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INTRODUCTION TO THE NONLINEAR DYNAMICS ARISING FROM MAGNETIC MULTIPOLES*

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ABSTRACT

We derive a Hamiltonian describing transverse particle motion in a storage ring. After a transformation to "action-angle" variables we show how to apply Green's function techniques to do Lie transform perturbation theory on this Hamiltonian. Two examples are worked out to second order: (1) normal and skew quadrupole field errors and (2) normal sextupoles. A brief discussion of the single resonance term Hamiltonian includes derivations of the two invariants and calculation of the resonance width for one degree of freedom systems. Finally, we generalize Courant's treatment of modulational diffusion as an illustrative application of Chirikov's criterion to a multi-resonance problem.

keywords: accelerator theory, Deprit's algorithm, Hamiltonian dynamics, Lie transforms, magnetic multipoles, modulational diffusion, nonlinear dynamics, perturbation theory, resonance theory.

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And why [did] people just naturally assume that [you'd] know what they're talking about. ... Every other man spoke a language entirely his own, which he had figured out by private thinking; he had his own ideas and peculiar ways. If you wanted to talk about a glass of water, you had to start back with God creating the heavens and the earth; the apple; Abraham; Moses and Jesus; Rome; the Middle Ages; gunpowder; the Revolution; back to Newton; up to Einstein; then war and Lenin and Hitler. After reviewing this and getting it all straight again you could proceed to talk about a glass of water. "I'm fainting, please get me a little water." You were lucky even then to make yourself understood. And this happened over and over and over with everyone you met. You had to translate and translate, explain and explain, back and forth, and it was the punishment of hell itself not to understand or be understood, not to know the crazy from the sane, the wise from the fools, ...

— Saul Bellow,
Seize the Day

SECTION I   INTRODUCTION

Virtually all calculations in accelerator theory are ultimately concerned with the stability of some equilibrium situation. Among them is a class of problems dealing with stability of particle orbits under small variations in magnetic fields. Even small imperfections in otherwise linear magnetic fields can produce important dynamical effects, and in this paper we shall introduce some of the ideas and methods that aid in analyzing those effects. A short survey cannot do justice to such a far-reaching subject, so we shall adopt a few *RULES OF ENGAGEMENT* in order to limit the scope of this inquiry:

(1) Only single particle dynamics will be considered; all coherent effects are to be ignored vigorously.

(2) Except for the treatment of synchrobetatron oscillations in the last section, longitudinal motion will be neglected; we shall consider only the transverse dimensions.

(3) The particle orbit is on-momentum: $\delta p = 0$.

(4) In keeping with (2) and (3), we shall treat static fields only. In particular, there is neither RF nor acceleration.

Even with these restrictions, we are confronted with an impressive expanse of phenomena to be explored. They can be classified vaguely according to severity, the three major scenarios being:

MILD. Depending on whether its tunes are commensurate, either (1) an orbit is homeomorphic to a circle or (2) its closure is homeomorphic to an n-dimensional torus (the celebrated "invariant torus") embedded in a 2n-dimensional phase space. Small nonlinear terms do no more than (a) distort these tori from their linear configurations and (b) introduce amplitude dependent tunes.

MODERATE. Nonlinearities with the appropriate harmonic structure produce locally unstable orbits, signalling the presence of a resonance. This situation is characterized by the existence of *separatrices*, surfaces which partition phase space into regions of inequivalent dynamics.

SEVERE. At large enough amplitudes the tori break up completely and the dynamics is characterized by *chaos*. The distribution of orbits on a Poincare map, seen as a final picture, seems almost random, although it is not unusual for *islands of stability* to float on this chaotic sea.

Generally, all three types of behavior are present simultaneously but manifested in different regions of phase space. Most of what concerns us in this paper falls into either the mild or moderate category.

In Section 2 we shall open these deliberations by tracing how magnetic field nonlinearities are introduced into a Hamiltonian commonly used to describe transverse orbital dynamics. Section 3 is devoted to calculating orbit distortion using perturbation theory, with special attention given to automated Lie transform methods. Resonant orbits are discussed in Section 4, while Section 5 touches briefly upon the question of multiple resonances and its connection with modulational instability.

