Fermi National Accelerator Laboratory

FERMILAB-Pub-86/60-A
April 1986

Geometrical Properties of an Internal Local Octonionic Space in Curved Space Time.

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Abstract

A geometrical treatment on a flat tangent space local to a generalized complex, quaternionic, and octonionic space-time is constructed. It is shown that it is possible to find an Einstein-Maxwell-Yang-Mills correspondence in this generalized (Minkowskian) tangent space.

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Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction.

We saw recently(1) that it is possible to interpret the General Relativity theory, the non-symmetric theory(2) and the unification of K. Borchsenius(3) as being performed upon algebras that follow the Hurwitz theorem, namely: the real algebra for General Relativity, the complex algebra for the non-symmetric theory and the quaternion algebra for the Borchsenius theory. We obtain from there, the final generalization for a theory using the octonion algebra. We then conclude that formally the geometrical objects and field equations obtained in each algebra maintain essentially their forms when we go from real algebra to the octonions. In the same manner, we will see that it is possible to construct (local) tangent spaces to curved space-time, corresponding to each of the above algebra and that again, the geometrical objects and field equations maintain their forms when we go from $\mathbb{R}$ to $\mathbb{O}$. On the other hand, we must observe that the geometrical objects in these tangent spaces are more deeply affected by the presence of internal space (that follows the $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{O}$ algebra) than that of curved space-time, this because of the manner in which are defined the tangent vectors on each point of the curved space-time.

In sections 2, 3 and 4 we study the properties of geometrical objects in the tangent space associated to the complex, quaternionic and octonionic algebra. Also, we obtain there the corresponding field equations to that obtained in the reference (1). In section 5 we finally consider some properties about the transformation law in this tangent space, associated to quaternionic and octonionic internal spaces. We then show how it is possible to relate them to the electromagnetic and Yang-Mills fields.

2. The complex tangent-space.

According to the correspondence principle, there exists in each point of curved space-time a local tangent space with the structure of a flat space-time, with metric given by the Minkowski tensor η_{ab}. By the symmetry property of the metric of General Relativity, the line element is given by $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \eta_{ab}\, dx^a\, dx^b$, where $g_{\mu\nu} = g_{\nu\mu}$. The geometrical properties of this tangent space is described in literature(4) and it is easy to conclude that it follows the real numbers algebra and by this will be referred here as a real tangent space.

In the Einstein non-symmetric theory (or in the Moffat-Boal theory(5)) the metric of curved space-time is no longer symmetric and real, but has the property $\overset{*}{g}_{\mu\nu} = g_{\nu\mu}$. If we now define,

$$g_{\mu\nu} = e^{*a}_{\nu}\, e^{b}_{\mu}\, \eta_{ab} \quad , \qquad (2.1)$$

where the objects e^a_μ are now complex vierbeins, we will have for the line element,

$$ds^2 = dx^\mu\, dx^\nu\, g_{\mu\nu} = dx^{*a}\, dx^b\, \eta_{ab} \qquad (2.2)$$

The metric of tangent space is again determined by the Minkowski tensor, η_{ab}. Therefore, in (2.2),

$$dx^a = e^a_\mu\, dx^\mu \quad , \quad dx^{*a} = e^{*a}_{\mu}\, dx^\mu \quad . \qquad (2.3)$$

There exists an inverse $g^{\mu\nu}$ such that $g^{\mu\nu}\, g_{\sigma\nu} = \delta^\mu_\sigma$, in this order of indexes. We have from (2.1),

$$g^{\mu\nu} = e^{*\mu}_{a}\, e^{\nu}_{b}\, \eta^{ab} \quad , \qquad (2.4)$$

where e^{μ}_{a} is the inverse of e^{a}_{μ}. Then, we obtain the following orthogonallity conditions for complex veirbeins:

$$
\begin{aligned}
e^{*a}_{\mu}\, e^{\mu}_{b} &= e^{a}_{\mu}\, e^{*\mu}_{b} = \delta^{a}_{b} \quad , \\
e^{*a}_{\mu}\, e^{\nu}_{a} &= e^{a}_{\mu}\, e^{*\nu}_{a} = \delta^{\nu}_{\mu} \quad .
\end{aligned}
\tag{2.5}
$$

The transformation law for vectors e^{a}_{μ} in the complex tangent space is defined by:

$$
e'^{a}_{\mu}(x) = L^{a}{}_{b}(x)\, e^{b}_{\mu}(x) \quad , \quad e^{*'a}_{\mu}(x) = L^{*a}{}_{b}(x)\, e^{*b}_{\mu}(x) \quad , \tag{2.6}
$$

where $L^{a}{}_{b}$ are pseudo-unitary rotation matrices, i.e.,

$$
L^{\dagger}\, \eta\, L = \eta \ . \tag{2.7}
$$

The interval covariant derivative of the vectors e^{a}_{μ} is given by:

$$
e^{a}_{\mu\,\|\nu} = e^{a}_{\mu,\nu} + \Lambda_{\nu}{}^{a}{}_{b}\, e^{b}_{\mu} \quad , \tag{2.8}
$$

where now, the affinity Λ_{ν} is complex. Its internal transformation law is:

$$
\Lambda'_{\mu} = L\, \Lambda_{\mu}\, L^{-1} - L_{,\mu}\, L^{-1} \quad , \quad \Lambda^{*'}_{\mu} = L^{*}\, \Lambda^{*}_{\mu}\, L^{*-1} - L^{*}_{,\mu}\, L^{*-1} \quad . \tag{2.9}
$$

One of Einstein field equations for the case of non-symmetric theory obtained through a variational principle, is $g_{\underset{+-}{\mu\nu}\,;\,\alpha} = 0$, where the Schroedinger connection for the curved space-time was used, $\Theta^{\rho}{}_{\mu\alpha}$, $\Theta_{\mu} = \Theta^{\rho}{}_{\underset{\vee}{\mu\rho}} = 0$. (In general, $\Theta^{\rho}{}_{\mu\alpha}$ is a non-symmetric affinity for which it is valid the property $\Theta^{*\rho}{}_{\mu\alpha} = \Theta^{\rho}{}_{\alpha\mu}$.) This equation along with (2.1), implies in the equations for the vierbeins:

$$e^{a}_{\underset{+}{\mu}|\alpha} = e^{a}_{\mu,\alpha} - \Theta^{\rho}_{\ \mu\alpha}\, e^{a}_{\rho} + \overset{(1)}{\Lambda}_{\alpha}{}^{a}{}_{b}\, e^{b}_{\mu} = 0 \quad ,$$

$$e^{*a}_{\underset{-}{\mu}|\alpha} = e^{*a}_{\mu,\alpha} - \Theta^{\rho}_{\ \alpha\mu}\, e^{*a}_{\rho} + \overset{(2)}{\Lambda}{}_{\alpha}{}^{a}{}_{b}\, e^{*b}_{\mu} = 0 \quad . \tag{2.10}$$

Therefore, in order to have only one independent equation in (2.10), the affinity $\overset{(1)}{\Lambda}_{\mu}$ is the complex of the affinity $\overset{(2)}{\Lambda}_{\mu}$, since then,

$$e^{a}_{\underset{+}{\mu}|\alpha} = (e^{*a}_{\underset{-}{\mu}|\alpha}) = 0 \quad . \tag{2.11}$$

Taking the inverse equation $g^{\mu\nu}_{\underset{+-}{}\,;\alpha} = 0$, we have the corresponding equations for the vierbeins:

$$e^{*\mu}_{\underset{+}{a}|\alpha} = (e^{\mu}_{\underset{-}{a}|\alpha})^{*} = e^{*\mu}_{a,\alpha} + \Theta^{\mu}_{\ \rho\alpha} e^{*\rho}_{a} - \Lambda_{\alpha}{}^{b}{}_{a}\, e^{*b}_{\mu} = 0. \tag{2.12}$$

From (2.11) and (2.12), we obtain the relation:

$$\Lambda_{\alpha}{}^{a}{}_{b} = e^{a}_{\mu}\, e^{*\mu}_{\underset{+}{b};\alpha} = -\, e^{a}_{\underset{+}{\mu};\alpha}\, e^{*\mu}_{b} \quad . \tag{2.13}$$

In the case of real tetrads, we re-obtain the results of General Relativity for the tangent space associated to the Riemann geometry.

