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N-POINT CORRELATIONS FOR BIASED GALAXY FORMATION

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Abstract

A general series expansion of arbitrary accuracy is given for the correlation function of 'biased' regions in terms of the mass autocorrelations. The first term of the expansion agrees with the limit of Kaiser (1984). It is shown that the 'biased' correlation function stays zero, whenever the mass correlation is zero. A simple formula to calculate the N-point correlations to arbitrary accuracy is given as well. Keeping the leading terms in our expansion we obtain the formulae of Politzer and Wise (1984). We calculate the maximum values of the N-point correlation functions at zero separation as a function of the threshold ν, and show that they satisfy the scaling law $\xi_\nu^{(N)} \propto \xi_\nu^{N-1}$.

1. INTRODUCTION

In order to explain the large cluster-cluster correlations, Kaiser(1984) suggested to associate the rich clusters with regions above a critical threshold of the overdensity. Later, in order to avoid simultanous problems with galaxy pair velocities and the fluctuations of the microwave background, the same idea of 'biasing', forming galaxies only in certain regions of high overdensity was used to enhance the galaxy correlations with respect to the mass autocorrelations (White 1985).

It has been known for years that the higher order galaxy correlation data have a certain symmetry (Groth and Peebles 1977, Fry and Peebles 1978), where the reduced N-point correlation function is $\propto \xi^{N-1}$. The three point correlation function is found to be $\zeta_{123} = Q(\xi_{12}\xi_{23} + \xi_{23}\xi_{13} + \xi_{13}\xi_{12})$, where $Q \approx 1$. Different hierarchies of that type were discussed by Fry (1984a). It is therefore of considerable interest whether the correlations arising via 'biasing' of Gaussian fluctuations satisfy these scaling relations.

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We assume that the fluctuations of the linear overdensity f are Gaussian with dispersion $\langle f^2 \rangle = \sigma^2$, correlation function $\langle f_1 f_2 \rangle = \xi(r_{12})$, and that all higher order correlations are zero. Here and throughout the rest of the paper we will use normalized variables $y = f/\sigma$, which have dispersion 1 and correlations $w(r) = \xi(r)/\sigma^2$. Then the 2-point correlation function of the regions where $f > \nu\sigma$ is given by the usual expression for the bivariate Gaussian :

$$1 + \xi_\nu = \left[\frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2} \int_\nu^\infty \int_\nu^\infty \frac{dy_1 dy_2}{2\pi\sqrt{(1-w^2)}} \exp\left(-\frac{y_1^2 + y_2^2 - 2wy_1y_2}{2(1-w^2)}\right) \tag{1}$$

In this context this was first calculated by Kaiser (1984). Due to the nonlinear occurence of $w(r)$ in the exponent only an approximate expression for ξ_ν was given there, using $\nu \gg 1$ and expanding the result up to first order in w, giving $\xi_\nu \simeq \nu^2 w(r)$. In an attempt to improve the accuracy of the approximation, Politzer and Wise (1984) expanded the exponent up to linear order in $w(r)$, yielding

$$1 + \xi_\nu \simeq e^{\nu^2 w(r)} \tag{2}$$

This result was also generalized for the N-point correlation functions of the biased regions. All these expansions are only strictly valid when $\nu \gg 1$ and $w \ll 1$. It would be very advantageous to have a systematic way to calculate the correlation function to higher order terms, and compare the results to the above approximations.

A similar technique was used by Bardeen, Bond, Kaiser and Szalay (1986) to approximate the correlation function of density peaks above a certain level ν. As discussed there, it is expected that peaks are a more realistic description of the galaxies than all mass above a certain overdensity, however the mathematical treatment of peaks is unfortunately extremely complicated.

Due to the importance of the applications, we show how to calculate the N-point correlations to arbitrary accuracy, discuss the validity of the various approximations, provide the asymptotic values of the N-point correlation functions at their highest point at zero spatial separation, and discuss the scaling of these values. Similar calculations are in progress to treat the correlations of peaks (Bardeen, Bond, Jensen and Szalay 1986).

