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THE LARGE SCALE MICROWAVE BACKGROUND FLUCTUATIONS, GAUGE–INVARIANT FORMALISM

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Abstract

The explicitly gauge–invariant formula for the large–scale fluctuations of the temperature of the microwave background radiation is obtained. The formula is applicable to the wide class of cosmological models (i.e. multicomponent and nonflat) based on Robertson – Walker metrics. Some specific cases are briefly discussed and in the case of general flat models the amplitudes of the multipole moments of the temperature pattern are obtained as functions of the baryonic perturbation quantities.

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I. INTRODUCTION

The observed isotropy of the microwave background radiation (MBR) puts stringent limits on parameters of cosmological models. The requirement of reconciliation of low level MBR fluctuations with the amplitude of the density perturbations necessary to produce galaxies today is a powerful tool for eliminating cosmological scenarios.

In order to calculate the small–scale MBR fluctuations one has to take the dynamics of the decoupling into account and the theory of this process is now relatively well known [1]. When dealing with fluctuations on scales larger than the horizon at the decoupling we can assume that decoupling occurs instantanuously. The distance that photons can travel is smaller than the characteristic length of the perturbation and the details of the decoupling process cannot influence the MBR pattern. However, there is another problem with the large scales. Because of the freedom of making gauge transformations (i.e. changing the correspondence between the points in the physical spacetime and the points in the undisturbed background) the perturbations of physical quantities are different in different gauges and can contain the spurious gauge modes.

The most elegant way of treating cosmological perturbations was proposed by Bardeen [2] and is based on using the gauge–invariant quantities that within the horizon are perturbations of well known physical quantities. Since the paper by Bardeen some developments of his formalism were done e.g. on the case of uncoupled fluids in the flat universe [3] and the system of fluid + collisionless gas [4] or on the case of matter described by the massive field [5]. The elements of the formalism were used to find the large–scale anisotropy of MBR by Abbott and Wise [6, 7] and Bond and Efstathiou [8].

We should realize that any calculations involving the scales larger than the horizon performed in a particular gauge have to be accompanied by the proof that the results are gauge invariant. Writing the formulae in gauge–invariant quantities after deriving them in choosen gauge is not sufficient, even if the results look correct.

In this paper we present the calculation leading to formulae for the large–scale anisotropy of the MBR in the general (nonflat,multicomponent) cosmological model based on the Robertson–Walker metric. The formulae for scalar, vector and tensor perturbations are given and those for scalar ones are analysed in some specific models of the Universe. For the general flat models we decompose the large–scale MBR fluctuations into multipoles and calculate the amplitude of the l-th multipole as a function of the baryonic density and velocity perturbation fields.

In Sec.II we define perturbation quantities used later and give the equations of evolution of the background and the equations of motion for the perturbations. In Sec.III the MBR anisotropies are found. The results for some specific cases are discussed in Sec.IV. Sec.V presents the formalism of the multipole decomposition of the MBR pattern. Finally, Sec.VI contains concluding remarks.

II. DEFINITIONS AND EQUATIONS OF MOTION

We try to follow the notation of original Bardeen's paper [2]. The background Robertson–Walker metric is:

$$ds^2 = g_{ij}\ dx^i dx^j = S^2(\tau)(\ -d\tau^2 + {}^3g_{\alpha\beta}\ dx^\alpha dx^\beta) \tag{1}$$

where $i, j ... = 0\ 1\ 2\ 3$, $\alpha, \beta ... = 1\ 2\ 3$. The derivative with respect to the conformal time will be denoted by a dot, the covariant derivative with respect to g_{ij} by a semicolon and with respect to ${}^3g_{\alpha\beta}$ by a vertical bar.

We assume that the universe contains N ideal fluids with the unperturbed energy–momentum tensors:

$$T_{a0}^{\ 0} = -E_{a0}, \qquad T_{a\beta}^{\ \alpha} = P_{a0}\ \delta_\beta^\alpha \tag{2}$$

where $a = 1...N$ and E_{a0} and P_{a0} are the background energy density and pressure of the ideal fluid a. The more careful approach would be to describe the collisionless components by the means of a distribution function [4].

We denote (c_{Sa}^2 is the speed of sound):

$$w_a = \frac{P_{a0}}{E_{a0}}, \qquad c_{Sa}^2 = \frac{dP_{a0}}{dE_{a0}}$$

and assume that the fluids are coupled only by gravity.

The equations governing the evolution of the background are:

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{1}{3}\ S^2 \sum_{a=1}^{N} E_{a0} - K \tag{3a}$$

$$\left(\frac{\dot{S}}{S}\right)^{\bullet} = -\frac{1}{6}\ S^2 \sum_{a=1}^{N} (E_{a0} + 3P_{a0}) \tag{3b}$$

$$\frac{\dot{E}_{a0}}{E_{a0} + P_{a0}} = -3\ \frac{\dot{S}}{S} \tag{3c}$$

where $K = -1, 0, 1$ is the scalar of curvature in open, flat and closed universe respectively (units $c = 8\pi G = 1$). We can also incorporate the nonzero cosmological constant in the model: $E_{a0} = -P_{a0} = \Lambda$.