We must also have: $\eta_{\underset{+-}{ab}\|\mu} = 0$, where we must have in mind that the "minus" sign corresponds to the complex conjugate of the affinity Λ_{μ} , because of (2.11):

$$\eta_{\underset{+-}{ab}\|\mu} = \eta_{ab,\mu} - \Lambda_{\mu}{}^{c}{}_{a}\, \eta_{cb} - \Lambda^{*}{}_{\mu}{}^{c}{}_{b}\, \eta_{ac} = 0 \quad . \tag{2.14}$$

η_{ab} lowers indexes, and we have therefore that Λ_{μ} is anti-Hermitian with respect to index of the tangent space. This results in the equality:

$$\Lambda_{\mu\,ab} = \Lambda_{\mu\,\underset{\vee}{ab}} + i\,\Lambda_{\mu\,\underline{ab}} \quad . \tag{2.15}$$

Calculating the difference $e^a_{\mu,\nu\gamma} - e^a_{\mu,\gamma\nu}$, we obtain the equality:

$$R^\rho{}_{\mu\nu\gamma}\, e^a_\rho - S_{\nu\gamma}{}^a{}_b\, e^b_\mu = 0 \quad , \tag{2.16}$$

where $R^\rho{}_{\mu\nu\gamma}$ is the curvature in the non-Riemannian space-time, written in terms of non-symmetric affinity, and $S_{\nu\gamma}{}^a{}_b$ is the curvature in the complex tangent space:

$$S_{\nu\gamma} = \Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - \left[\Lambda_\nu , \Lambda_\gamma \right] \quad . \tag{2.17}$$

$S_{\nu\gamma}$ is antissymmetric in the world indexes and anti-Hermitian in the internal indexes. Therefore, this implies that it must be written in the form:

$$S_{\nu\gamma ab} = S_{\nu\gamma \underset{\vee}{ab}} + i\, S_{\nu\gamma \underline{ab}} \quad . \tag{2.18}$$

With this we complete the resumed geometrical treatment of the complex tangent space. We will see in the following a similar treatment for the case of quaternionic and octonionic tangent spaces.

3. The quaternionic tangent space.

In the Borchsenius theory[3], we deal with a vector space with representation via Pauli matrices. This vector space can be reinterpreted as quaternions if we take $i^{-1}\sigma_i = \omega_i$, i = 1,2,3, σ_i being the Pauli matrices and ω_i quaternions, i.e., they satisfy the relations

$$\omega_i \, \omega_j = \varepsilon_{ijk} \, \omega_k - \delta_{ij} \, \omega_o \quad , \tag{3.1}$$

where ω_o is the unity element of quaternions algebra, $\omega_o = \sigma_o$. The "metric" in this matrix, or quaternionic, space-time, has the symmetry property,

$$G^{\dagger}_{\mu\nu} = G_{\nu\mu} \quad , \tag{3.2}$$

which generalises the conditions used for $g_{\mu\nu}$. The Hermitian conjugation operation is carried out over the quaternionic internal space, ou Q-space. Let it be then,

$$G_{\mu\nu} = E^{\dagger a}_{\nu} \, E^{b}_{\mu} \, \eta_{ab} \quad , \tag{3.3}$$

where E^a_μ are quaternionic vierbeins,

$$\begin{aligned} E^a_\mu &= k^a_{\mu_o} \, (x) \, \omega_o + k^a_{\mu_i} \, (x) \, \omega_i \quad , \\ E^{\dagger a}_{\mu} &= k^{*a}_{\mu_o} \, (x) \, \omega_o - k^{*a}_{\mu i} \, (x) \, \omega_i \quad . \end{aligned} \tag{3.4}$$

According to correspondence principle, the line element in the quaternionic space-time and in the quaternionic tangent space obtained with the vierbeins (3.4) is defined as:

$$ds^2 = \frac{1}{2} \, \mathrm{Tr} \, (G_{\mu\nu} \, dx^\mu \, dx^\nu) = \frac{1}{2} \, \mathrm{Tr} \, (\eta_{ab} \, d\boldsymbol{x}^{\dagger a} \, d\boldsymbol{x}^b) \quad , \tag{3.5}$$

where for simplicity sake and principally aiming to physical interpretation, we take the metric in Q-tangent space with Minkowski's structure η_{ab}. From (3.2) and (3.4), we have:

$$d\boldsymbol{x}^a = E^a_\mu \, dx^\mu \quad , \qquad d\boldsymbol{x}^{\dagger a} = E^{\dagger a}_{\mu} \, dx^\mu \tag{3.6}$$

There exists an inverse $G^{\mu\nu}$ such that, $G_{\mu\alpha} G^{\mu\nu} = G^{\nu\mu} G_{\alpha\mu} = \delta_{\alpha}^{\nu} \omega_o$. We define then,

$$G^{\mu\nu} = E_a^{\dagger\mu} E_b^{\nu} \eta^{ab} \quad , \tag{3.7}$$

and from that, we obtain the corresponding orthogonality relations for the Q-vierbeins:

$$\begin{aligned} E_\mu^b E_c^{\dagger\mu} &= E_c^\mu E_\mu^{\dagger b} = \delta_c^b \omega_o \\ E_\alpha^{\dagger a} E_a^\nu &= E_a^{\dagger\nu} E_\alpha^a = \delta_\alpha^\nu \omega_o \end{aligned} \tag{3.8}$$

The transformation law of tangent vectors $E_\mu^a(x)$ in $\mathbb{Q}$-tangent space is:

$$E'^{a}_{\mu} (x) = \mathbb{L}^a{}_b (x) E_\mu^b (x) \tag{3.9}$$

$\mathbb{L}^a{}_b$ is therefore a quaternion:

$$\mathbf{L}^a{}_b = L^a{}_{b_o} (x) \omega_o + L^a{}_{b_i} (x) \omega_i \tag{3.10}$$

In (3.9) and (3.10) we are maintaining the quaternions space fixed. $\mathbf{L}^a{}_b$ are then (quaternionic) rotation matrices in the $\mathbb{Q}$-tangent space. For the invariance under $\mathbb{L}$-transformation of the line element ds^2, we must have the relation:

$$\mathbb{L}^{\dagger a}{}_c \eta_{ab} \mathbb{L}^b{}_d = \eta_{cd} \omega_o \quad . \tag{3.11}$$

Again, the Hermitian conjugation operation being carried out in the Q-space,

$$\mathbb{L}^{\dagger a}{}_b = L^{*}{}_o{}^a{}_b \omega_o - L^{*}{}_i{}^a{}_b \omega_i \quad .$$

We can define on the $\mathbb{Q}$-tangent space, the operation of covariant differentiation, as example, for a vector $E = (E^1_a, \ldots, E^4_a)$. We have,

$$E^\mu_{a\,\|\nu}(x) = \partial_\nu E^\mu_a(x) - \Lambda_\nu{}^b{}_a E^\mu_b(x) \quad . \qquad (3.12)$$

The affinity $\Lambda_\nu{}^a{}_b$ is in general a quaternion,

$$\Lambda_\nu{}^a{}_b = \lambda_\nu{}^a{}_{b_0}\,\omega_o + \lambda_\nu{}^a{}_{b_i}\,\omega_i \qquad (3.13)$$

As in the case of the metric on the $\mathbb{Q}$-tangent space, we impose on the affinity Λ_ν the condition

$$\Lambda_\nu{}^a{}_b \equiv \lambda_\nu{}^a{}_b\,\omega_o \quad . \qquad (3.14)$$

Again, the reason for this restriction aims to a possible physical interpretation of this object.

