_I\beta) \mp \theta'_I\bar{\beta} \\ \theta'_I &\rightarrow \mp i\bar{\beta} + \beta z'_I + \theta'_I \end{aligned} \qquad (5.6)$$

Here the z'_I's are the variables before rescaling. We see that the following change of varibles after (5.4) eliminates θ'_1 and $\bar{\theta}'_1$ from the integrand.

$$\begin{aligned} z'_I &= \omega_I \pm \bar{\beta}\chi_I - i\omega_I\chi_I\beta \quad ,2 \le I \le N \\ &\theta'_1 = \mp i\bar{\beta}, \\ \theta'_I &= \chi_I(1+\frac{i}{2}(\pm)\beta\bar{\beta}) \mp i\bar{\beta} + \beta\omega_I \ ,2 \le I \le N \end{aligned} \tag{5.7}$$

Note that the above transformations mix analytic variables (z'_I, θ'_I) with antianalytic variables $(\bar{z}_I, \bar{\theta}_I)$.Rescaling ω_I's by $\sqrt{\lambda}$, we obtain a final formula:

$$\begin{aligned} A^{grav}_{N,disk} &= (N-1)\int_0^1 d\lambda\lambda^{N-2}\int_{\lambda|\omega_{I'}|^2\le 1}\prod_{I'=3}^{N} d^2\omega_{I'}\int\prod_{J'=2}^{N} d\theta_{J'}d\bar{\theta}_{J'}\prod_{K=1}^{N} \\ &\varsigma_K^{\mu\nu} d\eta_{\mu K} d\bar{\eta}_{\nu K}\prod_{I<J}|\omega_I-\omega_J|^{\alpha' k_I\cdot k_J}\prod_{I<J}|1-\lambda\omega_I\bar{\omega}_J|^{\alpha' k_I\cdot k_J} \\ &exp\left(I[\omega_I,\bar{\omega}_I,\theta_I,\bar{\theta}_I,\eta_I,\bar{\eta}_I,\lambda]\right) \quad , \end{aligned} \tag{5.8}$$

where $I[\omega_I,\cdots]$ is the same factor as seen in (5.5) except that θ_1 is set to zero in addition to $\omega_1 = 0, \omega_2 = 1$. Note also that $\lambda's$ in (5.8) and (5.5) are not the same object.

In this expression, it is no longer possible to find, in the integrand, a product of terms all of which are in T_1 when we perform θ_I and $\bar{\theta}_I(2 \le I \le N)$ and η_I and $\bar{\eta}_I(1 \le I \le N)$ integrations. After picking out terms in T_1 as many as possible, one is left with one analytic variable (either θ_I or η_I) and one antianalytic variable (either$\bar{\theta}_I$ or $\bar{\eta}_I$). To saturate this integration, one has to pick up one term from the last exponential in (5.5). The leading divergences, therefore, go like $\int d\lambda\lambda^{N-2}\lambda^{-(N-1)} = \int d\lambda/\lambda$ and these are the only infinities of (5.8) terms (5.9) Separating these terms from the rest of the terms, we can write (5.8) as

$$A^{grav}_{N,disk} = \int_0^1 \frac{d\lambda}{\lambda}F(\lambda; k_I, \varsigma_I) + \text{finite terms} \tag{5.9}$$

with F(x) some function regular and nonvanishing at $\lambda = 0$.

The analysis made above for the disk diagram goes through for the projective plane by a mere change of a sign in the Neumann's function $(cf.(A.8))$

$$N_{RP^2} = \frac{1}{2\pi}ln|z-z'+i\theta\theta'||1+\bar{z}z'\pm\bar{\theta}\theta'| \tag{5.10}$$