Perturbations can be classified according to their transformation properties under spatial coordinate transformations of the background spacetime as scalar, vector and tensor perturbations. The time– and spatial–dependent parts of perturbation quantities can be separated thanks to the homogeneity and isotropy of the background. The spatial parts may be decomposed into the solutions of the generalized Helmholtz equations. For now, we will restrict our analysis to the case of the perturbations described by the single mode.

A. Scalar perturbations

Scalar harmonics $Q(x^\mu)$ are solutions of the equation:

$$Q^{|\alpha}{}_{|\alpha} + k^2 Q = 0 \tag{4}$$

For a flat Robertson–Walker universe the Q's are conveniently taken to be plane waves.

The vector and traceless tensor quantities are constructed by:

$$Q_\alpha = -\frac{1}{k} Q_{|\alpha} \tag{5a}$$

$$Q_{\alpha\beta} = \frac{1}{k^2} Q_{|\alpha\beta} + \frac{1}{3}\ {}^3g_{\alpha\beta}\ Q \tag{5b}$$

The metric perturbations are written as:

$$g_{00} = -S^2(1 + 2AQ) \tag{6a}$$

$$g_{0\alpha} = -S^2 B Q_\alpha \tag{6b}$$

$$g_{\alpha\beta} = S^2[(1 + 2H_L Q)^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta}] \tag{6c}$$

Let u_a^i be the four–velocity of the rest frame of fluid a relative to the coordinate frame (the rest frame is the frame in which the energy flux of fluid a vanishes). We assume that to 0-th order all u_a^i, $a = 1...N$ are the same. The 1-st order perturbations of the velocity of the fluid a are:

$$u_a^0 = \frac{1}{S}(1 - AQ) \tag{7a}$$

$$u_a^\alpha = \frac{1}{S} v_a Q^\alpha \tag{7b}$$

The perturbations in the energy–momentum tensor are:

$$T_{a0}^{\ 0} = -E_{a0}(1 + \delta_a Q) \tag{8a}$$

$$T_{a0}^{\ \alpha} = -(E_{a0} + P_{a0}) v_a Q^\alpha \tag{8b}$$

$$T_{a\alpha}^{\ 0} = (E_{a0} + P_{a0})(v_a - B) Q_\alpha \tag{8c}$$

$$T_{a\beta}^{\ \alpha} = P_{a0}[(1 + \pi_{La} Q)\delta_\beta^\alpha + \pi_{Ta} Q_\beta^\alpha] \tag{8d}$$

and the entropy perturbation is:

$$\eta_a = \pi_{La} - \frac{c_{Sa}^2}{w_a}\delta_a \tag{9}$$

The general gauge transformation of the wavenumber k is:

$$\tilde{\tau} = \tau + T(\tau) Q(x^\mu) \tag{10a}$$

$$\tilde{x}^\alpha = x^\alpha + L(\tau) Q^\alpha(x^\mu) \tag{10b}$$

The gauge–invariant perturbation quantities are π_{Ta}, η_a and:

$$\Phi_A = A + \frac{1}{k}\dot{B} + \frac{1}{k}\frac{\dot{S}}{S}B - \frac{1}{k^2}(\ddot{H}_T + \frac{\dot{S}}{S}\dot{H}_T) \tag{11a}$$

$$\Phi_H = H_L + \frac{1}{3}H_T + \frac{1}{k}\frac{\dot{S}}{S}B - \frac{1}{k^2}\frac{\dot{S}}{S}\dot{H}_T \tag{11b}$$

$$\epsilon_a = \delta_a + 3\frac{1 + w_a}{k}\frac{\dot{S}}{S}(v_a - B) \tag{11c}$$

$$v_{Sa} = v_a - \frac{1}{k}\dot{H}_T \tag{11d}$$

The equations of the evolution of perturbations derived from the Einstein equations are:

$$\frac{2(k^2 - 3K)}{S^2}\Phi_H = \sum_{a=1}^{N} E_{a0}\epsilon_a \tag{12a}$$

$$-\frac{k^2}{S^2}(\Phi_A + \Phi_H) = \sum_{a=1}^{N} P_{a0}\pi_{Ta} \tag{12b}$$

And from the conservation equations $T_{a\,j;i}^{\,i} = 0$ we obtain $(a = 1...N)$:

$$(E_{a0}\epsilon_a S^3)^{\cdot} + \frac{3(E_{a0}+P_{a0})S^3}{k}\left\{\left[\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^{\cdot}\right] v_{Sa} + k\dot{\Phi}_H + \frac{1}{3}k^2 v_{Sa}\right\} +$$

$$-3S^3\frac{\dot{S}}{S}\left[(E_{a0}+P_{a0})\Phi_A - \frac{2}{3}\left(1-\frac{3K}{k^2}\right)P_{a0}\pi_{Ta}\right] = 0 \qquad (13a)$$

$$\dot{v}_{Sa} + \frac{\dot{S}}{S}v_{Sa} = k\Phi_A + \frac{k}{1+w_a}(c_{Sa}^2\epsilon_a + w_a\eta_a) - \frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{w_a}{1+w_a}\pi_{Ta} \qquad (13b)$$

The equation (13a) for the case $K = 0$ was first derived by Abbott and Wise [3].