2. N-POINT CORRELATIONS

Here we consider the N-point correlation functions of the regions above the threshold ν. As Politzer and Wise, we introduce the symbol P_N as

$$P_N = \frac{1}{(2\pi)^{N/2}(\det \mathbf{M})^{1/2}} \int_{\nu}^{\infty} ... \int_{\nu}^{\infty} d^N y \;\; \exp(-\frac{1}{2}\mathbf{y}\mathbf{M}^{-1}\mathbf{y}) \tag{3}$$

where $\mathbf{M}$ is the correlation matrix, which we split to the unit matrix $\mathbf{E}$ and the cross-correlation $\mathbf{W}$, with elements $w_{ij} = w(r_{ij})$, $w_{ii} = 0$. The unreduced N-point correlation function can then be expressed as

$$1 + \xi_{\nu}^{(N)} = \frac{1}{P_1^N} P_N \tag{4}$$

In order to calculate the integral in P_N we write the joint distribution using its Fourier transform. This enables us to avoid the major source of complication: the inversion of $\mathbf{M}$.

$$P_N = \frac{1}{(2\pi)^N} \int_{\nu}^{\infty} ... \int_{\nu}^{\infty} d^N y \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} d^N x \;\; \exp(-i\mathbf{y}\mathbf{x}) \exp(-\frac{1}{2}\mathbf{x}\mathbf{M}\mathbf{x}) \tag{5}$$

Using $Q = -\frac{1}{2}\mathbf{x}\mathbf{W}\mathbf{x}$, the last term of the integrand can be expanded:

$$\exp(-\frac{1}{2}\mathbf{x}\mathbf{M}\mathbf{x}) = \exp(-\frac{1}{2}\mathbf{x}\mathbf{x}) \sum_{m=0}^{\infty} \frac{Q^m}{m!} \tag{6}$$

Q can be expressed as a differential operator :

$$Q \exp(-i\mathbf{y}\mathbf{x}) = \sum_{k<l}^{N} (\frac{\partial}{\partial y_k} w_{kl} \frac{\partial}{\partial y_l}) \exp(-i\mathbf{y}\mathbf{x}) = \hat{Q} \exp(-i\mathbf{y}\mathbf{x}) \tag{7}$$

Since $\hat{Q}$ contains derivatives by y_k, it can be moved outside the x integrals, and we can do the reverse Fourier transform, but with a diagonal correlation matrix. The result is an expansion of P_N with powers of $\hat{Q}$.

$$P_N = \sum_{m=0}^{\infty} \frac{1}{(2\pi)^{N/2}} \int_{\nu}^{\infty} \ldots \int_{\nu}^{\infty} d^N y \, \frac{\hat{Q}^m}{m!} \exp(-\frac{1}{2}\mathbf{y}\mathbf{y}) \tag{8}$$

Using the multinomial expansion for $\hat{Q}^m$, the N-point correlation function can be written as a sum over m. The m_{kl} are such, that $m_{kl} = 0$, if $k \geq l$, their sum is $\sum_{k,l} m_{kl} = m$, and the m_k are defined as $m_k = \sum_l (m_{kl} + m_{lk})$.

$$1 + \xi_{\nu}^{(N)} = \sum_{m=0}^{\infty} \frac{w_{12}^{m_{12}}}{m_{12}!} \frac{w_{13}^{m_{13}}}{m_{13}!} \ldots A_{m_1} A_{m_2} \ldots A_{m_N} \tag{9}$$

The expansion coefficients are given by

$$A_0 = 1; \quad A_n = \frac{2xH_{n-1}(x)2^{-n/2}}{\sqrt{\pi} x e^{x^2} \mathrm{erfc}(x)}; \quad x = \frac{\nu}{\sqrt{2}} \tag{10}$$

As $\nu \to \infty$ the Hermite polynomials are dominated by the leading term $H_n(x) \approx (2x)^n$ and the denominator of A_n is approaching 1, so $A_n \to \nu^n$. Therefore

$$1 + \xi_{\nu}^{(N)} = \sum_{m=0}^{\infty} \frac{(\nu^2 w_{12})^{m_{12}}}{m_{12}!} \frac{(\nu^2 w_{13})^{m_{13}}}{m_{13}!} \ldots = e^{\nu^2 (w_{12} + w_{13} + \ldots)} \tag{11}$$

precisely the expression of Politzer and Wise, hereafter PW. For the special case of 2-point correlations

$$\xi_{\nu} = \sum_{m=1}^{\infty} \frac{w^m}{m!} A_m^2 \tag{12}$$

Taking only the first term, in the high ν limit we obtain $\xi_{\nu} \approx \nu^2 w$, Kaiser's formula. Eq.(12) also indicates that whenever $w = 0$, ξ_{ν} must be 0 as well.