We obtain the formula for the N particle amplitudes by replacing $1 - \bar{z}_I z_J$ in (5.2) by $1 + \bar{z}_I z_J$. The invariance group for the projective plane is the group $SU(2)$. Making a change of variables similar to (5.4), but this time corresponding to an element of $SU(2)$, we obtain

$$A^{grav}_{N,RP^2} = \frac{1}{(vol)'} \int_0^1 d\lambda \lambda^{N-2} \int_{\lambda|w_{I'}|^2 \leq 1} \prod_{I'=3}^{N} d^2\omega_{I'} \prod_{J'=1}^{N}$$

$$\varsigma^{\mu\nu}_{J'} \int d\theta_{J'} d\bar{\theta}_{J'} d\eta_{\mu J'} d\bar{\eta}_{\nu J'} \prod_{I<J} |\omega_I - \omega_J|^{\alpha' k_I \cdot k_J} \prod_{I<J} |1 + \lambda \omega_I \bar{\omega}_J|^{\alpha' k_I \cdot k_J}$$

$$exp\left(i\frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} k_I \cdot k_J \frac{\theta_I \theta_J}{(\omega_I - \omega_J)} + \frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} (k_I \cdot \eta_J + k_J \cdot \eta_I) \frac{(\theta_I - \theta_J)}{(\omega_I - \omega_J)} \right.$$

$$\left. - \frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} \frac{\eta_I \cdot \eta_J}{(\omega_I - \omega_J)} - \frac{\alpha'}{4\lambda} \sum_{I,J}^{N} \frac{\eta_I \cdot \eta_J \theta_I \theta_J}{(\omega_I - \omega_J)^2} \right)$$

$$exp\left(i\frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} k_I \cdot k_J \frac{\bar{\theta}_I \bar{\theta}_J}{(\bar{\omega}_I - \bar{\omega}_J)} + \frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} (k_I \cdot \bar{\eta}_J + k_J \cdot \bar{\eta}_I) \right.$$

$$\left. \frac{(\bar{\theta}_I - \bar{\theta}_J)}{\bar{\omega}_I - \bar{\omega}_J} - \frac{\alpha'}{4\sqrt{\lambda}} \sum_{I,J}^{N} \frac{\bar{\eta}_I \cdot \bar{\eta}_J}{\bar{\omega}_I - \bar{\omega}_J} - \frac{\alpha'}{4\lambda} \sum_{I,J}^{N} \frac{\bar{\eta}_I \cdot \bar{\eta}_J \bar{\theta}_I \bar{\theta}_J}{(\bar{\omega}_I - \bar{\omega}_J)^2} \right)$$

$$exp\left(+ \frac{\alpha' \sqrt{\lambda}}{2} \sum_{I,J}^{N} \frac{\eta_I \cdot k_J \theta_I \bar{\omega}_J}{1 + \lambda \omega_I \bar{\omega}_J} - \frac{\alpha' \sqrt{\lambda}}{2} \sum_{I,J} \right.$$

$$\left. \frac{\bar{\eta}_J \cdot k_I \bar{\theta}_J \omega_I}{1 + \lambda \omega_I \bar{\omega}_J} + \frac{\alpha'}{2} \sum_{I,J}^{N} \frac{\eta_I \cdot \bar{\eta}_J \theta_I \bar{\theta}_J}{(1 + \lambda \omega_I \bar{\omega}_J)^2} \right)$$

$$exp\left(\mp \frac{\alpha'}{2} \sum_{I,J}^{N} \frac{k_I \cdot k_J \theta_I \bar{\theta}_J}{1 + \lambda \bar{\omega}_I \omega_J} \mp \frac{\alpha'}{2} i \sum_{I,J}^{N} \frac{\eta_I \cdot k_J \bar{\theta}_J}{1 + \lambda \omega_I \bar{\omega}_J} \right.$$

$$\left. \mp \frac{\alpha'}{2} i \sum_{I,J}^{N} \frac{\bar{\eta}_J \cdot k_I \theta_I}{1 + \lambda \omega_I \bar{\omega}_J} \mp \frac{\alpha'}{2} \sum_{I,J}^{N} \frac{\eta_I \cdot \bar{\eta}_J}{1 + \lambda \omega_I \bar{\omega}_J} \right) \tag{5.11}$$

with $\omega_1 = 0$ and $\omega_2 = 1$. The sign ambiguities again disappear.

