Fermi National Accelerator Laboratory

FERMILAB-Conf-86/101-T
July, 1986

Cosmic Time Gauge in Quantum Cosmology and Chaotic Inflation Model[1]

AKIO HOSOYA
Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, Illinois 60510
and
Department of Physics,
Osaka University
Toyonaka, Japan

Abstract

We propose a cosmic time gauge formalism in quantum cosmology to get an equation for the Schrödinger type. Its application to the chaotic inflaton scenario reveals that the uncertainty in the scale factor grows exponentially as the universe inflates.

[1]This paper is dedicated to Professor Yoshio Yamaguchi on his 60th birthday

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

I. Introduction

The quantum theory of geometry[1] is still in its infancy and suffers from many conceptual difficulties as well as technical ones. One of the conceptual difficulties comes from general coordinate invariance built in to general relativity. The arbitrariness of the coordinate choice makes the meaning of time obscure.

Previously many people [2] have attempted to identify the "time" in the Wheeler-DeWitt equation.[3] Even in the simplified minisuperspace model[2], the concept of time remains unclear.

The problem of time in quantum cosmology is not only an academic one but a potentially practical one. Many people now believe that there was once an exponentially expanding era in history of universe to solve the horizon, flatness problems etc.[4] At the moment the most probable inflationary universe scenario seems to be the chaotic universe scenario advocated by Linde.[5] There, the initial conditions are essentially given by the consideration of quantum gravity era of universe and the classical motion of the "inflaton" scalar field and the scale factor of the universe are described by the cosmic time parameter. In order to study the quantum era before the classical era of the universe we have to find out the right description of the time development of the wave function of universe.

In this paper dedicated to Professor Yoshio Yamaguchi for his 60th birthday, the author would like to make a proposal which hopefully demystifies the wave equation of universe[6] by reducing quantum cosmology to a down-to-earth Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}(t)\psi \quad . \tag{1}$$

Here the time is the cosmic time in the Robertson-Walker metric and $\mathcal{H}(t)$ is a time dependent Hamiltonian which we shall define in what follows. An approximate solution is also given for the chaotic universe model.

II. A Model and Hamiltonian Constraint

Let us consider the $\lambda\phi^4$ model in the Robertson-Walker universe having the chaotic inflationary universe scenario[5] in mind. The action is given by

$$\begin{aligned} S = & - \frac{1}{16\pi G}\int \sqrt{-g}d^4xR \\ & + \int \sqrt{-g}d^4x[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4}\phi^4] \end{aligned} \tag{2}$$

For the metric of closed universe

$$ds^2 = \mathcal{N}^2dt^2 - a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right\}$$

and a homogeneous scalar field ϕ, the action S in Eq.(2) reduces to

$$S = \int dt\left[\frac{1}{2g^2}\frac{\mathcal{N}}{a}\left\{-\frac{\dot{a}^2a^2}{\mathcal{N}^2} + a^2\right\} + 2\pi^2a^3\mathcal{N}\left\{\frac{1}{2\mathcal{N}^2}\dot{phi}^2 - \frac{\lambda}{4}\phi^4\right\}\right] \tag{3}$$

Here $\mathcal{N}$ is the lapse function [7] and $g^2 = \frac{2G}{3\pi}$.

By varying the action S with respect to the lapse function $\mathcal{N}$, we obtain the Hamiltonian constraint

$$H = -\frac{g^2\pi_a^2}{2a} - \frac{a}{2g^2} + \frac{1}{2\pi^2a^3}\frac{\pi_\phi^2}{2} + 2\pi^2a^3\cdot\frac{\lambda\phi^4}{4} \approx 0\,, \tag{4}$$

where π_a and π_ϕ are conjugate momenta to a and ϕ, respectively. In the conventional quantization prescription, this constraint is replaced by an equation for the quantum state

$$H\psi = 0 \tag{5}$$

with $\pi_a \to -\partial/\partial a$ and $\pi_\phi \to -i\partial/\partial\phi$.

The Wheeler-DeWitt equation (5) has some peculiarities. First it does not contain the "time" at all. How can it describe the history of universe? The second, which is intrinsically related to the first, is that it is impossible to construct a positive definite conserved probability, since Eq.(5) is a second order differential equation of the Klein-Gordon type. So we are in trouble with the probabilistic interpretation of the wave function of universe.

III. Time in Quantum Cosmology

One may realize that the passage from the first class constraint(4) to the wave equation is not the only possibility in the quantum theory of constrained systems.[8] The other possibility, which is perhaps more familiar to particle physists, is fixing the gauge, $\chi \approx 0$ in such a way $\{\chi, H\}_{P.B.} \neq 0$.[9] The gauge fixing condition is arbitrary unless $\{\chi, H\}_{P.B.} = 0$. However, the canonical choice is the one which satisfies

$$\{\chi, H\}_{P.B.} = 1 \tag{6}$$

Since in this case there is no Faddeev-Popov complexity. From Eq.(6) it is clear that χ plays a role of time, which is conjugate to the Hamiltonian.