The derivative (3.12) is such that $E^\mu_{a\,\|\nu}$ transforms as a vector in the $\mathbb{Q}$-tangent space. The transformation law for the affinity Λ_ν must then be:

$$\Lambda'_\nu = \mathbb{L}\,\Lambda_\nu\,\mathbb{L}^{-1} - \mathbb{L}_{,\nu}\,\mathbb{L}^{-1} \qquad (3.15)$$

where $\mathbb{L}^{-1}$ is the inverse matrix with respect to the a,b,... indexes on the $\mathbb{Q}$-tangent space.

According to the properties of the objects $E^\mu_a(x)$, we may have a total covariant derivative:

$$E^{\mu}_{\overset{+}{a}|\nu} = \partial_\nu E^\mu_a + \Omega^\mu{}_{\rho\nu} E^\rho_a - \Lambda_\nu{}^c{}_a E^\mu_c + \left[\Gamma_\nu, E^\mu_a \right] \quad , \qquad (3.16)$$

where $\Gamma_\nu = -\vec{C}_\nu \cdot \vec{\omega}$ is the affinity in the quaternionic internal space[6] (with real $\vec{C}_\nu$), and $\Omega^\mu{}_{\rho\nu}\,\omega_o$ is the non-symmetric affinity of curved space-time.

K. Borchsenius showed that one of the field equations obtained in his unified theory[7] through a variational principle is: $G_{\underset{+-}{\mu\nu}|\alpha} = 0$, (its inverse: $G^{\mu\nu}_{+-}{}_{|\alpha} = 0$) in matrix or quaternionic notation. In this case the Schroedinger connection for the curved space-time was used,

$$\Omega^\mu{}_{\rho\nu} = \Theta^\mu{}_{\rho\nu} - \frac{2}{3}\,\delta^\mu_\rho\,\Omega_\nu \quad , \quad \Omega_\nu \equiv \Omega^\rho{}_{\underset{\vee}{\nu\rho}} \quad . \tag{3.17}$$

Here $\Theta^\mu{}_{\rho\nu}$ is the Schroedinger connection. We can obtain similar expressions for the Q-vierbeins. This is given in the following table:

$$G_{\underset{+-}{\mu\nu}|\alpha} = 0, \longleftrightarrow \quad E^{\dagger a}{}_{\underset{-}{\mu}|\alpha} = (E^a{}_{\underset{+}{\mu}|\alpha}) = 0 \tag{3.18}$$

$$E^a{}_{\underset{+}{\mu}|\alpha} = E^a_{\mu,\alpha} - E^a_\rho\,\mathbb{\Gamma}^\rho{}_{\underset{\bullet}{\mu\alpha}} + \Lambda_\alpha{}^a{}_c\,E^c_\mu = 0 \ ,$$

$$\mathbb{\Gamma}^\rho{}_{\underset{\bullet}{\mu\alpha}} = \Theta^\rho{}_{\mu\alpha}\,\omega_o + \delta^\rho_\mu\,\Gamma_\alpha \qquad \Theta^\rho{}_{\mu\alpha} = \text{Schroedinger connection}$$

$$\Lambda_\alpha{}^a{}_c = \lambda_\alpha{}^a{}_c\,\omega_o + \delta^a_c\,\Gamma_\alpha \qquad \Gamma_\alpha = \vec{C}_\alpha \cdot \vec{\omega} = -\Gamma^\dagger_\alpha \quad .$$

$$G^{\mu\nu}_{+-}{}_{|\alpha} = 0 \ . \longleftrightarrow \quad E^{\dagger\underset{+}{\mu}}{}_{a|\alpha} = (E^{\underset{-}{\mu}}{}_{a|\alpha}) = 0 \tag{3.19}$$

$$E^{\underset{-}{\mu}}_{a|\alpha} = E^\mu_{a,\alpha} + E^\rho_a\,\mathbb{\Gamma}^{\dagger\mu}{}_{\underset{\bullet}{\rho\alpha}} - \Lambda^\dagger{}_\alpha{}^c{}_a\,E^\mu_c = 0 \quad ,$$

$$\mathbb{\Gamma}^{\dagger\mu}{}_{\underset{\bullet}{\rho\gamma}} = \Theta^\mu{}_{\gamma\rho}\,\omega_o - \delta^\mu_\rho\,\Gamma_\gamma \ , \text{ because } \Theta^{*\mu}{}_{\rho\gamma} = \Theta^\mu{}_{\gamma\rho}$$

$$\Lambda^\dagger{}_\gamma{}^a{}_b = \Lambda^*{}_\gamma{}^a{}_b\,\omega_o - \delta^a_b\,\Gamma_\gamma \quad .$$

Again we must have here,

$$(\eta_{\underset{+-}{ab}}\ \omega_o)_{\|\gamma} = (-\ \Lambda_\gamma{}^c{}_a\ \eta_{cb} - \Lambda^*{}_\gamma{}^c{}_b\eta_{ac})\ \omega_o = 0 \quad , \tag{3.20}$$

since, from (3.18), the "plus" sign correspond to the affinity Λ_γ and the "minus" sign to Λ^*_γ, as the covariant derivative index is concerned. As η_{ab} lowers indexes, we have by (3.20) that $\Lambda_\gamma = (\Lambda_{\gamma ab})$ is anti-Hermitian. In the same way as in (2.15), to make this result true we must have for the affinity of the quaternionic tangent space:

$$\Lambda_{\gamma ab} = \lambda_{\gamma ab}\ \omega_o = (\lambda_{\gamma \underset{\vee}{ab}} + i\ \lambda_{\gamma \underline{ab}})\omega_o \quad . \tag{3.21}$$

We can obtain an expression for $\mathbb{\Lambda}_\gamma{}^a{}_b$ from (3.19), in the table above, namely:

$$\begin{aligned} \mathbb{\Lambda}_\gamma{}^a{}_b &= E^a_\mu\ E^{\dagger\mu}{}_{b,\gamma} + E^a_\mu\ \mathbb{\Gamma}^\mu{}_{\rho\dot\gamma}\ E^{\dagger\rho}_b \quad , \\ &= E^a_\mu\ E^{\dagger\mu}_{\underset{+}{b};\gamma} + E^a_\mu\ \Gamma_\gamma\ E^{\dagger\mu}_b \end{aligned} \tag{3.22}$$

and another, from (3.18):

$$\begin{aligned} \mathbb{\Lambda}_\gamma{}^a{}_b &= -\ E^a_{\mu,\gamma}\ E^{\dagger\mu}_b + E^a_\rho\ \mathbb{\Gamma}^\rho{}_{\mu\dot\gamma}\ E^{\dagger\mu}_b \quad , \\ &= -\ E^a_{\underset{+}{\mu};\gamma}\ E^{\dagger\mu}_b + E^a_\mu\ \Gamma_\gamma\ E^{\dagger\mu}_b \end{aligned} \tag{3.23}$$

The expression for the curvature on the quaternionic tangent space can be obtained from the comutator of total covariant derivatives. This give us:

$$E^a_\rho\ \mathbb{R}^\rho{}_{\mu\nu\gamma} - \mathbb{S}_{\nu\gamma}{}^a{}_c\ E^c_\mu = 0 \quad , \tag{3.24}$$

where $\mathbb{R}^{\rho}{}_{\mu\nu\gamma}$ is the total curvature[8], written with the "affinities" $\Gamma^{\rho}{}_{\mu\nu}$, and $S_{\nu\gamma}$ is the total curvature on the $\mathbb{Q}$-tangent space, written for the "affinities" Λ_{γ},

$$\begin{aligned} S_{\nu\gamma}{}^{a}{}_{c} &= (\Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - \left[\Lambda_{\nu}, \Lambda_{\gamma}\right])^{a}{}_{c} \quad , \\ &= S_{\nu\gamma}{}^{a}{}_{c}\, \omega_{o} + \delta^{a}{}_{c}\, P_{\nu\gamma} \quad , \end{aligned} \tag{3.25}$$

where $S_{\nu\gamma}$ is the curvature written with the affinities Λ_{ν} and $P_{\nu\gamma}$ is the $\mathbb{Q}$-curvature[9], written for the $\mathbb{Q}$-affinities, Γ_{γ}.