Going through the same argument which led to (5.9) in the disk case, we reach

$$A^{grav}_{N,RP^2} = c \int_0^1 \frac{d\lambda}{\lambda} F(-\lambda; k_I, \varsigma_I) + \text{finite terms} . \tag{5.12}$$

The constant c is a phase factor which we cannot determine for sure in the present framework. We, however, see that, with the choice $c = -1, A^{grav}_{N disk} + A^{grav}_{N,RP^2}$ is regulated by a principal value prescription.

Since the relative phase between $A^{grav}_{N,disk}$ and A^{grav}_{N,RP^2} in the final amplitude is not determined by our current consideration anyway, we here choose $c = -1$.

We have evaluated the N particle amplitudes for the super Riemann surface generalization of the disk and the projective plane. Each of them constitutes a consistent expession for the correlation function in the two dimensional local field theory defined on a given super Riemann surface. A principle of string theory, however, requires the sum over surfaces. In the absence of an entirely consistent formulation of the second quantized string theory, one has to appeal to various consistency requirements to fix the relative weights among diagrams.

In particular, we have not yet incorporated in the amplitudes the coupling constants g, κ mentioned at the beginning of this section or the group theory factor. Let us imagine a spacetime process which corresponds to the disk diagram with N external closed string states. (See Figure 3). After a series of closed string exchange interactions, a transition has to occur from a closed string state to an open string state, subsequently followed by a transition to another closed string state. The final closed string interactions then take place. For $SO(n)$ group, there must be a factor n due to n species of intermediate open string states, which is a conventional Chan Paton factor. *Eq.* (5.2) is therefore multiplied by $ng^2\kappa^{N-2}\alpha'^{-2N+1}$. (Incidentally, nonplaner diagrams with external open string states contain g only despite the fact that they contain a graviton pole).

As for RP^2, the corresponding spacetime process involves a closed string exchange interaction which is a "self-rearrangement" of a single closed string (see Figure 4) together with $(N-2)$ ordinary exchange interactions. One therefore has to multiply the RP^2 amplitudes by $(\kappa/\alpha'^2)^{N-1}$.

We now realize that the final expression with the coupling constants properly inserted is finite if and only if $\alpha'\kappa = ng^2$ and the relative sign is chosen as stated before. This, combined with the infinity cancellation of one-loop diagrams, yields $\alpha' k = 32g^2$ for the gauge group $SO(32)$.

Finally, we should mention that, working out overall normalization factor is sometimes clumsy in the functional integral formulation. On the other hand, *eq.* (5.8), for instance, can undoubtedly be checked by the conventional operator (oscillator) formalism. We hope to report on this in the near future.

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APPENDIX A:

In this Appendix we sketch the methods used to obtain the Green's and Neumann's functions (Green's functions of the second kind)of the super-Laplace's equation which are needed for the calculation of superstring tree diagrams. We begin by noting the form of the superdistance, $z - z' + i\theta\theta'$, and the generalization of Mōbius transformations:

$$z \rightarrow \frac{az+b}{cz+d}, \theta \rightarrow \frac{\pm\theta}{cz+d}, ad-bc=1 \tag{A.1}$$

We remark that there is a genuine sign ambiguity in the equation above. In what follows, we denote pairs of superspace variables by $(z,\theta),(w,\chi)$,etc. For simplicity of notation we shall label Green's functions using only the commuting variables.

Our basic idea is to use a superspace generalization of the method of images. Recall that our super-Laplacian is $4D\bar{D}$, with $D = -\partial_\theta + i\theta\partial_z, \bar{D} = \partial_{\bar{\theta}} - i\bar{\theta}\partial_{\bar{z}}$, and that the two-dimensional superspace delta function is $\delta^{(2)}(z-z')(\theta-\theta')(\bar{\theta}-\bar{\theta}')$. Let us introduce sources at the point (w,χ) and at the "conjugate point" $(\tilde{w},\tilde{\chi})$. Let $g(z,z')$ be the Green's function of the ordinary Laplace's equation in a given region. Then it is easy to check[14] that a solution of the super-Laplace's equation with delta-function sources at (w,χ) and $(\tilde{w},\tilde{\chi})$ is given by