It is perhaps instructive to digress here and recall the old story of the "time operator" in quantum mechanics. Pauli[10] pointed out that the "time operator" T which satisfies the commutation relation with the Hamiltonian, [T,H]=i is actually ill-defined in the Hilbert space. His argument is based on the observation that the lower boundedness of Hamiltonian contradicts with the existence of an energy shift operator $e^{i\epsilon T}$. For example, a singular expression $T = (p^{-1}q + qp^{-1})/2$ would be obtained in the case of the Hamiltonian $H = p^2/2$ for a free point particle which is obviously positive definite. A counter example is $T = p/E$ for an unbounded Hamiltonian $H = p^2/2 + Eq$ which describes a point particle in a uniform electric field, E. Here it has to be pointed out that the Hamiltonian (4) of quantum cosmology is not bounded due to the negative sign in front of the kinetic term of scale factor. This may suggest that quantum cosmology is in a unique position in which the time operator is well-defined and plays a significant role.

Going back to the gauge condition $\chi \approx 0$, let us consider its formal construction so that $\{\chi, H\}_{P.B.} = 1$ holds in a general framework of analytical dynamics. Let a canonical set of dynamical variables be $(p_i, q_i), i = 1, 2, \cdots N$.

A general solution of Hamilton's equations

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ i &= 1, 2, \cdots N, \end{aligned} \tag{7}$$

contains 2N integration constants corresponding to initial values of p's and q's. One of 2N constants corresponds to the arbitrariness of initial time t_0,

$$\begin{aligned} p_i &= p_i(t - t_0; c_1, c_2, \cdots c_{2N-1}) \\ q_i &= q_i(t - t_0; c_1, c_2, \cdots c_{2N-1}) \quad , i = 1, 2, \cdots N. \end{aligned} \tag{8}$$

Eliminating the integrating constants $c_1, c_2 \cdots c_{2N-1}$ we may solve Eq.(8) in terms of $t - t_0$,

$$t - t_0 = T(p_i, q_i) \tag{9}$$

By construction, $T(p,q)$ satisfies the Poisson bracket $\{T, H,\}_{P.B.} = 1$. It may be amusing to reproduce $T's$ in the previous examples by following the procedure $(7) - (9)$. It is now obvious that we can take as a gauge condition.

$$\chi(p,q) = T(p,q) - t \approx 0 \tag{10}$$

It is suggestive to point out that the "time" has been manufactured from dynamical variables through dynamics. This may sound philosophically deep.

Now that we have chosen a gauge, let us consider the dynamics in the restricted phase space under the constraint $H(p,q) = 0$ and the gauge condition $\chi(p,q,t) = 0$. First find a canonical transformation from a set (p_i, q_i), $i = 1, 2, \cdots N$ to a new set $(p*_i, q*_i)$, $i = 1, 2, \cdots N$ and (H, χ), such that

$$\sum_{i=1}^{N} p_i dq_i - H dt = \sum_{i=1}^{N-1} p_i^* dq_i^* + H d\chi - \mathcal{H} dt + d\Phi(q, q^*, \chi, t) \quad . \tag{11}$$

Here Φ is a generator of the canonical transformation. It is clear from Eq. (11) that in the restricted phase space the time development is described by a new Hamiltonian

$$\mathcal{H}(p*, q*, t) = \frac{\partial \Phi(q, q*, \chi, t)}{\partial t} \Big|_{q=q(p*,q*,t), H=\chi=0} \tag{12}$$

We are now in a position to quantize our constrained system by employing the new Hamiltonian (12) and setting commutation relations, $[p_i^*, q_j^*] = i\delta_{ij}$. Or in the Schrödinger representation, the time development of the state is dictated by

$$i\partial_t \psi(q^* t) = \mathcal{H}(-i\partial/\partial q^*, t)\psi(q^*, t) \tag{13}$$

There may well remain operator ordering ambiguities for which we do not have any general prescription.

In the next section we are going to apply the quantization method prescribed here to the model given in the previous section.

IV. Quantum Chaotic Universe Model

Let us go back to our model given in Eq.(4). The classical equations for ϕ and a are given by Hamilton's equations with the Hamiltonian (4) and are the standard ones,

$$\begin{aligned} \ddot{\phi} &+ 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} = -\lambda\phi^3 \,, \\ H^2 &= \frac{\dot{a}^2}{a^2} = \frac{4}{g^2}(\frac{1}{2}\ddot{phi}^2 + V) - \frac{1}{a^2} \,. \end{aligned} \tag{14}$$

(Here we took $\mathcal{N} = 1$.)