3. ASYMPTOTIC VALUES

In the limit of small separations the correlation coefficients can be expanded as $w_{ij} = 1 - \alpha r_{ij}^2/2 = 1 - C_{ij} r^2$. Here $i, j = 1, 2, \ldots, N$ and r is a small parameter measuring the linear size of the geometrical configuration spanned by the N points. As $r \to 0$, the C_{ij} remain constant. The N-point correlation matrix reduces to the following form :

$$\mathbf{M} = \mathbf{M_0} - r^2 \mathbf{C}; \quad (M_0)_{ij} = 1; \quad C_{ii} = 0; \quad C_{ij} = C_{ji} \tag{13}$$

In order to calculate the N point correlation $\xi_\nu^{(N)}$ we must first evaluate the integral P_N. Because $\mathbf{M_0}$ is singular, it is not possible simply to expand the exponential around $r = 0$, instead we shall try to isolate the singularity by replacing the y_i's with a more suitable set of coordinates. Let us define new orthonormal coordinates $z_1, z_2, \ldots z_N$ with $z_1 = (y_1 + y_2 + \ldots + y_N)/\sqrt{N}$ and $z_2, \ldots, z_N$ arbitrary. Let $\mathbf{U}$ denote the orthogonal matrix relating the y_i's and the z_i's, i.e. $z_i = U_{ij} y_j$. Now the integral (3) may be written as

$$\frac{1}{(2\pi)^{N/2} (\det \tilde{\mathbf{M}})^{1/2}} \int \ldots \int d^N z \, \exp(-\frac{1}{2} \mathbf{z} \tilde{\mathbf{M}}^{-1} \mathbf{z}) \tag{14}$$

where $\tilde{\mathbf{M}} = \mathbf{U M U^T}$. One finds that $\tilde{\mathbf{M}}$ has the following structure to leading order in r^2

$$\tilde{\mathbf{M}} = \begin{pmatrix} N & r^2 \mathbf{R} \\ r^2 \mathbf{R}^T & r^2 \mathbf{S} \end{pmatrix} \quad ; \qquad \det \tilde{\mathbf{M}} = N r^{2(N-1)} \det \mathbf{S} \tag{15}$$

where $\mathbf{R}$ is a $1 \times (N-1)$ and $\mathbf{S}$ is an $(N-1) \times (N-1)$ matrix. Therefore to lowest order

$$\tilde{\mathbf{M}}^{-1} = \begin{pmatrix} 1/N & -\mathbf{R S}^{-1}/N \\ -\mathbf{S}^{-1} \mathbf{R^T}/N & \mathbf{S}^{-1}/r^2 \end{pmatrix} \tag{16}$$

Using these expressions we see that in the limit $r \to 0$ the integral over $z_2, \ldots, z_N$ reduces to an integral over an $(N-1)$ dimensional δ-function, so we are left with

$$P_N = \frac{1}{\sqrt{2\pi N}} \int\limits_{\sqrt{N}\nu}^{\infty} dz_1 \, \exp(-z_1^2/2N) = \frac{1}{2} \mathrm{erfc}(\frac{\nu}{\sqrt{2}}) = P_1 \tag{17}$$

ie.

$$1+\xi_\nu^{(N)} = \frac{1}{\left[\frac{1}{2}\mathrm{erfc}(\frac{\nu}{\sqrt{2}})\right]^{N-1}} = (1+\xi_\nu)^{N-1} \tag{18}$$

This result means, that if we have N normal Gaussian variates, which are all very strongly correlated, the number of the degrees of freedom is reduced from N to 1.

For the case of the 2-point function we found the behaviour of P_2 in the neighbourhood of $w = 1$:

$$P_2 = P_1 - \frac{1}{2\pi} e^{-\frac{1}{2}\nu^2}\sqrt{1-w^2} \tag{19}$$

This shows, that ξ_ν approaches $w = 1$ with an infinite derivative. Although a similar behaviour is expected for the higher order correlation functions, we did not succed in finding such a simple expression.