4. The octonionic tangent space.

Again we will take here the split octonions algebra[10] because of its convenience for a possible physical interpretation. Another advantage when we take the split $\mathbb{O}$-algebra is its isomorphism with the Zorn matrix algebra[11]. Therefore, an octonion P is written in the split $\mathbb{O}$-algebra as,

$$P = a\, u^{*}_{o} + b\, u_{o} - n_{k}\, u^{*}_{k} + m_{k}\, u_{k} \ , \qquad k = 1,2,3 \ , \tag{4.1}$$

where $\{u^{*}_{o}\ , \vec{u}^{*}, u_{o}\ , \vec{u}\ \}$ is the split $\mathbb{O}$-base, and

$$Z\ (P) = \begin{pmatrix} a & -\vec{n} \\ \vec{m} & b \end{pmatrix} \quad , \tag{4.2}$$

is the representation of P by means of a Zorn matrix. We will take a representation of this algebra by means of the use of the Pauli matrices, as in the case of quaternions defined in (3.1). In this way,

$$Z\,(u^*_o) = \begin{pmatrix} 1.\omega_o & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \quad , \quad Z\,(u_o) = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & 1.\omega_o \end{pmatrix}$$

$$Z\,(\vec{u}^*) = \begin{pmatrix} 0_2 & -1.\vec{\omega} \\ 0_2 & 0_2 \end{pmatrix} \quad , \quad Z\,(u) = \begin{pmatrix} 0_2 & 0_2 \\ 1.\vec{\omega} & 0_2 \end{pmatrix} \tag{4.3}$$

$(\omega_o, \vec{\omega})$ defined in Section-3.

The octonionic conjugation of P is defined as

$$\bar{P} = b\,u^*_o + a\,u_o - n_k\,u^*_k - m_k\,u_k \tag{4.4}$$

and the Hermitian conjugation of P is defined as[12]

$$P^\dagger = \bar{P}^* = b^*\,u^*_o + a^*\,u_o - n^*_k\,u^*_k - m^*_k\,u_k \tag{4.5}$$

Due to the isomorphism of the split $\mathbb{O}$-algebra with the Zorn matrices algebra, we will not differenciate from now on, its designations.

The $\mathbb{O}$-"metric" is written as:

$$\mathbb{G}_{\mu\nu}\,(x) = \begin{pmatrix} s_{\mu\nu_o}\,\omega_o & -\,s_{\mu\nu_k}\,\omega_k \\ r_{\mu\nu_k}\,\omega_k & r_{\mu\nu_o}\,\omega_o \end{pmatrix} = \mathbb{G}_{\mu\nu}\,(s,r). \tag{4.6}$$

Taking[13] $r_{\mu\nu_o} = s_{\mu\nu_o} = g_{\underline{\mu\nu}} + i\kappa\,F_{\overset{\mu\nu}{\vee}}$, $g_{\underline{\mu\nu}}$ being the symmetric metric and $F_{\overset{\mu\nu}{\vee}}$ the Maxwell tensor, we have the symmetry property,

$$\mathbb{G}^\dagger_{\mu\nu}\,(s,r) = \mathbb{G}_{\nu\mu}\,(s,r)\ . \tag{4.7}$$

The quaternionic vierbein (3.4) is generalized to the octonionic vierbein, or $\mathbb{O}$-vierbein, as:

$$H^a_\mu\,(x) = \begin{pmatrix} k^a_{\mu_o}\,\omega_o & -\,k^a_{\mu_i}\,\omega_i \\ \ell_{\mu_i}\,\omega_i & \ell^a_{\mu_o}\,\omega_o \end{pmatrix} = H^a_\mu\,(k,\ell) \tag{4.8}$$

Let it be then,

$$\mathbb{G}_{\mu\nu} = H^{\dagger a}_{\nu} H^{b}_{\mu} \eta_{ab} \qquad (4.9)$$

where η_{ab} is the Minkowski tensor. There exists an inverse of $\mathbb{G}_{\mu\nu}$:

$$\mathbb{G}^{\mu\nu} = H^{\dagger\mu}_{a} H^{\nu}_{b} \eta^{ab} \quad , \qquad (4.10)$$

such that,

$$\mathbb{G}_{\mu\alpha}(s,r)\, \mathbb{G}^{\mu\nu}(s,r) = \mathbb{G}^{\nu\mu}(s,r)\, \mathbb{G}_{\alpha\mu}(s,r) = \delta^{\nu}_{\alpha}(u_o + u^{*}_{o}) \ , \qquad (4.11)$$

in this order. We can obtain from (4.9), (4.10), and (4.11) the orthogonality relations for $\mathbb{O}$-vierbeins. Then,

$$H^{a}_{\mu} H^{\dagger\mu}_{b} = H^{\mu}_{b} H^{\dagger a}_{\mu} = \delta^{a}_{b}(u_o + u^{*}_{o}) \quad ,$$
$$\qquad (4.12)$$
$$H^{\dagger a}_{\alpha} H^{\nu}_{a} = H^{\dagger\nu}_{a} H^{a}_{\alpha} = \delta^{\nu}_{\alpha}(u_o + u^{*}_{o}) \ .$$

The above relations are such that the following trace is true:

$$\mathrm{Tr}\,(\mathbb{G}_{\mu\alpha}\, \mathbb{G}^{\mu\nu}) = \mathrm{Tr}\left[(H^{\dagger a}_{\alpha} H^{b}_{\mu})(H^{\dagger\mu}_{c} H^{\nu}_{d})\, \eta_{ab}\, \eta^{cd}\right]$$

where the position of the parenthesis is no longer important, if the objects are fixed in their positions.