In the interesting regime ($M_p \lesssim \phi \lesssim \lambda^{-\frac{1}{4}} M_p$, $\lambda \sim 10^{-12}, M_p$: Planck mass $\sim g^{-1}$), the scalar field is slowly varying, so we can neglect the $\ddot{\phi}$ term in Eq. (14) and the $\ddot{phi}^2$ term in Eq. (15). We can also ignore the spatial curvature term $-1/a^2$ in Eq.(15) in that regime.[5] Then ϕ satisfies a first order linear equation!

$$\ddot{phi} \doteq -\sqrt{\frac{\lambda}{6\pi}} M_p \phi = -\frac{\sqrt{\lambda}}{3g}\phi \ . \tag{15}$$

In the regime under consideration, it is sufficient to consider a simplified Hamiltonian constraint instead of the original one (4),

$$H' = -\frac{g^2 \pi_a^2}{2a} - \frac{\sqrt{\lambda}}{3g}\phi\pi_\phi + 2\pi^2 a^3 \frac{\lambda\phi^4}{4} \approx 0 \,. \tag{16}$$

From now on we shall confine ourselves to this new constrained system which is supposed to be a good approximation to the original one before the oscillation of the scalar field starts.

For H', Eq. (16) becomes exact. According to the prescription in the previous section, the canonical gauge condition is simply given by

$$\chi = -\frac{3g}{\sqrt{\lambda}} log(\phi/\phi_0) - t \approx 0 \,. \tag{17}$$

Namely log ϕ is a "clock"[2].

Perhaps for such a simple system we do not need the heavy artillery of canonical transformation in §3 to get a new Hamiltonian. Eqs. 17 and 18 immediately give

$$\begin{aligned} \mathcal{H}(\pi_a, a, t) &= -\frac{g^2\pi_a^2}{2a} + 2\pi^2 a^3 \frac{\lambda\phi^4}{4} , \\ \phi &= \phi_0 e^{-\mu t} , \\ \mu &\equiv \frac{\sqrt{\lambda}}{3g} = \sqrt{\frac{\lambda}{6\pi}} M_p^{-1} . \end{aligned} \tag{18}$$

Let us choose the symmetrization prescription for the operator ordering, $\pi_a^2/a \to -a^{-\frac{1}{2}}\ \partial_a a^{-\frac{1}{2}}$ and write $x = \frac{2}{3}a^{\frac{3}{2}}$, $m = g^{-2}$ and

$$k(t) = \frac{9}{4}\pi^2\lambda\phi_0^4 \cdot e^{-4\mu t} . \tag{19}$$

We obtain a time dependent Schrödinger equation

$$-i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \frac{k(t)}{2}x^2\right]\Psi \tag{20}$$

Apart from the minus sign on the left hand side, this is a quantum mechanics of an upside-down harmonic oscillator with a time dependent spring constant given by Eq. (20). (We may extend the region of x so that $-\infty < x < \infty$, or set $\psi(0) = 0$. Here we do not discuss the latter possibility).

For $|\mu t| \ll 1$, the potential is just that of the upside-down harmonic oscillator discussed by Guth and Pi[11] in a completely different context. The wave packet initially localized at $x = 0 (a = 0)$ would quickly spread out while the average value of $a = (\frac{3}{2}x)^{2/3}$ would show an exponential growth, corresponding to Linde's inflation.

For $|\mu t| \gtrsim 1$ but within the range $M_p \lesssim \phi \lesssim \lambda^{-\frac{1}{4}}M_p$, the potential flattens out and the growth of $< x^2 > \propto < a^3 >$ becomes proportional to t as one can see in any quantum mechanics text book. Fortunately Eq. (21) is exactly soluble. We just give an expression for $|\psi|^2$ in the case of a gaussian distribution at $t = 0$,

$$|\psi(t)|^2 = \frac{1}{\sqrt{A_1^2 + \alpha^2 A_2^2}}\sqrt{\frac{\alpha}{2\pi}} exp\left[-\frac{\alpha}{A_1^2 + \alpha^2 A_2^2}\frac{4}{9}\left(\frac{a}{g}\right)^3\right] \tag{21}$$

Here α is an initial spread of the wave function and A_1 and A_2 are solutions of

$$\left(\frac{d^2}{d\tilde{t}^2} - \omega^2(\tilde{t})\right)A_i = 0 \quad (i = 1, 2) , \tag{22}$$

with

$$
\begin{aligned}
A_1(0) &= \frac{d}{d\tilde{t}}A_2(0) = 1\ , \\
\frac{d}{d\tilde{t}}A_1(0) &= A_2(0) = 0\ , \\
\omega^2(\tilde{t}) &= \lambda\phi_0^4 M_p^{-4} exp\left[-4\frac{\sqrt{\lambda}}{3\pi}\tilde{t}\right]\ , \\
\tilde{t} = t/g &= \sqrt{\frac{2}{3\pi}}M_p^{-1}t\ , \qquad (23) \\
& \qquad (24)
\end{aligned}
$$

(time in Planck's unit).