B. Vector perturbations.

In this mode the quantities being scalars under spatial coordinate transformations remain unperturbed. The vector harmonics $Q^{(1)\alpha}$ are solutions of:

$$Q^{(1)\alpha|\beta}_{|\beta} + k^2 Q^{(1)\alpha} = 0 \qquad (14)$$

The tensor quantity is obtained by:

$$Q^{(1)\alpha\beta} = -\frac{1}{2k}(Q^{(1)\alpha|\beta} + Q^{(1)\beta|\alpha}) \qquad (15)$$

In analogy to the scalar perturbations we have:

$$\begin{aligned}
g_{0\alpha} &= -S^2 B^{(1)} Q^{(1)}_\alpha \\
g_{\alpha\beta} &= S^2({}^3g_{\alpha\beta} + 2H_T^{(1)} Q^{(1)}_{\alpha\beta}) \\
u_a^\alpha &= \frac{1}{S} v_a^{(1)} Q^{(1)\alpha} \\
T_{a\,\alpha}^{\,0} &= (E_{a0}+P_{a0})(v_a^{(1)} - B^{(1)})Q^{(1)}_\alpha \\
T_{a\,0}^{\,\alpha} &= -(E_{a0}+P_{a0})v_a^{(1)} Q^{(1)\alpha} \\
T_{a\,\beta}^{\,\alpha} &= P_{a0}(\delta^\alpha_\beta + \pi_{Ta}^{(1)} Q_\beta^{(1)\alpha})
\end{aligned} \qquad (16)$$

The allowed gauge transformation is $\tilde{x}^\alpha = x^\alpha + L^{(1)}(\tau)Q^\alpha(x^\mu)$ and the gauge–invariant quantities are $\pi_{Ta}^{(1)}$ and:

$$\Psi = B^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} \tag{17a}$$

$$v_{Sa}^{(1)} = v_a^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} \qquad or \qquad v_{Ca} = v_{Sa}^{(1)} - \Psi \tag{17b}$$

The equations of motion are:

$$\frac{k^2 - 2K}{2S^2}\Psi = \sum_{a=1}^{N}(E_{a0} + P_{a0})v_{Ca} \tag{18a}$$

$$\dot{v}_{Ca} + \frac{\dot{S}}{S}(1 - 3c_{Sa}^2)v_{Ca} = -\frac{k}{2}\frac{w_a}{1 + w_a}\pi_{Ta}^{(1)} \tag{18b}$$

C. Tensor perturbations.

Now only tensor quantities are perturbed and the tensor harmonics are solutions of:

$$Q^{(2)}{}^{\alpha\beta|\gamma}_{|\gamma} + k^2 Q^{(2)\alpha\beta} = 0 \tag{19}$$

We have:

$$\begin{aligned} g_{\alpha\beta} &= S^2({}^3g_{\alpha\beta} + 2H_T^{(2)}Q_{\alpha\beta}^{(2)}) \\ T_{a\beta}^{\ \alpha} &= P_{a0}(\delta_\beta^\alpha + \pi_{Ta}^{(2)}Q_\beta^{(2)\alpha}) \end{aligned} \tag{20}$$

The quantities $H_T^{(2)}$ and $\pi_{Ta}^{(2)}$ are automatically gauge–invariant. The equation of motion is:

$$\ddot{H}_T^{(2)} + 2\frac{\dot{S}}{S}\dot{H}_T^{(2)} + (k^2 + 2K)H_T^{(2)} = S^2\sum_{a=1}^{N}P_{a0}\pi_{Ta}^{(2)} \tag{21}$$

III. THE TEMPERATURE OF MBR IN THE PERTURBED UNIVERSE

Our aim here is to find the MBR pattern $T_R(\theta,\phi)$ (subscript $_R$ denotes reception and subscript $_E$ – emission) in a cosmological model based on a Robertson–Walker metric if the solutions of the perturbation equations for density and velocity fields are known. We assume that the decoupling is instantanuous and that we are interested in large angular scales only.

The temperature of the MBR coming from a given direction is determined by the redshift that photons acquired from emission till reception. The density and velocity perturbations of the fluids filling the universe cause the perturbations in the metric that in turn influence the motion of light, resulting in redshift differences of photons coming from different directions. (The interaction of light and matter can also have the direct form of scattering, if the intergalactic medium was reionized in the epoch of galaxy or star formation).

In order to obtain the pattern of the MBR in a perturbed universe we should in principle integrate the Boltzmann equation for a generalized, gauge–invariant distribution function of photons through the decoupling phase and further, until today. However, details of the decoupling are important only for small angular scales, and in this case the use of gauge–invariant quantities is not necessary. For large scales the opposite is true. By large angular scales we mean scales larger than those subtended today by the light rays emitted at decoupling from two points at seperation equal to size of observable universe at decoupling (i.e. twice the distance to the horizon). This angular scale is [9]:

$$\theta_H \simeq 3^\circ \left(\frac{1300}{z}\,\Omega_R\right)^{\frac{1}{2}}$$

where z is the redshift of the decoupling and Ω_R is the density parameter today.

Therefore a suitable approximation is to follow the movement of the single photon emitted from the last scattering hypersurface using the gauge–invariant quantities.