The line element in the $\mathbb{O}$-space-time and in the space of $\mathbb{O}$-vierbeins is:

$$ds^2 = \frac{1}{4}\,\mathrm{Tr}\,(dx^{\mu}\, dx^{\nu}\, \mathbb{G}_{\mu\nu}) = \frac{1}{4}\,\mathrm{Tr}\,(d\boldsymbol{x}^{\dagger a}\, d\boldsymbol{x}^{b}\, \eta_{ab}) \quad , \qquad (4.13)$$

due to the definition (4.9). Again we are taking the metric on the tangent space as being that of Minkowski. We have therefore,

$$dx^{a} = H^{a}_{\mu}\, dx^{\mu} \quad , \quad dx^{\dagger a} = H^{\dagger a}_{\mu}\, dx^{\mu} . \qquad (4.14)$$

The transformation law of the tangent vector H^{a}_{μ} in the octonionic tangent space is

$$H'^{a}_{\mu}(x) = \mathbb{L}^{a}{}_{b}(x)\, H^{b}_{\mu}(x) \quad , \qquad (4.15)$$

where $\mathbb{L}^{a}{}_{b}(x)$ is now an octonion:

$$\mathbb{L}^{a}{}_{b}(x) = n^{a}{}_{b_0}(x)\, u^{*}{}_{0} + n^{a}{}_{b_i}(x)\, u^{*}{}_{i} + m^{a}{}_{b_0}(x)\, u_{0} + m^{a}{}_{b_i}(x)\, u_{i} \quad . \qquad (4.16)$$

In the above expressions, we are maintaining the internal space of octonions fixed. $\mathbb{L}^{a}{}_{b}$ are octonionic rotation "matrices" in the octonionic tangent space. The invariance under $\mathbb{L}$-transformation of the line element gives the relation;

$$\mathbb{L}^{\dagger a}{}_{c}\, \eta_{ab}\, \mathbb{L}^{b}{}_{d} = \eta_{cd}(u_{0} + u^{*}{}_{0}) \quad , \qquad (4.17)$$

where

$$\mathbb{L}^{\dagger a}{}_{b} = m^{*a}{}_{b_0}\, u^{*}{}_{0} - n^{*a}{}_{b_i}\, u^{*}{}_{i} + n^{*a}{}_{b_0}\, u_{0} - m^{*a}{}_{b_i}\, u_{i} \; .$$

The derivative over the $\mathbb{O}$-tangent space of tangent vectors $H^{\mu}_{a}(x)$, for instance, is defined by

$$H^{\mu}{}_{a\|\nu}(x) = \partial_{\nu} H^{\mu}_{a}(x) - \Lambda_{\nu}{}^{b}{}_{a}(x)\, H^{\mu}_{b}(x) \quad , \qquad (4.18)$$

where again we restrict the affinity Λ_{ν} to be proportional to the unity element of the algebra:

$$\Lambda_{\nu}{}^{a}{}_{b} = q_{\nu}{}^{a}{}_{b}\,(u_{0} + u^{*}{}_{0}) \quad . \qquad (4.19)$$

The total derivative is given then by:

$$H^{\mu}_{\underset{+}{a}|\nu} = \partial_\nu H^{\mu}_{a} + \Omega^{\mu}{}_{\rho\nu} H^{\rho}_{a} - \Lambda_\nu{}^{c}{}_{a} H^{\mu}_{c} + \left[\Gamma_\nu , H^{\mu}_{a} \right] , \qquad (4.20)$$

where Γ_μ is now the affinity in the internal octonionic space,

$$\Gamma_\nu = - \begin{pmatrix} 0_2 & -\vec{L}_\nu . \vec{\omega} \\ \vec{K}_\nu . \vec{\omega} & 0_2 \end{pmatrix} , \qquad (4.21)$$

with $\vec{L}_\nu$ and $\vec{K}_\nu$ real, like in quaternionic case.

The transformation law of the affinity Λ_ν on the $\mathbb{O}$-tangent space is given now by

$$\Lambda'_\nu (x) = \mathbb{L}\, \Lambda_\nu\, \mathbb{L}^{-1} - \mathbb{L}_{,\nu}\, \mathbb{L}^{-1} , \qquad (4.22)$$

where

$$\mathbb{L}^{-1} = n^{-1}{}_{o}\, u^{*}{}_{o} + n^{-1}{}_{i}\, u^{*}{}_{i} + m^{-1}{}_{o}\, u_{o} + m^{-1}{}_{i}\, u_{i} ,$$

is the inverse of $\mathbb{L}$ with relation to the indexes on the tangent space.

One of the field equations obtained through a variational principle in the generalized octonionic theory[1] is: $\mathbb{G}_{\underset{+-}{\mu\nu}|\alpha} = 0$ (its inverse is: $\mathbb{G}^{\mu\nu}_{+-|\alpha} = 0$). Using (4.9) and (4.10) we obtain the corresponding equations for the octonionic vierbeins, which we write in the following table:

$\mathbb{G}_{\underset{+-}{\mu\nu}\|\alpha} = 0 , \longleftrightarrow$	$H^{\dagger a}{}_{\underset{-}{\mu}\|\alpha} = (H^{a}{}_{\underset{+}{\mu}\|\alpha})^{\dagger} = 0$	(4.23)
$H^{a}{}_{\underset{+}{\mu}\|\alpha} = H^{a}{}_{\mu,\alpha} - H^{a}_{\rho}\, \Gamma^{\rho}{}_{\mu\alpha} + \Lambda_\alpha{}^{a}{}_{c}\, H^{c}_{\mu} = 0$, $\Gamma^{\rho}{}_{\mu\alpha} = \Theta^{\rho}{}_{\mu\alpha}\, (u_o + u^{*}{}_{o}) + \delta^{\rho}_{\mu}\, \Gamma_\alpha$, $\Theta^{\rho}{}_{\mu\alpha}$ = Schroedinger connection , $\Lambda_\alpha{}^{a}{}_{c} = q_\alpha{}^{a}{}_{c}\, (u_o + u^{*}{}_{o}) + \delta^{a}_{c}\, \Gamma_\alpha$, $\Gamma_\alpha = \vec{L}_\alpha . \vec{u}^{*} + \vec{K}_\alpha . \vec{u}$		
$\mathbb{G}^{\mu\nu}_{+-\|\alpha} = 0 \longleftrightarrow$	$H^{\dagger}{}^{\underset{+}{\mu}}{}_{a\|\alpha} = (H^{\underset{-}{\mu}}{}_{a\|\alpha})^{\dagger} = 0$	(4.24)

$$\boxed{\begin{array}{l} H^{\mu}_{\underset{=}{a}|\alpha} = H^{\mu}_{a,\alpha} + H^{\rho}_{a}\, \Gamma^{\dagger\mu}{}_{\rho\dot{\alpha}} - \Lambda^{\dagger}{}_{\alpha}{}^{c}{}_{a}\, H^{\mu}_{c} = 0 \quad , \\[2ex] \Gamma^{\dagger\rho}{}_{\dot{\mu}\alpha} = \Theta^{*\rho}{}_{\mu\alpha}\,(u_o + u^{*}{}_{o}) - \delta^{\rho}_{\mu}\, \Gamma_{\alpha} \quad , \quad \Theta^{*\rho}{}_{\mu\alpha} = \Theta^{\rho}{}_{\alpha\mu} \quad , \\[2ex] \Lambda^{\dagger}{}_{\alpha}{}^{a}{}_{c} = q^{*}_{\alpha}{}^{a}{}_{c}\,(u_o + u^{*}{}_{o}) - \delta^{a}_{c}\, \Gamma_{\alpha} \quad . \end{array}}$$

Considering that again,

$$\left[n_{\underset{=}{a}\underline{b}}\,(u_o + u^{*}{}_{o})\right]_{\|\gamma} = (-\Lambda_{\gamma}{}^{c}{}_{a}\, n_{cb} - \Lambda^{*c}_{\gamma}{}_{b}\, n_{ac})\,(u_o + u^{*}{}_{o}) = 0 \ , \qquad (4.25)$$

the affinity $\Lambda_{\gamma} = (\Lambda_{\gamma\dot{a}b})$ continues being anti-Hermitian, which result,

$$\Lambda_{\gamma ab} = q_{\gamma ab}\,(u_o + u^{*}{}_{o}) = (q_{\gamma a\underset{\vee}{b}} + i\, q_{\gamma a\underline{b}})\,(u_o + u^{*}{}_{o}) \ . \qquad (4.26)$$