Eq.(18) has the right scaling as far as the powers of ξ_ν are concerned, but the amplitudes are very small. For the three point correlations one can see, that we obtain the asymptotic value for Q, satisfying the equality in Eq.(55) of Fry (1984b).

$$Q = \frac{1}{3} - \frac{1}{3\xi_\nu} \tag{20}$$

Typically the expansion in Eq.(9) converges fairly rapidly, the worst case occurs when all correlations are equal. In this special case it is possible to show, by modifying a derivation in Kendall and Stuart (1952), that

$$P_N = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\; e^{-(x-t)^2} \left\{\frac{1}{2}\mathrm{erfc}(bx)\right\}^N \;;\quad w = \frac{b^2}{1+b^2}\;;\quad t = \frac{\nu}{\sqrt{2w}} \tag{21}$$

The asymptotic $w \to 1$ corresponds to $b \to \infty$, in which case $\{\frac{1}{2}\mathrm{erfc}(bx)\}^N$ converges to a step function yielding the previous result, $P_N \to P_1$. For a large w this single integral may be easier to calculate than summing up the series in Eq.(9).

4. NUMERICAL RESULTS

We have numerically evaluated the biased 2-point correlation function ξ_ν as given by Eq.(12) and compared it to the approximations of Kaiser and PW for various values of ν as a function of w. Generally the convergence is very fast, except when $w \approx 1$. The graphs on Fig.1. show the thresholds $\nu = 1.0, 2.0, 3.0$ respectively, with w on logarithmic scale. For small ν both approximations yield systematically low values, caused by taking $\sqrt{\pi} x e^{x^2} \mathrm{erfc}(x) = 1$ in the normalization; the PW curve has the correct shape otherwise. This global error decreases rapidly as higher values of ν are considered, but for larger w the remaining terms in the Hermite polynomials become also significant. In order to show this effect, on Fig.2. we have plotted $\log\,(1 + \xi_\nu)$ against w, where the PW approximation is a straight line with a slope of ν^2. It is obvious, that the real correlation function has a less steep behaviour, although it still seems to be roughly exponential in shape. The Kaiser approximation behaves reasonably only if $w < 0.1$ for $\nu > 2$, otherwise its errors are considerable. The PW approximation is much better, but still if $w > 0.5$, the deviations start to increase. On Fig.3. we have plotted the asymptotic value of ξ_ν for $w = 1$ as a function of ν. This should give reasonable estimates about how large the correlation function becomes at short distances.

We have calculated the 3-point function as well, for an equilateral triangle using Eq.(9). The convergence was not as good as for the 2-point functions, although for $w < 0.6$ it was still very fast. Beyond that point even summing up more than 30 terms was not enough, but the values of Q were converging towards the asymptotically expected 1/3. The results of this calculation are shown on Fig.4. for the cases $\nu = 1.0, 2.0, 3.0$. The dotted lines denote an extrapolation between the results of summing Eq.(11) for $w < 0.5$ and the asymptotic 1/3. Except for the very low w's where Q is rising, our results are in sharp contrast with the Politzer and Wise approximation, which suggests a cubic ξ term in the 3-point function, causing a runaway of Q for high ξ.

Several papers claim recently $Q \approx 1$ for a wide range of dynamic scales in 'biased' galaxy N-body simulations for various values of the threshold ν (Melott and Fry 1985, White et al 1985). We have shown, that this scaling is not a reflection of the biasing in the Gaussian initial conditions, though it may arise through dynamical evolution (Peebles 1980), or that peaks will be the more realistic description of the galaxies, but it is not yet established, whether the scaling is satisfied in that case.

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REFERENCES

Bardeen,J.M., Bond,J.R., Kaiser,N. and Szalay,A.S. 1986, Ap.J. in press.

Bardeen,J.M., Bond,J.R., Jensen,L.G. and Szalay,A.S. 1986, in progress.

Davis,M., Efsthathiou,G.P., Frenk,C.S. and White,S.D.M. 1984, Ap.J., **288**, 836.

Frenk, C.S., White, S.D.M. and Davis, M. 1983, Ap.J., **271**, 417.

Fry,J.N. 1984a, Ap.J., **277**, L5.

Fry,J.N. 1984b, Ap.J., **279**, 499.