The authors of these Accelerator Summer School lectures have been asked to sow problems throughout their texts; you will come across twelve while reading this material. Rather than include more locally, let me suggest a

GLOBAL PROBLEM: Fill in all the pieces omitted from the derivations in subsequent sections. Verify all equations. Find all the errors, and report them to the author.

SECTION II A TRANSVERSE HAMILTONIAN

As stated above, the purpose of this section is to trace the introduction of field nonlinearities into an accelerator Hamiltonian. Now, it is tacitly acknowledged that a derivation in physics is not like one in mathematics. The latter proceeds by a series of logically acceptable steps so that if the premise is correct, the conclusion must be also. A physics derivation, on the other hand, seldom connects the "correct" to the "correct;" it usually connects the "correct" to the "useful." The physicist does this by piercing it liberally with subsidiary assumptions, approximations, or swindles, as needed. We shall not distinguish between these three types of qualifiers in what follows but simply label them all "conditions", or CONs for short. One of the major objectives of this section will be to expose CONs as they surface so that the reader can relax them more easily when he needs to generalize our final expressions.[1] That objective is dropped in subsequent sections, where the reader must discover for himself when he is being hoodwinked.

Transverse phase space

We shall follow the approach used by everyone else for introducing transverse dynamical variables. (Courant & Snyder 1958)[2] Begin by defining the *reference orbit*, $\vec{r}_o(s)$: a closed, periodic orbit, parametrized by arc length, s, and possessing the periodicity of the lattice,

$$\vec{r}_o(s+2\pi R) = \vec{r}_o(s). \qquad (1)$$

A travelling frame of basis vectors $\{ \hat{u}_1(s), \hat{u}_2(s), \hat{u}_3(s) \}$ is constructed on this orbit using standard techniques from differential geometry.

$$\hat{u}_3 = d\vec{r}_o/ds \ ; \quad \hat{u}_1 = -\rho \, d\hat{u}_3/ds \ ; \quad \hat{u}_2 = \hat{u}_3 \times \hat{u}_1 \qquad (2)$$

---

1) It is essentially impossible for physicists to avoid either CONning or being CONned, but as far as possible, it is best to know when it is happening. No formalism should be accepted outside of its CONtext.

2) How scarce are the papers in accelerator theory which do not refer to this document!

We adopt the conventions that (1) $\hat{u}_3$ lies along the beam current, and (2) $\rho \geq 0$ (in which case, $\hat{u}_1$ points outward). The radius of curvature, $\rho(s)$, and torsion, $\omega(s)$, of the orbit are defined by the formulas of Serret and Frenet. (Coxeter 1969)

$$\begin{aligned} d\hat{u}_1/ds &= \frac{1}{\rho}\,\hat{u}_3 + \omega\,\hat{u}_2 \\ d\hat{u}_2/ds &= -\omega\,\hat{u}_1 \\ d\hat{u}_3/ds &= -\frac{1}{\rho}\,\hat{u}_1 \end{aligned} \tag{3}$$

Now define transverse position variables x_1 and x_2 relative to this orbit.

$$\vec{r} = \vec{r}_o(s) + x_1\hat{u}_1(s) + x_2\hat{u}_2(s) \tag{4}$$

We want x_1 and x_2 to fit into a canonical system of variables $(x_1,x_2,s;\ p_1,p_2,p_s;\ t)$. This is accomplished, starting from (canonical) Euclidean variables $(\vec{r};\ \vec{q};\ t)$, by applying the generating function

$$\begin{aligned} F(\vec{q};\ x_1,x_2,s) &\equiv \vec{q}\cdot\vec{r}(x_1,x_2,s) \\ &= \vec{q}\cdot[\ \vec{r}_o(s) + x_1\hat{u}_1(s) + x_2\hat{u}_2(s)\ ]\ . \end{aligned} \tag{5}$$

Momenta conjugate to (x_1,x_2,s) are then obtained immediately.

$$\begin{aligned} p_1 &= \vec{q}\cdot\hat{u}_1 \\ p_2 &= \vec{q}\cdot\hat{u}_2 \\ p_s &= (\ 1 + x_1/\rho\ )\ \vec{q}\cdot\hat{u}_3 \ + \ \omega(\ x_1p_2 - x_2p_1\ ) \\ &\xrightarrow[\omega=0]{} (\ 1 + x_1/\rho\ )\ \vec{q}\cdot\hat{u}_3 \end{aligned} \tag{6}$$

That last expression for p_s follows from

<u>CON 1</u>: <u>No torsion</u>. We assume that $\omega = 0$ on the reference orbit.