The light moves along the null–like geodesics $x^i(\lambda)$ where λ is the affine parameter. The null vector tangent to the geodesic is:

$$k^i = \frac{dx^i}{d\lambda}, \qquad k^i = (\nu, P^\alpha), \qquad k_i k^i = 0 \tag{22}$$

We write :

$$\nu = \overset{0}{\nu}(1 - M) \qquad\qquad P^\alpha = \overset{0}{P}{}^\alpha + \overset{1}{P}{}^\alpha \tag{23}$$

where $\overset{0}{\nu}M$ and $\overset{1}{P}{}^\alpha$ are the first order corrections to componenets of vector k^i. The equation of motion is:

$$\frac{Dk^i}{d\lambda} = \frac{dk^i}{d\lambda} + \Gamma^i_{kl}k^k k^l = 0 \tag{24}$$

The temperature of the MBR now observed is:

$$\frac{T_R}{T_E} = \frac{1}{1+z} = \frac{(k^i u_{bi})_R}{(k^i u_{bi})_E} \tag{25}$$

where u_{bi} is the 4–vector of velocity of observer at rest relative to baryonic fluid (observers are made of baryons). We take no account of local, gravity–induced motions e.g. of the Galaxy, that result in additional dipole term.

It is convenient to introduce a new parameter $s(\lambda)$ (the derivative with respect to s will be denoted by a prime) such that:

$$\frac{d\lambda}{ds} = \frac{S^2}{S_R^2}, \qquad \frac{d}{ds} = \frac{S^2}{S_R^2}\frac{d}{d\lambda} = \frac{S^2}{S_R^2}\left(\nu\frac{\partial}{\partial\tau} + P^\alpha\frac{\partial}{\partial x^\alpha}\right) \tag{26}$$

From (22) and (24) we obtain in the 0-th order:

$$\overset{0}{\nu} = \frac{S_R^2}{S^2} \qquad\qquad \overset{0}{\nu}{}^2 = \overset{0}{P}{}^\alpha \overset{0}{P}_\alpha \tag{27}$$

Defining $\overset{0}{P}{}^{\alpha} = -\overset{0}{\nu} R^{\alpha}$ where R^{α} is the spatial unit vector ($R^{\alpha} R_{\alpha} = 1$) in direction of observation and using the normalization $\overset{0}{\nu}_R = 1$ we obtain the solutions for the lightlike geodesics to 0-th order:

$$\tau = \tau_E + s \tag{28a}$$

$$x^{\alpha} = R^{\alpha}(\tau_R - \tau_E - s) \tag{28b}$$

$$\frac{dR^{\alpha}}{ds} = {}^3\Gamma^{\alpha}_{\beta\gamma} R^{\beta} R^{\gamma} \tag{28c}$$

where

$$\frac{d}{ds} = \frac{\partial}{\partial \tau} - R^{\alpha} \frac{\partial}{\partial x^{\alpha}}, \qquad R^{\alpha} = -\frac{dx^{\alpha}}{ds}$$

The coordinates of the emission event are: $s = 0$, $\tau = \tau_E$, $x^{\alpha}_E = R^{\alpha}_E(\tau_R - \tau_E)$ and of the reception event: $s = \tau_R - \tau_E$, $\tau = \tau_R$, $x^{\alpha}_R = 0$.

The equations to 1-st order look differently for scalar, vector and tensor perturbations.

A. Scalar perturbations.

Using (22) and (24) to 1-st order we obtain:

$$M' = \dot{A} Q + 2kAQ_{\alpha} R^{\alpha} + \frac{k}{3} BQ + \dot{H}_L Q + (\dot{H}_T - kB) Q_{\alpha\beta} R^{\alpha} R^{\beta} \tag{29}$$

and from (25):

$$\frac{T_E}{T_R} = \frac{S_R}{S_E} \left\{ 1 + \int_E^R \left[\left(\dot{H}_L + \frac{k}{3} v_b \right) Q - (\dot{v}_b - kA - \dot{B}) Q_{\alpha} R^{\alpha} + \right.\right.$$
$$\left.\left. - (k v_b - \dot{H}_T) Q_{\alpha\beta} R^{\alpha} R^{\beta} \right] ds \right\} \tag{30}$$

where the integral is along the 0-th order lightlike geodesic.

The above formula is gauge invariant because S transforms under (10a) as $S(\tilde{\tau}) = S(\tau)\cdot(1 + \frac{\dot{S}}{S}TQ)$ and then we should be able to express it using only the gauge–invariant variables.

In our simplified model of decoupling the emission of radiation occurs on the hypersurface of the constant density of free electrons that couple to photons by Thomson scattering. This density is a function of the local temperature and density of baryons and thus for general perturbations the hypersurface of emission is neither the hypersurface of constant temperature nor that of constant baryon density. In the presence of perturbations the emission in a given point is space occurs in the moment $\tau_{E_0} + \Delta\tau$ where $\Delta\tau$ is a function of perturbations. Denoting the density of free electrons at the emission by n_{eE} and the moment of emission in the 0-th order by the subscript $_{E_0}$ we have on the hypersurface of last scattering:

$$n_{eE} = const = n_e(\tau_{E_0} + \Delta\tau) = n_{e0}(\tau_{E_0} + \Delta\tau)\cdot(1 + \delta_e Q) \tag{31}$$