Fry,J.N. and Peebles,P.J.E. 1978, Ap.J., **221**, 19.

Groth,E.J. and Peebles,P.J.E. 1977, Ap.J.Suppl., **217**, 385.

Kaiser,N. 1984, Ap.J., **284**, L9.

Kendall,M. and Stuart,A. 1977, *The Advanced Theory of Statistics*, Macmillan Press.

Melott,A.L. and Fry,J.N. 1985. Ap.J. in press.

Peebles,P.J.E. 1980, *The Large Scale Structure of the Universe*, Princeton University Press.

Politzer,H.D. and Wise,M.B. 1984, Ap.J., **285**, L1.

White,S.D.M. Proc. Inner Space/Outer Space, The University of Chicago Press.

FIGURE CAPTIONS

Fig.1. The different approximations to the correlation function of the biased regions for thresholds $\nu = 1, 2, 3$ as a function of the normalized mass correlation $w = \xi(r)/\xi(0)$. This graph shows the approximations for low values of w. The solid lines are the results of this paper, with errors less than 10^{-3}.

Fig.2. The correlation function of the biased regions for $\nu = 1, 2, 3$ plotted against w on linear scale, to show ξ_ν for large w. The Politzer-Wise approximation is a straight line in this plot. Note the deviations at the high end.

Fig.3. The $w = 1$ asymptotic values of the 2-point correlation function $\xi_\nu = \frac{1}{P_1}$, shown for a range of ν.

Fig.4. The 3-point correlation coefficient Q is plotted as a function of the mass correlation w for $\nu = 1, 2, 3$. The calculation was made for an equilateral triangle, using the expansion given by Eq.(9). The dotted line is an extrapolation between the results of the expansion and the asymptotic value.