Multipole field representation

Forget temporarily about the curvilinear coordinate system we just introduced, and go back to a Euclidean system, — say (x,y,z), with frame $\{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \}$. We want to represent in this system the field inside a magnet. To simplify the final expression, introduce three more CONs.

CON 2: Translation symmetry. We are going to neglect edge effects or any other variations that might occur in traversing the length of the magnet.

CON 3: $\partial/\partial t = 0$. We shall deal here with static fields only. (See the fourth rule of engagement.)

CON 4: $\vec{B}\cdot\hat{e}_3 = 0$. Neglect all longitudinal components of the magnetic field. In particular, no solenoids are allowed. (This is in agreement with CON 1, since *non-trivial* torsion would require a longitudinal field.)

With these provisos, there is a gauge in which the scalar potential vanishes and the vector potential is a longitudinal, harmonic, vector function of the transverse coordinates. Its power series representation can be written economically in terms of a complex variable $\zeta \equiv x + iy$.

$$\begin{aligned} \vec{A} = A_3(x,y)\hat{e}_3 \quad &\Rightarrow \quad \nabla\cdot\vec{A} = 0 \\ &\Rightarrow \quad \nabla^2\vec{A} = 0 \\ &\Rightarrow \quad A_3 \sim \left.\begin{matrix}\mathrm{Re} \\ \mathrm{Im}\end{matrix}\right\} (x + iy)^n \end{aligned} \tag{7}$$

If we think of A_3 as being the real part of some complex function $\Gamma(\zeta)$, then the transverse magnetic field is obtained by differentiation.

$$\begin{aligned} A_3 &= \mathrm{Re}[\ \Gamma(\zeta)\ ]\ , \\ \vec{B} &= \nabla \times \vec{A} \quad \Rightarrow \quad B_2 + iB_1 = -\frac{d\Gamma(\zeta)}{d\zeta} \end{aligned} \tag{8}$$

Multipole coefficients appear as the coefficients in a power series expansion for $\Gamma(\zeta)$. The Fermilab convention — or at least, what was the Fermilab convention at the time of Tevatron construction — for labelling these is as follows.

$$
\begin{aligned}
B_2 + iB_1 &= B_o \sum_{n=0}^{\infty} (b_n + ia_n)\, \zeta^n \\
\Gamma(\zeta) &= -B_o \sum_{n=1}^{\infty} \frac{1}{n} (b_{n-1} + ia_{n-1})\, \zeta^n
\end{aligned}
\tag{9}
$$

The b_n's and a_n's are respectively the *normal* and *skew* multipole coefficients. Notice that the pole number is 2(n+1); for example, b_2 and a_2 correspond to sextupole excitation. B_o is a normalizing field, usually taken to be the value of the bending dipole field, in which case $b_0=1$.

Now, we have chosen to express the dynamics in a coordinate system that is not Euclidean but curved — $\rho\neq 0$ inside a dipole — so we must deal with this complication. It turns out that naively replacing $\hat{e}_3 \rightarrow \hat{u}_3$, $x \rightarrow x_1$, and $y \rightarrow x_2$ works well provided that (1) the accelerator is large enough, and (2) we simultaneously replace $b_1 \rightarrow b_1 - b_0/\rho$. To partially justify this assertion, consider what would happen if we replaced CON 4 with

<u>CON 4</u>′: <u>$\vec{B}\cdot\hat{u}_3 = 0$</u>. This says again that $\vec{B}$ has no longitudinal component, but "longitudinal" is now defined to be along the beam direction.