so:

$$\Delta\tau = -\frac{n_{e0}}{\dot{n}_{e0}}\,\delta_e Q \tag{32}$$

where δ_e is a perturbation of density of electrons and in general:

$$n_e = f(E_b)g(T) \tag{33}$$

Using this functional form we can express $\Delta\tau$ as a function of the perturbations of baryons and photons ($E_\gamma = \sigma T^4$). The result is:

$$\frac{\dot{S}}{S}\Delta\tau = \frac{1}{3+D}\delta_b Q + \frac{D}{4(3+D)}\delta_\gamma Q \tag{34}$$

where:

$$D = \left(\frac{f\frac{dg}{dT}T}{\frac{df}{dE_b}gE_b}\right)_E \tag{35}$$

In the simplest model of decoupling we can use the Saha formula for the fractional ionization [10] and we obtain $D \simeq \frac{3}{2} + \frac{B}{k_B T_E}$ where $B = 13.6\ eV$ and for $T_E = 3500K, \quad D \simeq 47$.

At the moment of emission:

$$S_E = S_{E_0}\left(1 + \frac{\dot{S}}{S}\Delta\tau\right)_E \tag{36}$$

Now we can rewrite (30) in gauge–invariant quantities. We define:

$$\left(\frac{\delta T}{T}\right)_R = \frac{T_R - T_{R_0}}{T_{R_0}}, \qquad T_{R_0} = \frac{S_{E_0} T_E}{S_{R_0}} \tag{37}$$

to obtain:

$$\begin{aligned}\left(\frac{\delta T}{T}\right)_R = \left(\frac{\delta T}{T}\right)_E - \left[\left(\Phi_H - \frac{1}{k}\frac{\dot{S}}{S}v_{Sb}\right)Q\right]_E^R + \\ -\int_E^R \left[\left(\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A\right)Q_{|\alpha}R^\alpha - \frac{v_{Sb}}{k}Q_{|\alpha\beta}R^\alpha R^\beta\right] ds\end{aligned} \tag{38}$$

where:

$$\left(\frac{\delta T}{T}\right)_E \equiv \left[\left(\frac{1}{3+D}\epsilon_b + \frac{D}{4(3+D)}\epsilon_\gamma\right)Q - \frac{1}{k}\frac{\dot{S}}{S}\frac{D}{3+D}(v_{S\gamma} - v_{Sb})Q\right]_E \tag{39}$$

and we used $w_b = 0, \quad w_\gamma = \frac{1}{3}$ and the fact that any part of T_R independent of the direction of observation can be incorporated into the definition of T_{R_0}.

This is the explicitly gauge–invariant form of the temperature fluctuations in the general case.

In the most popular (and suggested by many theories of the very early Universe) case of adiabatic perturbations we have $\delta_\gamma = \frac{4}{3}\delta_b$ and the initial fluctuations of temperature reduce to $\frac{1}{3}\epsilon_b Q$ at the emission. We will analyse further this case only, and, as we will see this term can be dropped. However, we should emphasize that in nonadiabatic models the initial fluctuations given by the formula (39) can be important – for example the term $\sim \epsilon_\gamma$ is responsible for the increase of the large–scale angular fluctuations in the model of isocurvature axion perturbations analysed by Efstathiou and Bond [8].

We can put $c_{Sb}^2 = 0$ if we are interested in scales larger than the baryon Jeans mass. This is the case because for $\lambda \geq (ct)_E$ this Jeans mass is much less than the mass in a sphere of diameter $\sim \lambda$. The equations of motion (13) for the baryonic perturbations are then:

$$\dot{\epsilon}_b + 3\left[\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^{\bullet}\right]\frac{v_{Sb}}{k} + 3\dot{\Phi}_H - 3\frac{\dot{S}}{S}\Phi_A + kv_{Sb} = 0 \tag{40a}$$

$$\dot{v}_{Sb} + \frac{\dot{S}}{S}v_{Sb} = k\Phi_A \tag{40b}$$

We can use these equations to rewrite (38) in the form:

$$\left(\frac{\delta T}{T}\right)_R = \left(\frac{1}{3}\epsilon_b Q\right)_E + \int_E^R \left[\frac{1}{3}(\dot{\epsilon}_b + kv_{Sb})Q + \frac{v_{Sb}}{k}Q_{|\alpha\beta}R^\alpha R^\beta\right] ds \tag{41}$$

The first term represents the influence of the density perturbations at the emission and can be dropped because in our case is much smaller than the integral term (see Sec.IV). The integral term describes how the motion of light is influenced by the geometry perturbations generated by all density and anisotropic pressure perturbations present in the model (see (12)) and described by the baryonic quantities ($|\frac{k}{S}v_{Sb}|$ is the magnitude of shear of the baryonic velocity field). Some specific cases of (41) are described in the next section.

B. Vector perturbations.

An analysis similar to the one performed before gives us the result:

$$\left(\frac{\delta T}{T}\right)^{(1)}_R = \int\limits_E^R (\dot{v}_{Cb} Q^{(1)}_\alpha R^\alpha + k v^{(1)}_{Sb} Q^{(1)}_{\alpha\beta} R^\alpha R^\beta) ds \tag{42}$$

C. Tensor perturbations.

$$\left(\frac{\delta T}{T}\right)^{(2)}_R = -\int\limits_E^R \dot{H}^{(2)}_T Q^{(2)}_{\alpha\beta} R^\alpha R^\beta ds \tag{43}$$

The vector and tensor cases will not be analysed further.