From (4.23) and (2.24) in the above table, we obtain for the octonionic "affinity" Λ_{γ} , the expressions,

$$\Lambda_{\gamma}{}^{a}{}_{b} = -\, H^{a}_{\mu,\gamma}\, H^{\dagger\mu}_{b} + (H^{a}_{\rho}\, \Gamma^{\rho}{}_{\mu\dot{\gamma}})\, H^{\dagger\mu}_{b} - \{\, \Lambda_{\gamma}{}^{a}{}_{c} \ , H^{c}_{\mu} \ , H^{\dagger\mu}_{b} \,\} \ , \qquad (4.27)$$

and

$$\Lambda_{\gamma}{}^{a}{}_{b} = H^{a}_{\mu}\, H^{\dagger\mu}{}_{b,\gamma} + H^{a}_{\rho}\,(\Gamma^{\rho}{}_{\mu\dot{\gamma}}\, H^{\dagger\mu}_{b}) + \{\, \Lambda_{\gamma}{}^{c}{}_{b} \ , H^{a}_{\mu} \ , H^{\dagger\mu}_{c} \,\} \ . \qquad (4.28)$$

We define now, the objects,

$$\mathbb{R}^{\lambda}{}_{\mu\nu\gamma} = \Gamma^{\lambda}{}_{\mu\nu,\gamma} - \Gamma^{\lambda}{}_{\mu\gamma,\nu} + \Gamma^{\lambda}{}_{\rho\gamma}\, \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\rho\nu}\, \Gamma^{\rho}{}_{\mu\gamma} \qquad (4.29)$$

and

$$\mathbb{S}_{\nu\gamma}{}^{a}{}_{b} = \Lambda_{\nu}{}^{a}{}_{b,\gamma} - \Lambda_{\gamma}{}^{a}{}_{b,\nu} - \left[\, \Lambda_{\nu} \ , \Lambda_{\gamma} \right]^{a}{}_{b} \quad , \qquad (4.30)$$

as being the $\mathbb{O}$-"curvatures" of $\mathbb{O}$-space-time, written with the $\mathbb{O}$-"affinities", $\mathbb{\Gamma}^{\rho}{}_{\mu\gamma}$, and of $\mathbb{O}$-tangent space, written with the $\mathbb{O}$-"affinities", $\mathbb{A}_{\nu}{}^{a}{}_{b}$, respectively. These are such that the following expression are true:

$$\mathrm{Tr}\left[H^{a}_{\lambda}\, \mathbb{R}^{\lambda}{}_{\mu\nu\gamma} - \mathbb{S}_{\nu\gamma}{}^{a}{}_{c}\, H^{c}_{\mu} \right] = 0 \quad ,$$

obtained from the trace of the curl $(\partial_{\gamma}\, \partial_{\nu} - \partial_{\nu}\, \partial_{\gamma})\, H^{a}_{\mu} = 0$.

The "curvature" $\mathbb{S}_{\nu\gamma}$ may be written as,

$$\mathbb{S}_{\nu\gamma}{}^{a}{}_{c} = S_{\nu\gamma}{}^{a}{}_{c}\, (u_{o} + u^{*}{}_{o}) + \delta^{a}_{c}\, \mathbb{P}_{\nu\gamma} \quad , \qquad (4.31)$$

where $S_{\nu\gamma}{}^{a}{}_{c}$ is the curvature written in terms of affinities $\mathbb{A}_{\nu}$, and $\mathbb{P}_{\nu\gamma}$ is the $\mathbb{O}$-curvature(17), written with the $\mathbb{O}$-affinities, $\mathbb{\Gamma}_{\nu}$.

Thus the above analysis completes the geometrical treatment of tangent spaces on the quaternionic and octonionic algebra. In the Table I are resumed the principal geometrical objects obtained in the above calculations, where we can observe that these maintain essentially their form when we go from real tangent space to the complex one, and from there, to the quaternions and octonions, the last one being a a non-associative algebra. In the next section we will analyse the possible physical considerations about these spaces.

5. Physical considerations about the transformation law in a tangent space associated to quaternionic and octonionic internal spaces.

In the General Relativity if we consider a vector in the vierbein space, namely, V^{a}, its transformation law is given by:

$$V'^{a} = L^{a}{}_{b} V^{b} \qquad (5.1)$$

where $L = (L^{a}{}_{b})$ is the transformation matrix which characterize local rotations in the vierbein space (SO(3,1)). Under an infinitesimal transformation of first order, the expansion of L is given by

$$L \cong 1\!\!1 + \varepsilon \ , \qquad L^{-1} \cong 1\!\!1 - \varepsilon \ , \qquad (5.2)$$

where the $\varepsilon = \varepsilon\,(x)$ matrices are antisymmetric and are characteristic of an infinitesimal rotation.

We saw in Section-2 that the transformation matrix, L, for the non-symmetric theory is complex. Therefore, under an infinitesimal transformation we have in this case,

$$L \cong 1\!\!1 + \varepsilon + i\mu \quad , \qquad L^{-1} \cong 1\!\!1 - \varepsilon - i\mu \quad , \qquad (5.3)$$

where $\varepsilon = \varepsilon(x)$ are infinitesimal rotation matrices and $\mu = \mu(x)$ are symmetric infinitesimal matrices and must be related with electromagnetism. These last ones can be written as:

$$\mu_{ab} = (a + 1\!\!1 \tfrac{1}{4}\, \mathrm{tr}\ \mu)_{ab} \quad , \qquad (5.4)$$

where a is a symmetric trace free matrix. If we consider a particular transformation,

$$L \cong 1\!\!1 + i\,\tfrac{1}{4}\,K \quad , \qquad K = \mathrm{tr}\ \mu \ , \qquad (5.5)$$

the affinity Λ_α of the non-symmetric theory is transformed as:

$$\Lambda'_{\alpha} \cong \Lambda_{\alpha} - \tfrac{i}{4}\,K_{,\alpha} \qquad (5.6)$$

or,

$$\mathrm{tr}\,\Lambda'_\alpha \cong \mathrm{tr}\,\Lambda_\alpha - i\,K_{,\alpha} \quad . \tag{5.7}$$

which is similar to the gauge transformation of an electromagnetic potential. In the same way, the complex part on non-symmetric curvature $S_{\mu\nu}$, given in (2.17), will therefore, be related to the Maxwell electromagnetic field tensor.

On the quaternionic tangent space the transformation law of connection Λ_ν is given by (3.15). In terms of components it is written as:

$$\Lambda'_\nu = L_o\,\Lambda_\nu\,L_o^{-1} - L_i\,\Lambda_\nu\,L_i^{-1} - (L_{o,\nu}\,L_o^{-1} - L_{i,\nu}\,L_i^{-1}) \;, \tag{5.8}$$

plus the condition:

$$\begin{aligned} &L_o\,\Lambda_\nu\,L_k^{-1} + L_k\,\Lambda_\nu\,L_o^{-1} + \varepsilon_{ijk}\,L_i\,\Lambda_\nu\,L_j^{-1} + \\ &- (L_{o,\nu}\,L_k^{-1} + L_{k,\nu}\,L_o^{-1} + \varepsilon_{ijk}\,L_{i,\nu}\,L_j^{-1}) = 0 \end{aligned} \tag{5.9}$$

Under an infinitesimal transformation of first order of $\mathbb{L} = L_o\omega_o + L_i\omega_i$, the component L_o will have an expressiom similar to (5.3), of the type of non-symmetric theory, while the expansion of L_i is easily obtained when we consider that we must have, equivalently, in the quaternionic tangent space, the condition:

$$\mathbb{L}\,\mathbb{L}^{-1} = \mathbb{L}^{-1}\,\mathbb{L} = \mathbb{1}\omega_o \quad , \quad \mathbb{1} = (\delta^a_b) \tag{5.10}$$

In terms of componentes this is written as:

$$L_o\,L_o^{-1} - L_i\,L_i^{-1} = L_o^{-1}\,L_o - L_i^{-1}\,L_i = \mathbb{1} \;, \tag{5.11}$$

plus the condition:

$$L_o\,L_k^{-1} + L_k\,L_o^{-1} + \varepsilon_{ijk}\,L_i\,L_j^{-1} = L_o^{-1}\,L_k + L_k^{-1}\,L_o + \varepsilon_{ijk}\,L_i^{-1}\,L_j = 0 \;. \tag{5.12}$$

Therefore we obtain for the expansion of L_i in first order:

$$L_i \cong (\eta_i{}^a{}_b) \quad , \qquad L_i^{-1} \cong - (\eta_i{}^a{}_b) \tag{5.13}$$

where metric $\eta_i = \eta_i(x)$ are infinitesimal.