Then it is possible to choose a gauge for which

$$\vec{A} = A_3(x_1,x_2)\ \hat{u}_3(s) \ . \tag{10}$$

<u>WARNING</u>: A_3 is not the same function in Eq.(10) as in Eq.(7). I am redefining the symbol within the scope of this argument.

The one non-trivial Maxwell equation which remains to be satisfied is: $\nabla\times\vec{B} = -\nabla^2\vec{A} = 0$. Using our curvilinear coordinates, this becomes

$$(\rho+x_1)^2(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)A_3 \quad + \quad (\rho+x_1)\partial A_3/\partial x_1 \quad = \quad A_3 \quad . \tag{11}$$

Suppose we again expand A_3 in a power series,

$$A_3 = -B_o \sum_{\substack{m,n=0 \\ m+n>0}}^{\infty} (c_{mn} + e_{mn})\ x_1{}^m x_2{}^n \quad , \tag{12}$$

where the coefficients c_{mn} are precisely those we had before,

$$c_{mn} = \frac{1}{m+n} \begin{pmatrix} m+n \\ n \end{pmatrix} \begin{cases} (-1)^{n/2} & b_{m+n-1} \quad , \text{ even } n \\ (-1)^{(n+1)/2} & a_{m+n-1} \quad , \text{ odd } n \end{cases} \tag{13}$$

and are characterized by the condition,

$$(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2) \sum c_{mn}\ x_1^m\ x_2^n \quad = \quad 0 \ . \tag{14}$$

Applying Eq.(11) then provides the e_{mn} recursively:

$$\begin{aligned} &(m+2)(m+1)\rho^2 e_{m+2,\ n} \quad + \quad (n+2)(n+1)\rho^2 e_{m,\ n+2} \\ &+ (m+1)(2m+1)\rho e_{m+1,\ n} \quad + \quad 2(n+2)(n+1)\rho e_{m-1,\ n+2} \\ &+ (m+1)(m-1)e_{mn} \quad + \quad (n+2)(n+1)e_{m-2,\ n+2} \\ &= \quad -[\ (m+1)\rho c_{m+1,\ n} + (m-1)c_{mn}\ ] \quad , \end{aligned} \tag{15}$$

with initial conditions,

$$\forall n: \quad e_{1n} = e_{0n} = 0 \quad . \tag{16}$$

Specifically working out the first few of these, we get the following expression for A_3.

$$
\begin{aligned}
A_3 = - B_o \Big[& b_0 x_1 - a_0 x_2 + \tfrac{1}{2}(b_1 - b_0/\rho)\, x_1^2 - \tfrac{1}{2} b_1 x_2^2 - a_1 x_1 x_2 \\
& + (b_2/3 - b_1/6\rho + b_0/2\rho^2) x_1^3 - b_2 x_1 x_2^2 \\
& - (a_2 - a_1/2\rho + a_0/2\rho^2) x_1^2 x_2 + \tfrac{1}{3} a_2 x_2^3 \\
& + \cdots \text{ and so forth } \Big]
\end{aligned} \tag{17}
$$

The argument can now advance along two fronts: (I) In large synchrotrons $b_{n+1}\rho \gg b_n$ for $n\geq 1$, and $a_{n+1}\rho \gg a_n$ for $n\geq 0$. All the correction terms can be neglected, therefore, *with the exception* of b_0/ρ. (II) In large synchrotrons $x/\rho \ll 1$, so $b_1 x^3/\rho \ll b_1 x^2$, $b_2 x^4/\rho \ll b_2 x^3$, and so forth. Whichever one you choose to believe, the conclusion remains unchanged when more general schemes for handling curvature are used. Regardless of the scheme, the quadratic term, which is purely kinematic in origin, will always be $b_1 - b_0/\rho$. All of this effectively returns us to the representation of Eqs. (8) and (9) for the nonlinear terms and justifies our assertion.

CON 5: Large machine. The radius of curvature ρ must be large enough to justify neglecting the higher order terms in which $1/\rho$ appears.

PROBLEM 1: Relax CON 4 (or 4′) and write expressions for the vector potential up to third degree polynomials in x_1 and x_2.