$$\begin{aligned} G(z,w) &= \frac{1}{2\pi}D\bar{D}[g(z,w)(\theta-\chi)(\bar{\theta}-\bar{\chi})] \\ &= \frac{1}{2\pi}[ln|z-w+i\theta\chi| - ln|z-\tilde{w}+i\theta\tilde{\chi}|] \\ &\quad +\theta\bar{\theta}\chi\bar{\chi}\delta^{(2)}(z-w) \\ &\quad +\theta\bar{\theta}\tilde{\chi}\bar{\tilde{\chi}}\delta^{(2)}(z-\tilde{w}) \end{aligned} \tag{A.2}$$

It turns out that the delta-function terms in $(A.2)$ may be omitted for purposes of calculating amplitudes. To see this, recall that our only use of the Green's function G is as an inverse to the super-Laplacian (call it G^{-1}) for shifting variables in the Gaussian functional integral. At first sight, then, it looks as if one requires

$$\begin{aligned} G^{-1}G &= \delta^{(2)}(z-z')(\theta-\theta')(\bar{\theta}-\bar{\theta}') \\ &= (\theta\bar{\theta} - \theta\bar{\theta}' - \theta'\bar{\theta} + \theta'\bar{\theta}')\delta^{(2)}(z-z') \end{aligned} \tag{A.3}$$

In fact, the required expression is

$$\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' \tilde{X} G^{-1} G J = \int d\theta d\bar{\theta} d\theta d\bar{\theta}' \tilde{X} J \delta^{(2)}(z - z') \tag{A.4}$$

where and J is a superfield source. However, if one assumes that the auxiliary filed F has been eliminated (as is necessary for X to be a sum of analytic and antianalytic terms), then one can see that the $\theta'\bar{\theta}'$ term in (A.3) is not needed to satisfy the condition (A.4). In turn, this means that the delta-function terms in (A.2) can be omitted.

Thus, our fundamental solution of the two-dimensional super-Laplace's equation on the entire plane, having sources at two points (w, χ) and $(\tilde{w}, \tilde{\chi})$, is given by

$$\Omega_{w,\tilde{w}}(z) = \frac{1}{2\pi}[ln|z - w + i\theta\chi| - ln|z - \tilde{w} + i\theta\tilde{\chi}|] \ . \tag{A.5}$$

This is a superspace version of what is called the "third Abelian integral" on the sphere By choosing the coordinates $(\tilde{w}, \tilde{\chi})$ of the" image charge", one can obtain the Green's or Neumann's functions on various super-Riemann surfaces. As was discussed in the book mentioned above, the method of images can be viewed in terms of the "doubling" of the given surface. By this we mean the following.

Suppose we want a Green's function on a given surface M, which we regard at first as an abstract surface and later project onto the plane. Following the method of [15], we construct another surface F, called the "double" of M, by making two copies of M and glueing them together along their boundaries; the resulting super-Riemann surface is orientable. For example, we double the disk by glueing together two disks (or two hemispheres)-the resulting surface is a sphere. Each point on M has its corresponding "conjugate point" on the copy of M. The Green's and Neumann's functions are then computed as certain combinations of the fundamental solutions which are conformal invariants and have properties appropriate for a Green's function (symmetry under interchange of argument and parameter, vanishing of the Green's function on the boundary, etc.) On the sphere the fundamental solution $\Omega_{w,\tilde{w}}(z)$ is as given above; this is all we need for tree amplitudes. For one-loop calculations the double will be a torus, and the fundamental solution will be more complicated.

As an example, consider the case of the Neumann's function inside the unit disk.