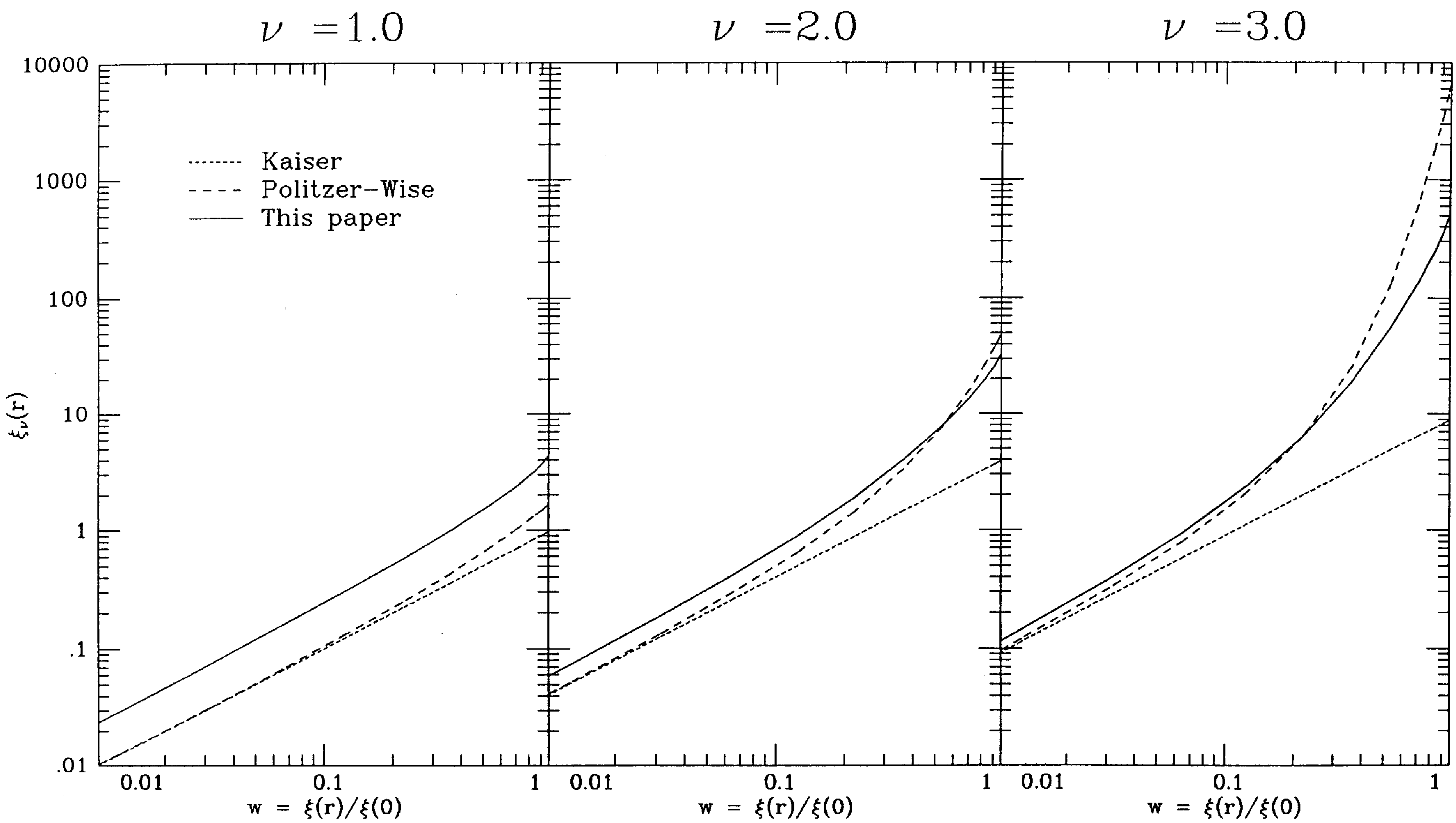