Hint: Choose a gauge in which the transverse dot product $x_1A_1+x_2A_2$ vanishes. There is then some function V such that $A_1=-x_2V$ and $A_2=x_1V$.

The Hamiltonian

We are now ready to construct a transverse Hamiltonian from the minimal coupling Hamiltonian,

$$
E = [(\vec{p} - e\vec{A})^2 c^2 + m^2 c^4]^{1/2} + eV \quad . \tag{18}
$$

CON 6: Ignore eV. This is related to CON 3, in which we restricted our deliberations to static fields. It also means that we are not going to treat coherent effects arising from wake fields. That, of course, is in agreement with the rules of engagement outlined in Section I.

After throwing away the electric term, square both sides and express vector dot products in terms of the conjugate momenta of Eq.(6).

$$
\begin{aligned}
E^2 &= (\vec{p} - e\vec{A})_\perp^2 \, c^2 + \frac{1}{(1 + x_1/\rho)^2} \, (p_s - eA_s)^2 c^2 + m^2 c^4 \\
&= p_\perp^2 + \left(\frac{p_s}{1 + x_1/\rho} - eA_3 \right)^2 c^2 + m^2 c^4
\end{aligned}
\tag{19}
$$

(The subscript $_\perp$ indicates transverse coordinates.) Notice that we have used our gauge condition $\vec{A}_\perp = 0$, which depends on CON 4′, to eliminate the transverse vector potential.

The next step is purely mathematical: change the roles of t and s. That is, we want to treat time as a dynamical variable and arc length as the "independent" variable. The variable conjugate to t is -E, and the new Hamiltonian is $-p_s$.

$$
\begin{aligned}
&(x_1, x_2, s;\; p_1, p_2, p_s;\; t) \longrightarrow (x_1, x_2, t;\; p_1, p_2, -E;\; s) \\
&H \equiv -p_s = -(1+x_1/\rho) \left(eA_3 + \frac{1}{c} \left(E^2 - m^2 c^4 - p_\perp^2 c^2 \right)^{1/2} \right)
\end{aligned}
\tag{20}
$$

The next CON is only a numerical approximation. By virtue of the static assumption, E is a constant of the motion. Its numerical value is approximated by ignoring the proton's transverse momentum and the magnetic fields. It is expressed in terms of the magnetic rigidity, $|B\rho|$, as follows.

CON 7: Numerical approximation. $E^2 - m^2 c^4 := p_3^2 \, c^2 \cong (eB\rho)^2 c^2$

(Note: p_3 is a number, not a variable!) This brings us to

$$H = -(1+x_1/\rho)\left[\, eA_3 + (p_3^2 - p_\perp^2)^{1/2} \,\right] \quad . \tag{21}$$

We now expand this in powers of $p_\perp/p_3$, keeping only the lowest order terms. Of course, this requires

<u>CON</u> <u>8</u>: <u>Small angles</u>. $p_\perp/p_3 \ll 1$. Further, $(p_\perp/p_3)^4 \ll$ any "small" nonlinear terms in eA_3/p_3 whose effects are considered important!

$$\begin{aligned} H &\simeq -(1+x_1/\rho)\left(eA_3 + p_3(1 - p_\perp^2/2p_3^2 + \cdots) \right) \\ &= - eB_o a_0 x_2 - eB_o a_1 x_1 x_2 \\ &\quad + \left(\frac{1}{2p_3} (p_1^2 + p_2^2) + \frac{eB_o}{2} (b_1 + b_0/\rho) x_1^2 - \frac{eB_o}{2} b_1 x_2^2 \right) \\ &\quad - eA_3^{(hot)} \end{aligned} \tag{22}$$

Note that we have again used CON 5 to approximate $a_1 + a_0/\rho \simeq a_1$. The superscript "hot" stands for "higher order terms."

We are going to ignore the first two terms, involving skew dipoles and quadrupoles. In principle we could continue to carry them along either treating them later as perturbations or incorporating them into the linear solution. To keep this development simple, however, it is easier just to drop them here and assume that the linear part of the Hamiltonian is decoupled — that is, horizontal and vertical motions are independent in the linear approximation.

<u>CON</u> <u>9</u>: <u>No skew linear elements</u>. $a_0 \equiv a_1 \equiv 0$.