The equation for the transformation law of Λ_ν in the quaternionic tangent space, in first order, is then:

$$\Lambda'_\nu \cong \Lambda_\nu - \left[\Lambda_\nu , \varepsilon + i\mu \right] + (\varepsilon + i\mu)_{,\nu} \tag{5.14}$$

like in non-symmetric theory, plus the condition:

$$\left[\Lambda_\nu , \eta_i \right] + \eta_{i,\nu} = 0 \ . \tag{5.15}$$

The solution of (5.15) is obtained expanding $\eta_i(x)$ and $\Lambda_\nu (x)$ in power series of x^ν ,

$$\begin{aligned} \eta_i (x) &= \eta_i (o) + x^\nu \phi_{\nu i} (o) \quad , \\ \Lambda_\nu (x) &= \Lambda_\nu (o) + x^\alpha S_{\alpha\nu} (o) \ . \end{aligned} \tag{5.16}$$

Replacing (5.16) in (5.15), we obtain:

$$\begin{aligned} &\phi_{\nu i} (o) = - \left[\Lambda_\nu (o) , \eta_i (o) \right] \quad , \\ &\left[S_{\alpha\nu} (o), \eta_i (o) \right] = - \left[\Lambda_\nu (o), \phi_{\alpha i}(o) \right] = - \left[\Lambda_\nu(o), \left[\Lambda_\alpha(o), \eta_i(o) \right] \right] \ , \\ &\left[S_{\alpha\nu} (o), \phi_{\beta i} (o) \right] = \left[S_{\alpha\nu} (o), \left[\Lambda_\beta(o), \eta_i (o) \right] \right] = 0 \end{aligned} \tag{5.17}$$

The solution of (5.15) is then:

$$\eta_i (x) = \eta_i (o) - x^\nu \left[\Lambda_\nu (o), \eta_i (o) \right] \quad , \tag{5.18}$$

where $n_i\,(o)$ is a constant infinitesimal matrix with the restrictions (5.17). In a first order, $n_i\,(o)$ characterizes an interference in the transformation law of the type of non-symmetric theory, described by L_o, because of association with the quaternionic internal space, having therefore, origin in the Yang-Mills fields. When we consider the curvature (3.25), $S_{\nu\alpha}$ is related as before, with gravitation and electromagnetism, while $P_{\nu\alpha}$, the quaternionic internal curvature, is proportional to the Yang-Mills field, $\vec{f}_{\nu\alpha}$.

In terms of components, the transformation law for octonionic Λ_ν, (4.22), is written as:

$$\begin{aligned}\Lambda'_\nu = \frac{1}{2}\Big[&n_o\,\Lambda_\nu\,n_o^{-1} + m_o\,\Lambda_\nu\,m_o^{-1} - (n_{o,\nu}\,n_o^{-1} - n_{i,\nu}\,m_i^{-1}) +\\ &- n_i\,\Lambda_\nu\,m_i^{-1} - m_i\,\Lambda_\nu\,n_i^{-1} - (m_{o,\nu}\,m_o^{-1} - m_{i,\nu}\,n_i^{-1})\Big]\,,\end{aligned} \tag{5.19}$$

along with the conditions:

$$\begin{aligned}&n_o\,\Lambda_\nu\,n_k^{-1} + n_k\,\Lambda_\nu\,m_o^{-1} + \varepsilon_{ijk}\,m_i\,\Lambda_\nu\,m_j^{-1} +\\ &- (n_{o,\nu}\,n_k^{-1} + n_{k,\nu}\,m_o^{-1} + \varepsilon_{ijk}\,m_{i,\nu}\,m_j^{-1}) = 0\,,\\ &m_o\,\Lambda_\nu\,m_k^{-1} + m_k\,\Lambda_\nu\,n_o^{-1} + \varepsilon_{ijk}\,n_i\,\Lambda_\nu\,n_j^{-1} +\\ &- (m_{o,\nu}\,m_k^{-1} + m_{k,\nu}\,n_o^{-1} + \varepsilon_{ijk}\,n_{i,\nu}\,n_j) = 0\,.\end{aligned} \tag{5.20}$$

Again, we must have for transformation matrices in the octonionic tangent space

$$\mathbb{L}\,\mathbb{L}^{-1} = \mathbb{L}^{-1}\,\mathbb{L} = \mathbb{1}\,(u_o + u^*_o)\,, \qquad \mathbb{1} = (\delta^a_b)\,, \tag{5.21}$$

where **L** is given in (4.16), which gives in terms of components:

$$\frac{1}{2}\,(n_o\, n_o^{-1} + m_o\, m_o^{-1} - n_i\, m_i^{-1} - m_i\, n_i^{-1}) =$$
$$(5.22)$$
$$= \frac{1}{2}\,(n_o^{-1}\, n_o + m_o^{-1}\, m_o - n_i^{-1}\, m_i - m_i^{-1}\, n_i) = \mathbb{1} \quad , \quad \mathbb{1} = (\delta^a_b) \ ,$$

plus the conditions:

$$n_o\, n_k^{-1} + n_k\, m_o^{-1} + \varepsilon_{ijk}\, m_i\, m_j^{-1} = m_o\, m_k^{-1} + m_k\, n_o^{-1}\, \varepsilon_{ijk}\, n_i\, n_j^{-1} =$$
$$(5.23)$$
$$= n_o^{-1}\, n_k + n_k^{-1}\, m_o + \varepsilon_{ijk}\, m_i^{-1} m_j = m_o^{-1}\, m_k + m_k^{-1}\, n_o + \varepsilon_{ijk}\, n_i^{-1}\, n_j = 0 \ .$$

This means that we must have once more the matrices n_o and m_o of the type L_o of the non-symmetric theory, and $\vec{m}$ and $\vec{n}$ being the "interference" due to the association of tangent space of tetrads with the octonionic internal space.

We will take here, looking for a more objective physical interpretation, $m_o = n_o = L_o$ of the non-symmetric theory. Therefore, in an infinitesimal expansion of first order of octonionic $\mathbb{L}$, we have:

$$m_o = n_o = L_o \cong \mathbb{1} + \varepsilon + i\mu \quad , \quad L_o^{-1} \cong \mathbb{1} - \varepsilon - i\mu \quad , \qquad (5.24)$$

and,

$$m_i \cong (\alpha_i{}^a{}_b) \quad , \quad m_i^{-1} \cong -\,(\alpha_i{}^a{}_b) \quad ,$$
$$(5.25)$$
$$n_i \cong (\beta_i{}^a{}_b) \quad , \quad n_i^{-1} \cong -\,(\beta_i{}^a{}_b) \quad .$$

α_i and β_i are infinitesimal matrices such that in the limit $\alpha_i \to \beta_i$ we re obtain the quaternionic case.