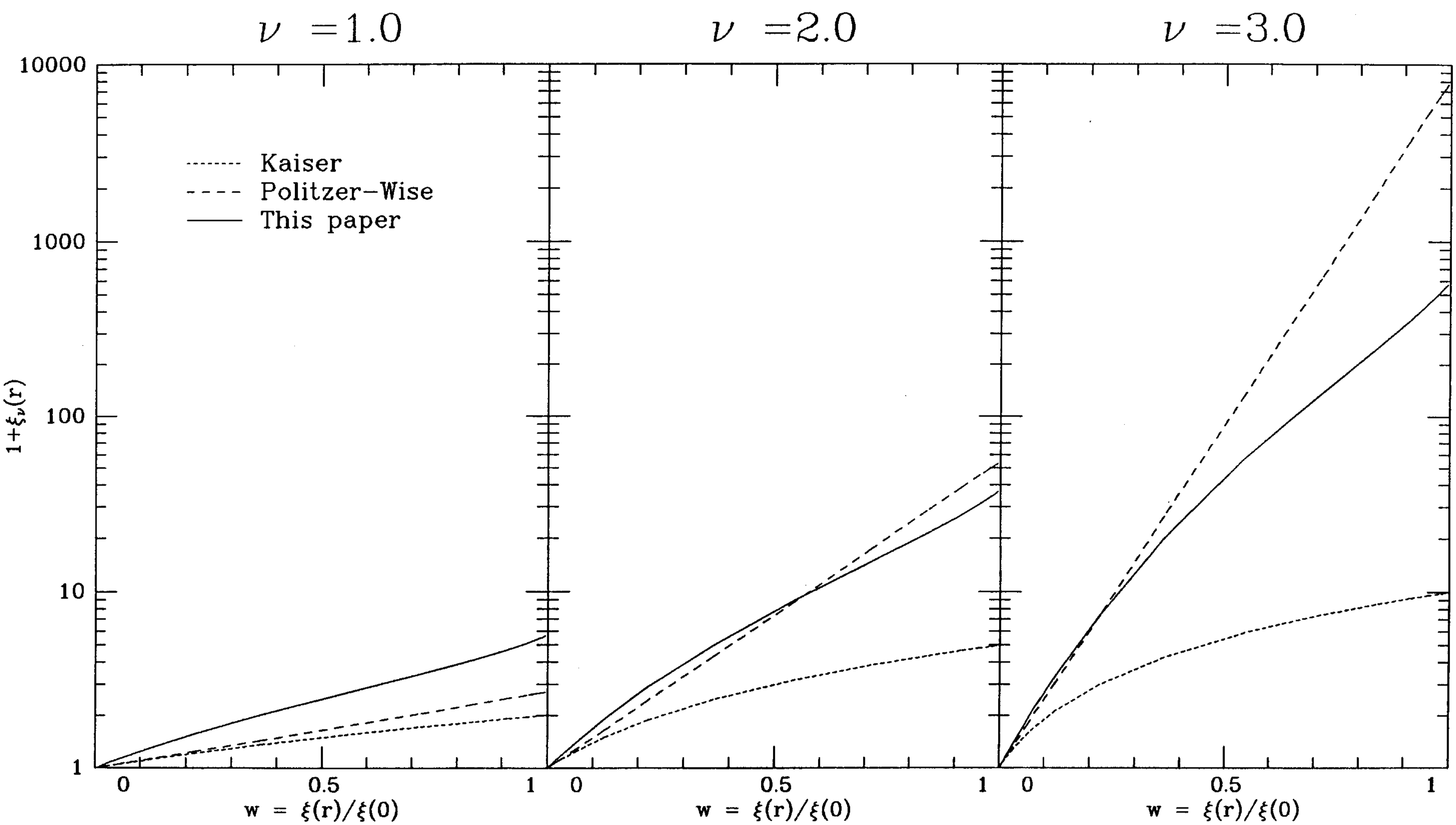