IV. SPECIFIC CASES OF THE SCALAR PERTURBATIONS

Although the general formula describing the MBR pattern in the presence of scalar perturbations (41) is very simple the troubles arise with its use because of the complicated form of the equations of motion in the general case (12, 13). However, we can simplify them in some specific models.

First of all we expect that the anisotropic stress and entropy perturbations in any component of the universe operate only at very early stages of the evolution. Thus we can omit them in our analysis. Now the equations of motion are (a=1...N):

$$(E_{a0}\epsilon_a S^3)^\cdot + \frac{3(E_{a0}+P_{a0})S^3}{k}\left\{\left[\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot\right] v_{Sa} + k\dot{\Phi}_H + \frac{1}{3}k^2 v_{Sa}\right\} +$$
$$+ 3S^3\frac{\dot{S}}{S}(E_{a0}+P_{a0})\Phi_H = 0 \tag{44a}$$

$$\dot{v}_{Sa} + \frac{\dot{S}}{S} v_{Sa} = -k\Phi_H + \frac{k c_{Sa}^2}{1+w_a}\epsilon_a \tag{44b}$$

$$\Phi_H = \frac{S^2}{2(k^2-3K)} \sum_{a=1}^{N} E_{a0}\epsilon_a \tag{44c}$$

The next simplification is obtained if we are interested in the models dominated by nonrelativistic components with $P_{a0} = 0$ (e.g. CDM models) since decoupling. If we omit the radiation and relativistic neutrinos in the equations of motion, we have (a=1...N):

$$\dot\epsilon_a \left(1 + \frac{3V_a}{2(k^2-3K)S}\right) + kv_{Sa}\left(1 - \frac{3K}{k^2} + \frac{3}{2k^2 S}\sum_{c=1}^{N} V_c\right) + \frac{3}{2(k^2-3K)S}\sum_{\substack{c=1\\ c\neq a}}^{N} V_c\dot\epsilon_c = 0 \tag{45a}$$

$$\dot v_{Sa} + \frac{\dot S}{S} v_{Sa} = -\frac{1}{2k\left(1-\frac{3K}{k^2}\right)S}\sum_{c=1}^{N} V_c\epsilon_c \tag{45b}$$

$V_a = E_{a0}S^3$ are constants closely related to the present density parameters Ω_{Ra} (H_R is the Hubble constant measured today):

$$\Omega_{Ra} = \frac{E_{a0R}}{3H_R^2} = \frac{V_a}{\sum\limits_{c=1}^{N} V_a - 3KS_R} \tag{46}$$

In the case $K = 0$ suggested by the inflationary scenarios we have $\Omega_a = const = \Omega_{Ra}$ and we denote the sum of all V_c by V :

$$\dot\epsilon_a \left(1 + \frac{3\Omega_a V}{2k^2 S}\right) + kv_{Sa}\left(1 + \frac{3V}{2k^2 S}\right) + \frac{3V}{2k^2 S}\sum_{\substack{c=1\\ c\neq a}}^{N} \Omega_c\dot\epsilon_c = 0 \tag{47a}$$

$$\dot v_{Sa} + \frac{\dot S}{S} v_{Sa} = -\frac{V}{2kS}\sum_{c=1}^{N} \Omega_c\epsilon_c \tag{47b}$$

Greatly simplified equations of motion are obtained from (45a,b) if $N = 1$ or all $\dot\epsilon_c$ $c = 1...N$ are equal. For baryons:

$$\dot\epsilon_b + \left(1 - \frac{3K}{k^2}\right) kv_{Sb} = 0 \tag{48a}$$

This is the case in most of the dark matter scenarios – on scales larger than the Jeans mass the baryons sink in the potential wells of the dark component. In a few expansion times all the ϵ_c's become equal. Then the second equation of motion is:

$$\dot{v}_{Sb} + \frac{\dot{S}}{S} v_{Sb} = -\frac{V}{2k\left(1 - \frac{3K}{k^2}\right)S}\,\epsilon_b \tag{48b}$$

Now we can write the formula (41) under the assumption of equal ϵ_c's:

$$\left(\frac{\delta T}{T}\right)_R = \left(\frac{1}{3}\epsilon_b Q\right)_E - \frac{1}{k^2 - 3K}\int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds \tag{49}$$

In the flat universe $K = 0$, the covariant derivatives become normal and Q is taken to be a plane wave with the wave vector k^α: $\quad Q = \exp(ik_\alpha x^\alpha)$. From (49) we obtain:

$$\left(\frac{\delta T}{T}\right)_R = \left(\frac{1}{3}\epsilon_b Q\right)_E + \frac{1}{k^2}\int_E^R \dot{\epsilon}_b (k_\alpha R^\alpha)^2 Q ds \tag{50}$$