A's can be expressed by Bessel functions of an imaginary argument. For an almost constant ω (and therefore ϕ), the $A's$ grow exponentially while for a small ω (and therefore large t), $A_1 \approx 1$ and $A_2 \approx \tilde{t}$.

What are implications of our result Eq. (22) for the chaotic universe scenario?

At least one thing is clear; the probability distribution of the scale factor becomes broader while its mean value follows the classical value as time goes on. This phenomenon is also easily understood from consideration of the uncertainty principle. Roughly speaking, the Hubble expansion rate is conjugate to the scale factor. If we sharply define the scale factor at the Planck time, the scale factor after some time would be extremely uncertain since the rate of the exponential expansion is very much uncertain., Perhaps it is important to point out that in contrast to the scale factor, the distribution of the scalar ϕ becomes more and more sharply peaked around the classical value as time goes on even if we take into account the zero point fluctuation of ϕ.[12] Hence inclusion of inhomogeneous component ϕ will not change a qualitative feature of the distribution of scale factor. It is not clear for the present author whether the broadening distribution of the scale factor alters the qualitative features of the global structure of the universe.[12]

V. Summary

In order to quantize a simplified version of general relativity (minisuperspace model), we have proposed a special gauge choice, $T(p,q) - t \approx 0$. Here the "time" T(p,q) is determined by classical dynamics. Then the quantum mechanics of the universe is formally given by Eq.(1)((13)) with the Hamiltonian (12).

Fortunately enough the problem is exactly solvable in the case of chaotic inflation model if one concentrates on the nonoscillation regime. The probabilistic distribution of the scale factor of universe is explicitly given. It spreads by a huge amount corresponding to the gigantic expansion of universe.

VI. Acknowledgements

The author would like to think Dr. R. Holman for reading the manuscript. He also thanks people in the theory group at Fermilab for their warm hospitality.

References

[1] Wheeler, J.A., in"Battle Recontes 1967", p273, DeWitt, C.M. and Wheeler, J.A., eds., Benjamin, New york, (1968).

[2] Blyth, W.F. and Isham, C.J., Phys. Rev. **D11** 768(1975), For reviews, e.g. MacCallum, M.A.H., in "Quantum gravity: an Oxford symposium", Isham, C.J., Penrose, R., and Sciama, D.W., eds., Claredon Press, Oxford, (1975), Ryan, M.P., "Hamiltonian Cosmology", Springer-Verlag, Heidelberg, (1972).

[3] DeWitt, B.S., Phys. Rev. **160**, 1113 (1967).

[4] Guth, A.H., Phys. Rev.**D23** 347 (1981), Einhorn, M.B., Sato, K.,Nucl. Phys. **180** 385 (1981), Linde, A.D., Phys.Lett. **108B** 389 (1982), Albrecht, A., and Steinhardt, P.J., Phys. Rev. Lett, **48** 1220 (1982).

[5] Linde, A.D., Rep. Prog. Phys. **47** 925 (1984).

[6] Hawking, S. W. and Hartle, J.B., Phys. Rev. **D28** 83(1984), Vilenkin, A., Phys. Rev. **D30** 508 (1984).

[7] Arnowitt, R., Deser, S. and Misner, C.M., in "Gravitation: An Introduction to Current Research", Witten, L. ed., Wiley New York, (1962).

[8] Dirac, P.A.M., "Lectures on Quantum Mechanics", Belfer Graduate School of Science, Yeshiva University, New York, (1964).

[9] Hanson, A., Regge, T. and Teitelboim, C., "Constrained Hamitonian Systems", Accaemia Nazionale dei Lincei, Rome, 1975.

[10] Pauli, W., "Handbuch der Physik", Fluügge ed., Springer Verlag, (1958) vol. 5/1 p60. Gotto, T.,Naka, S. and Yamaguchi, K., Prog. Theor. Phys. **64**, 1, (1981).

[11] Guth, A.H. and Pi, So-Young, Phys. Rev. **D32** 1899 (1985)

[12] Linde, A.D., "Eternally Existing, Self-Reproducing Inflationary Universe", Lebedev Institute Preprint, (1986).