ν =1.0
ν =2.0
ν =3.0
10000
1000
100
10
1
.1
.01
Kaiser
Politzer-Wise
This paper
ξ_ν(r)
0.01
0.1
1
w = ξ(r)/ξ(0)

FIGURE 1



ν =1.0
ν =2.0
ν =3.0
Kaiser
Politzer–Wise
This paper
1+ξν(r)
10000
1000
100
10
1
0
0.5
1
w = ξ(r)/ξ(0)

FIGURE 2

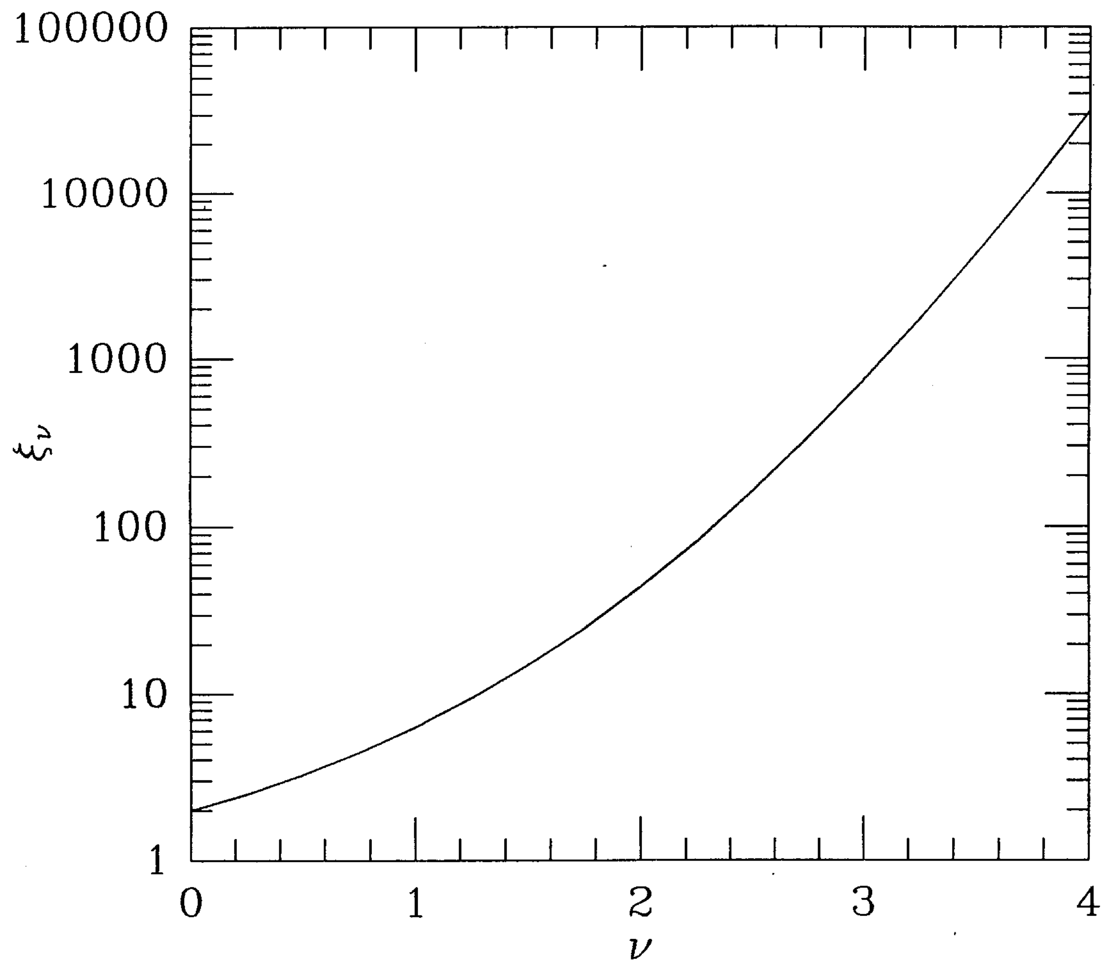


FIGURE 3

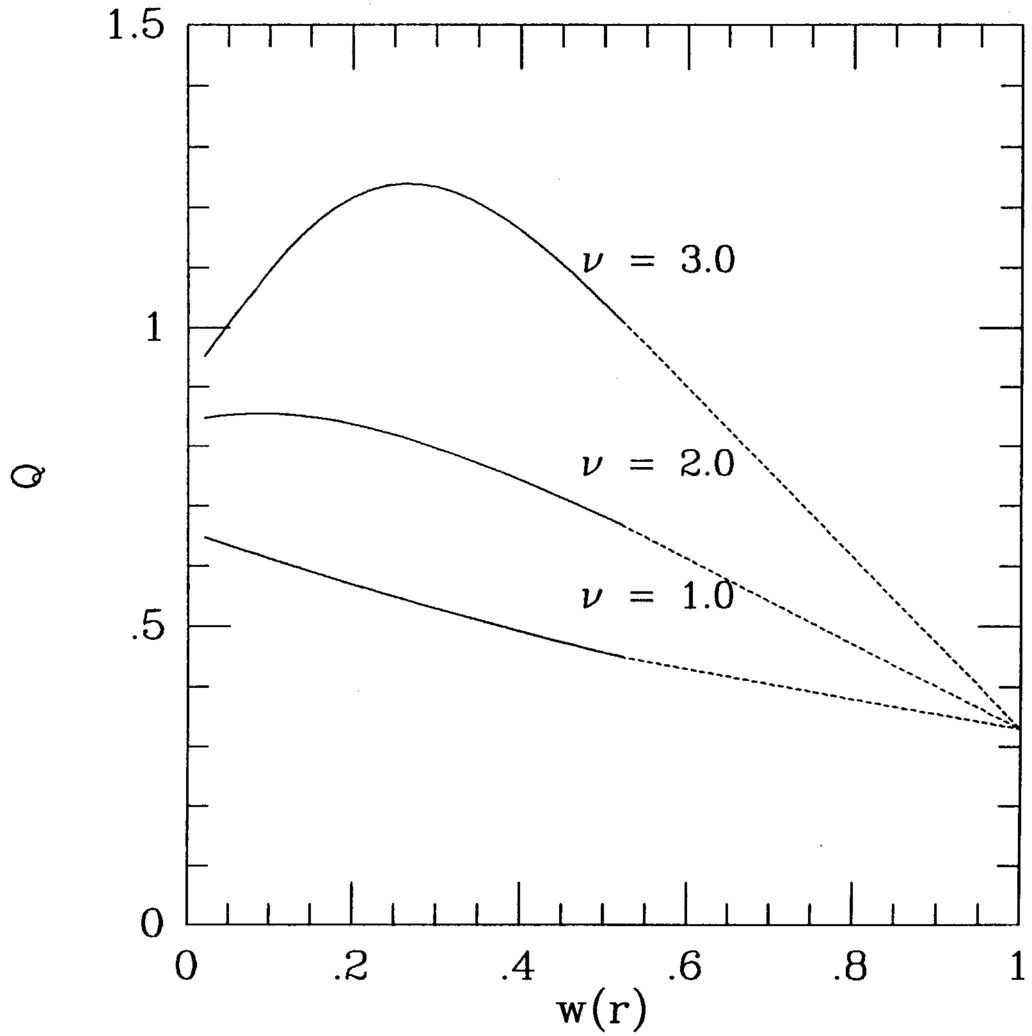


FIGURE 4