In this case the equation for density perturbations has the solutions:

$$\epsilon_b = A\tau^2 + B\tau^{-3} \tag{51}$$

For the growing mode $\epsilon_b = \epsilon_{bE}(\frac{\tau}{\tau_E})^2$ the integral in (50) can be integrated by parts:

$$\left(\frac{\delta T}{T}\right)_R = \frac{1}{3}\epsilon_{bE} Q_E + \frac{2\epsilon_{bE}}{k^2\tau_E^2}\left[\tau_R (R^\alpha Q_{,\alpha})_R - \tau_E (R^\alpha Q_{,\alpha})_E + Q_R - Q_E\right] \tag{52}$$

The comoving coordinate distance to the horizon at τ_E is equal τ_E. Then the criterion for the comoving scale k to be larger than the horizon at τ_E is $k\tau_E \ll 1$. In this regime we can drop the first term because it is much smaller than the integral term. (In fact when the first term is important, on scales $k\tau_E \geq 1$, it does not look so simple). We can incorporate the term $\sim Q_R$ into the definition of T_{R_0}. This gives:

$$\left(\frac{\delta T}{T}\right)_R = \frac{2\epsilon_{bE}}{k^2\tau_E^2}\left[\tau_R(R^\alpha Q_{,\alpha})_R - \tau_E(R^\alpha Q_{,\alpha})_E - Q_E\right] \tag{53}$$

This is the well known result of Sachs and Wolfe.

V. MULTIPOLE DECOMPOSITION

The anisotropies of the MBR calculated in the cosmological models can be compared with the limits from observations to restrict the parameters of these models. For the large–scale anisotropies the convenient quantities are the amplitudes of the multipoles in the decomposition of the function $\left(\frac{\delta T}{T}\right)_R(\theta,\phi)$. In further analysis we will restrict ourselves to the case of the flat universe.

We usually assume that the field of the density perturbations of all components consists of the sum of plane waves with random wave vectors and phases. The amplitude of the wave with the wave vector k^α is assumed to be the function of the modulus of the wave vector only, usually power law at the prescribed moment.

The power law behavior is understandable on scales smaller than the horizon but its continuation on scales larger than the horizon is not obvious. This is because there is no unique choice of gauge–invariant quantity for density perturbations. However, we argue that we can assume the continuation of the functional behavior for the used here quantity ϵ (this is ϵ_m in Bardeen's notation), because it is uniform on scales smaller and larger than the horizon (Bardeen's ϵ_g is not) and because it directly couples to the potential Φ_H (see (12a)).

Some spectra that behave as power laws at the time of horizon crossing at a given scale become substantially distorted (e.g. CDM spectra) at decoupling. Therefore we will assume the general functional dependence of the amplitude of density perturbations at a given scale at the emission. The single plane wave is then replaced by the sum:

$$\sum_{\vec{k}\neq 0} \epsilon_a(k) e^{i\vec{k}\vec{x}} \tag{54}$$

Our initial conditions are the density perturbations at decoupling. We assume that the equations of motion were solved and we can write the appriopriate form of $\epsilon_b(k)$ and $v_{Sb}(k)$ for all $\tau_E \leq \tau \leq \tau_R$.

Now the formula (41) for the flat universe can be written in the form:

$$\left(\frac{\delta T}{T}\right)_R = \sum_{\vec{k}\neq 0} \int_E^R \left[\frac{1}{3}\dot{\epsilon}_b(k) + k v_{Sb}(k)\left(\frac{1}{3} - \xi_{\vec{k}}^2\right)\right] e^{i\vec{k}\vec{x}} ds \tag{55}$$

where $\xi_{\vec{k}}$ is the cosine of the angle between the wave vector $\vec{k}$ and the direction of observation $\vec{R}$.

Our aim is to find the coefficients of the decomposition:

$$\left(\frac{\delta T}{T}\right)_R = \sum_{l,m} a_l^m Y_l^m(\Omega), \qquad a_l^m = \int Y^{*m}_l \left(\frac{\delta T}{T}\right)_R d\Omega \tag{56}$$

where $\Omega = (\theta, \phi)$.

We use the following mathematical formulae:

$$e^{i\vec{k}\vec{x}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kx) P_l(\xi_{\vec{k}}) \tag{57}$$

$$\xi_{\vec{k}}^2 P_l(\xi_{\vec{k}}) = \frac{1}{2l+1}[b_l P_{l+2}(\xi_{\vec{k}}) + c_l P_l(\xi_{\vec{k}}) + d_l P_{l-2}(\xi_{\vec{k}})] \tag{58}$$

$$P_l(\xi_{\vec{k}}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^{*m}_l(\Omega_{\vec{k}}) Y_l^m(\Omega) \tag{59}$$

where:

$$b_l = \frac{(l+1)(l+2)}{2l+3}, \qquad c_l = \frac{(l+1)^2}{2l+3} + \frac{l^2}{2l-1}, \qquad d_l = \frac{l(l-1)}{2l-1} \tag{60}$$

and $\Omega_{\vec{k}} = (\theta_{\vec{k}}, \phi_{\vec{k}})$ are the angular coordinates of the vector $\vec{k}$.