Again therefore, the transformation law of Λ_ν, of the type of (5.5), in the octonionic tangent space, is related with transformation of electromagnetic gauge. The "interference", in first order, being given by the equations:

$$\left[\Lambda_\nu, \alpha_i\right] + \alpha_{i,\nu} \cong 0 , \tag{5.26}$$

$$\left[\Lambda_\nu, \beta_i\right] + \beta_{i,\nu} \cong 0 , \tag{5.27}$$

where (5.24) was considered. We conclude then, from (5.26) and (5.27), that in first order, the "interference" α_i and β_i , because of the presence of the octonionic internal space, are at least proportional and of the Yang-Mills type, in this case. (We must remember that we are dealing with a split octonionic algebra.) With regard to the curvature (4.31), once again $S_{\nu\gamma}$ is related to gravitation and electromagnetism, because it consists of the non-symmetric part of the theory, while $\mathbb{P}_{\nu\gamma}$, the internal octonionic curvature, is of the type of Yang-Mills field, being however, of two types.

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(8) - See ref.(1), eq. (3.18).

(9) - See ref. (1), eq. (4.6).

(10) - See ref. (1), Appendix I.

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(12) - See ref. (1), eq. (5.21).

(13) - See ref. (1), eqs. (5.25) and (5.26).

Table I

	General Relativity Theory ($\mathbb{R}$ - algebra)	Non-symmetric theory ($\mathbb{C}$ - algebra)	Borchsenius or Quaternionic theory ($\mathbb{Q}$ - algebra)	Octonionic Theory ($\mathbb{O}$ - algebra)
Line element	$ds^2 = dx^a dx^b \eta_{ab}$ $dx^a = h^a_\mu dx^\mu$	$ds^2 = dx^{*a} dx^b \eta_{ab}$ $dx^a = e^a_\mu dx^\mu ; dx^{*a} = e^{*a}_\mu dx^\mu$	$ds^2 = \frac{1}{2} Tr\,(d\mathbf{x}^a d\mathbf{x}^b \eta_{ab})$ $d\mathbf{x}^a = E^a_\mu dx^\mu;\ d\mathbf{x}^{\dagger a} = E^{\dagger a}_\mu dx^\mu$	$ds^2 = \frac{1}{4} Tr\,(d\mathbf{x}^a d\mathbf{x}^b \eta_{ab})$ $dx^a = H^a_\mu d\mathbf{x}^\mu;\ d\mathbf{x}^{\dagger a} = H^{\dagger a}_\mu dx^\mu$
Metric relation	$g^{\mu\nu} = h^\mu_a h^\nu_b \eta^{ab}$ $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ $g_{\mu\nu} = g_{\nu\mu}$	$g^{\mu\nu} = e^{*\mu}_a e^\nu_b \eta^{ab}$ $g_{\mu\nu} = e^{*a}_\nu e^b_\mu \eta_{ab}$ $g^*_{\mu\nu} = g_{\nu\mu}$	$G^{\mu\nu} = E^{\dagger\mu}_a E^\nu_b \eta^{ab}$ $G_{\mu\nu} = E^{\dagger a}_\nu E^b_\mu \eta_{ab}$ $G^\dagger_{\mu\nu} = G_{\nu\mu}$	$\mathbf{G}^{\mu\nu} = H^{\dagger\mu}_a H^\nu_b \eta^{ab}$ $\mathbf{G}_{\mu\nu} = H^{\dagger a}_\nu H^b_\mu \eta_{ab}$ $\mathbf{G}^\dagger_{\mu\nu}(s,r) = \mathbf{G}_{\nu\mu}(s,r)$
Ortogonality relations	$h^\mu_a h^b_\mu = \delta^b_a$ $h^\mu_a h^a_\nu = \delta^\mu_\nu$	$e^{*b}_\mu e^\mu_a = \delta^b_a$ $e^{*b}_\mu e^\nu_b = \delta^\nu_\mu$	$E^\mu_a E^{\dagger b}_\nu = \delta^b_a \omega_o$ $E^{\dagger a}_\alpha E^\nu_a = \delta^\nu_\alpha \omega_o$	$H^\mu_a H^{\dagger b}_\mu = \delta^b_a (u_o + u^*_o)$ $H^{\dagger a}_\alpha H^\nu_a = \delta^\nu_\alpha (u_o + u^*_o)$
Pseudo-Orthogonality condition	$L^T \eta L = \eta$ "T" acting upon tangent space - $\mathbb{R}$	$L^\dagger \eta L = \eta$ "†" acting upon tangent space - $\mathbb{C}$	$\mathbb{L}^{\dagger a}{}_c \eta_{ab} \mathbb{L}^b{}_d = \eta_{cd} \omega_c$ "†" acting upon Quaternionic space - $\mathbb{Q}$	$\mathbb{L}^{\dagger a}{}_c \eta_{ab} \mathbb{L}^b{}_d = \eta_{cd} (u_o + u^*_o)$ "†" acting upon Octonionic space - $\mathbb{O}$
Tangent connection	$\Lambda_{\mu ab} = - \Lambda_{\mu ba}$	$\Lambda^*_{\mu ab} = - \Lambda_{\mu ba}$ $\Lambda_\mu = (\Lambda_{\mu ab} + i\, \Lambda_{\mu ab})$	$\Lambda_\mu = (\Lambda_{\mu ab})\, \omega_o$ $\lambda^*_{\mu ab} = - \lambda_{\mu ba}$	$\Lambda_\mu = (q_{\mu ab})(u_o + u^*_o)$ $q^*_{\mu ab} = - q_{\nu ba}$
Tangent Transformation law on Λ_μ	$\Lambda'_\mu = L \Lambda_\mu L^{-1} - L_{,\mu} L^{-1}$	$\Lambda'_\mu = L \Lambda_\mu L^{-1} - L_{,\mu} L^{-1}$ $\Lambda^{*\prime}_\mu = L^* \Lambda^*_\mu L^{*-1} - L^*_{,\mu} L^{*-1}$	$\Lambda_\mu = \mathbb{L} \Lambda_\mu \mathbb{L}^{-1} - \mathbb{L}_{,\mu} \mathbb{L}^{-1}$ $\mathbb{L} = L_o \omega_o + L_i \omega_i$ $i = 1,2,3$	$\Lambda_\mu = \mathbb{L} \Lambda_\mu \mathbb{L}^{-1} - \mathbb{L}_{,\mu} \mathbb{L}^{-1}$ $\mathbb{L} = n_o u^*_o + n_i u^*_i + m_o u_o + m_i u_i$ $i = 1,2,3$
Tangent Curvature	$S_{\mu\nu} = \Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - [\Lambda_\nu, \Lambda_\gamma]$	$S_{\nu\gamma} = \Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - [\Lambda_\nu, \Lambda_\gamma]$ $S_{\nu\gamma} = (S_{\nu\gamma ab} + i\, S_{\nu\gamma \underline{ab}})$	$\mathbf{S}_{\nu\gamma} = \mathbf{\Lambda}_{\nu,\gamma} - \mathbf{\Lambda}_{\gamma,\nu} - [\mathbf{\Lambda}_\nu, \mathbf{\Lambda}_\gamma]$ $= S_{\nu\gamma}\, \omega_o + P_{\nu\gamma}$ $\mathbf{\Lambda}_\nu = \Lambda_\nu + \Gamma_\nu$ $\Gamma_\nu = -\vec{C}_\nu \cdot \vec{\omega}$	$\mathbf{S}_{\nu\gamma} = \mathbf{\Lambda}_{\nu,\gamma} - \mathbf{\Lambda}_{\gamma,\nu} - [\mathbf{\Lambda}_\nu, \mathbf{\Lambda}_\gamma]$ $= S_{\nu\gamma}\, \omega_o + \mathbb{P}_{\nu\gamma}$ $\mathbf{\Lambda}_\nu = \Lambda_\nu + \mathbf{\Gamma}_\nu$ $\mathbf{\Gamma}_\nu = -\vec{L}_\nu \cdot \vec{u}^* - \vec{K}_\nu \cdot \vec{u}$