Using the Schiffer-Spencer formula,

$$N(z,z',w) = \frac{1}{2}\left[\Omega_{z'w}(z) + \Omega_{\tilde{z}'\tilde{w}}(\tilde{z}) + \Omega_{\tilde{z}'\tilde{w}}(\tilde{z}) + \Omega_{\tilde{z}'\tilde{w}}(z)\right] \tag{A.6}$$

There is dependence on an artificial third parameter w because it is actually necessary to consider the difference $N(z,z') - N(z,w)$ of two ordinary Neumann's functions in order to get a conformal invariant; the dependence on w will drop out during the calculation of amplitudes. The double is the sphere. The coordinates with and without tildes can be regarded as the locations of a "charge" and its "image charge" respectively in the usual method of images. We identify the conjugate point as $\tilde{z} = 1/\bar{z}, \tilde{\theta} = \mp i\bar{\theta}/\bar{z}$.

Note that there are two possibilities for choosing the odd coordinate,related to the sign ambiguity in the super-Möbius transformation (A1).

In calculating N we choose an arbitrary constant so that $ImN = 0$; we discard terms which do not depend on both arguments (z,θ) and (z',θ'). Such terms are cancelled in the amplitudes due to a momentum conservation condition. Then, for the disk we have two possible Neumann's functions,

$$N_{disk}(z,z') = \frac{1}{2\pi} ln|z - z' + i\theta\theta'||1 - \bar{z}z' \pm \bar{\theta}\theta'|. \tag{A.7}$$

The sign corresponds to the two choices of the conjugate point.

On the boundary of the disk, both of these reduce to

$$N_{disk}|_{\partial M} = \frac{1}{\pi} ln|z - z' + i\theta\theta'|$$

In the case of the superspace generalization of the projective plane, using the picture of RP^2 as a disk with opposite points on the boundary identified, we find that the conjugate points are $\tilde{z} = -1/\bar{z}, \tilde{\theta} = \mp i\bar{\theta}/\bar{z}$(again, there two choices). Again using $(A.6)$, the Neumann's functions are

$$N_{RP^2} = \frac{1}{2\pi} ln|z - z' + i\theta\theta'||1 + \bar{z}z' \pm \bar{\theta}\theta'| . \tag{A.8}$$

We also need the Green's function on the sphere. In this case the double can be pictured as two overlapping spheres connected by a flux tube. We use the Schiffer-Spencer formula

$$G(z,z',w,w') = Re[\Omega_{z'w}(z) - \Omega_{z'w}(w')] . \tag{A.9}$$

Here the artificial points (w, χ) and (w', χ') can be thought of as the points of intersection of the flux tube with each of the spheres; naturally, they drop out of amplitude calculations. Upon substituting for $\Omega_{w,\bar{w}}(z)$ and dropping all terms which are irrelevant for the amplitudes, we get

$$G_{sphere}(z, z') = \frac{1}{2\pi} ln|z - z' + i\theta\theta'| \ . \qquad (A.10)$$

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Figure Captions

Fig. 1: A disk with closed string states attached.

Fig. 2: RP^2 with closed string states attached.

Fig. 3: A spacetime process which corresponds to the disk diagram with N external closed string states.

Fig. 4: A spacetime process wich corresponds to the RP^2 diagram with N external closed string states.

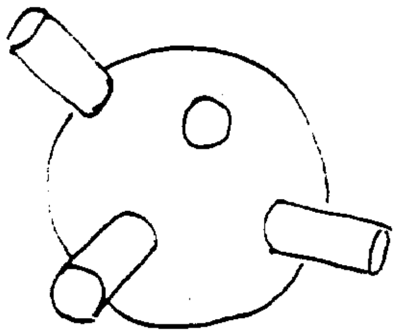

Figure 1

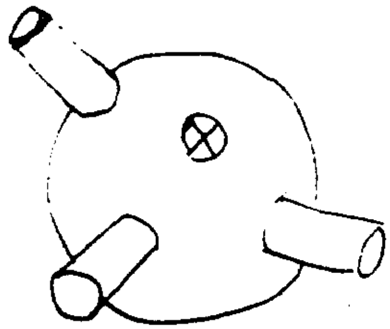

Figure 2

Figure 3

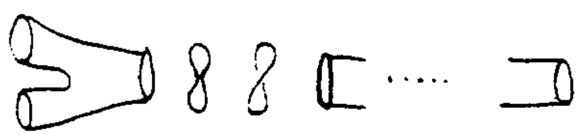

Figure 4