In (56) we decompose the plane waves into a series of Legendre polynomials, which in turn are represented by the spherical harmonics. We then integrate over $d\Omega$. The result is:

$$a_l^m = 4\pi i^l \sum_{\vec{k}\neq 0} A_l(k) Y^{*m}_{\ l}(\Omega_{\vec{k}}) \tag{61}$$

where:

$$\begin{aligned} A_l(k) = \int_E^R \Bigg(& \frac{1}{3}\Big[\dot{\epsilon}_b(k) + k v_{Sb}(k)\Big] j_l(kx) + \\ & + \frac{k v_{Sb}(k)}{2l+1}\Big[d_l j_{l-2}(kx) - c_l j_l(kx) + b_l j_{l+2}(kx)\Big]\Bigg) ds \end{aligned} \tag{62}$$

This result can be generalized on the case of nonadiabatic models by simply adding to a_l^m the contribution from the decomposition of initial fluctuations (39).

The quantity used to compare with observations is:

$$(a_l)^2 = \langle |a_l^m|^2 \rangle \tag{63}$$

(in fact for a given l all a_l^m are statistically independent with the same expectation values). We use the assumption that the phases of $\epsilon_b(k)$ and $v_{Sb}(k)$ are random and change the sum over $\vec{k}$ into the integral:

$$\sum_{\vec{k}\neq 0} \longrightarrow \int k^2 dk \frac{d\Omega_{\vec{k}}}{4\pi} \tag{64}$$

to obtain:

$$(a_l)^2 = 4\pi \int_0^{k_{max}} k^2 |A_l(k)|^2 dk \tag{65}$$

The cutoff k_{max} reflects the limits of our simplified model. They can be related to the horizon size and the scale of the Silk damping of baryonic perturbations, but another requirement for the small scales is not to be smaller than the scales that are nonlinear today. Anyway we should expect the results to be practically cutoff–independent for small l (if k_{max} is large enough) because small scale perturbations averaged over large scales give the result ~ 0.

The formula (62) is simplified in the case of the universe dominated by the nonrelativistic matter such that all ϵ_c, $c = 1...N$ are equal (or $N = 1$) because then $\dot{\epsilon}_b + kv_{Sb} = 0$ and:

$$A_l(k) = -\frac{1}{2l+1}\int_E^R \dot{\epsilon}_b(k)\Big[d_l j_{l-2}(kx) - c_l j_l(kx) + b_l j_{l+2}(kx)\Big]ds \tag{66}$$

For the growing mode of the density perturbations $\epsilon_b = \epsilon_{bE}(\frac{\tau}{\tau_E})^2$ the integral can be calculated [6] (or equivalently we can integrate the Sachs – Wolfe formula). The result is ($s_R = \tau_R - \tau_E$):

$$\begin{aligned}(a_l)^2 = \frac{16\pi}{(2l+1)^2\tau_E^4}\int_0^{k_{max}} k^{-2}|\epsilon_{bE}(k)|^2\Big[&(2l+1)j_l(ks_R)\\ &+ k\tau_E(lj_{l-1}(ks_R) - (l+1)j_{l+1}(ks_R))\Big]^2 dk\end{aligned} \tag{67}$$

and the first integral term dominates the others.

The results of observations on medium angular scales are represented in the form of the angular correlation function of fluctuations:

$$W(\theta) = \frac{1}{8\pi^2}\int\int d\Omega_1 d\Omega_2 \left[\frac{\delta T}{T}(\Omega_1)\right]_R \left[\frac{\delta T}{T}(\Omega_2)\right]_R \delta_D(\cos\theta_{12} - \cos\theta) \tag{68}$$

which can be expressed using the multipole coefficients as:

$$W(\theta) = \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)(a_l)^2 P_l(\cos\theta) \tag{69}$$

VI. CONCLUSIONS

The gauge–invariant approach [2] solves many ambiguities around behavior of the large scale perturbations. Its application to the large–scale fluctuations of the MBR temperature is presented in this paper. The explicitly gauge–invariant formula for these fluctuations (38) is obtained for the wide class of cosmological models based on the Robertson – Walker metrics. The only assumptions used in its derivation are that all constituents of the universe can be described as perfect fluids and that the decoupling of matter and radiation occurs instantanuously on the hypersurface of the last scattering. However, relaxing these assumptions will not substantially change the results.

For the most popular model of the flat universe we obtain the multipole coefficients of the decomposition of the MBR pattern into spherical harmonics (65). They can be explicitly calculated if the equations of motion for baryonic perturbations were solved.

The multipole coefficients and the angular correlation function of fluctuations found in a model can be compared with the results of observations to provide valuable constraints on parameters of the model. The higher multipole moments were not observed yet and we know only the upper limits of the two first moments.

Unfortunately the dipole moment ($l = 1$) is influenced by the nonlinear, gravity induced motions of our Galaxy that cannot be reliably subtracted. However, the observations [11, 12] indicate that the intrinsic dipole moment of MBR can be $a_1 \leq 10^{-4}$ [12].

The recent observational limits on the quadrupole ($l = 2$) moment compiled by Bond and Efstathiou [8] give the value $a_2 \leq 10^{-4}$.

Any comparison of $W(\theta)$ calculated in the model with the observations requires additional information about the characteristic of the antenna used (see [8] for examples). The dipole momentum is usually excluded.

The reheating of intergalactic medium during star or galaxy formation could influence the pattern of the MBR and this effect will be addressed in another paper.

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