The linear equations of motion derived, from the quadratic part of the Hamiltonian, are then the Kerst-Serber equations. (Kerst & Serber 1941)

$$d^2x_1/ds^2 = \frac{n-1}{\rho^2}\, x_1 \quad , \quad d^2x_2/ds^2 = -\frac{n}{\rho^2}\, x_2 \quad ;$$
$$n = -eB_ob_1\rho^2/p_3 = -B_ob_1\rho/B \tag{23}$$

There are two more transformations to be made — more a matter of convenience than anything else — before introducing action-angle variables:

(1) Change the transverse momentum variables by dividing by p_3. With this, the momenta will be interpreted as transverse angles. The new Hamiltonian is simply the old divided by p_3.

$$\text{new } \vec{p}_\perp := \text{old } \vec{p}_\perp/p_3 \quad \Leftrightarrow \quad \text{new } H := \text{old } H/p_3 \tag{24}$$

(2) Change the "independent" coordinate from arc length, s, to angle, θ, the relationship between the two being $ds = Rd\theta$. The Hamiltonian must be multiplied by R.

<u>WARNING</u>: <u>NOTATION</u> <u>SHIFT</u>. We shall use the old symbols, p and H, to represent the new momenta and Hamiltonian rather than inventing new ones.

Putting the pieces together, we get the following result.

$$H = \frac{R}{2}\left(p_1^2 + p_2^2 \right) + \frac{1}{2}\frac{B_oR}{B\rho}\left[(b_1 + b_0/\rho)x_1^2 - b_1x_2^2 \right]$$
$$- \frac{R\, A_3^{(hot)}}{B\rho} \tag{25}$$

"Action-angle" formulation

Everyone reading these words knows how to handle the quadratic part of the Hamiltonian, or should by this time, the matter having been rather frequently mentioned in these summer schools. (Courant 1982, Collins 1983, Edwards 1985) This is "merely" the linear machine. Its solution introduces *lattice functions* $\beta_k(\theta)$, $\psi_k(\theta)$, and $\alpha_k(\theta)$, all of which are related via

$$\beta_k d\psi_k = ds \quad , \qquad d\beta_k = -2\alpha_k ds \quad , \text{ for } k=1,2. \tag{26}$$

The linear tunes, ν_1 and ν_2, are defined by the condition

$$\psi_k(\theta+2\pi) = \psi_k(\theta) + 2\pi\nu_k .$$

We shall make use of the auxiliary functions

$$\tilde{\psi}_k(\theta) = \psi_k(\theta) - \nu_k\theta \ , \qquad k = 1,2 \quad , \tag{27}$$

which are 2π-periodic. That property makes possible a canonical transformation to "action-angle"[3] variables, $(\delta_1,\delta_2;\ I_1,I_2)$,

$$\begin{aligned} x_k &= [\ 2I_k\beta_k(\theta)\ ]^{1/2} \sin(\ \tilde{\psi}_k(\theta) + \delta_k\ ) \\ q_k &\equiv [\ 2I_k\beta_k(\theta)\ ]^{1/2} \cos(\ \tilde{\psi}_k(\theta) + \delta_k\ ) \\ &= \alpha_k(\theta)\ x_k \ + \ \beta_k(\theta)\ p_k \qquad\qquad , \ k = 1,2. \end{aligned} \tag{28}$$

which can be accomplished via generating functions (Snowdon 1969),

$$F_k(x_k,\ \delta_k;\ \theta) = (x_k^2/2\beta_k(\theta))(\ -\ \alpha_k(\theta) + \cot[\ \tilde{\psi}_k(\theta) + \delta_k\ ]\ )\ . \tag{29}$$

Written in action angle variables, the quadratic part of our Hamiltonian becomes simply

$$\begin{aligned} H^{(quad)} &= \nu_1 I_1 + \nu_2 I_2 \\ &\equiv \underline{\nu}\cdot\underline{I} \quad , \end{aligned} \tag{30}$$

---

3) I_k is not necessarily a true action when the equations of motion are nonlinear. Perhaps "polar variables" or "amplitude-angle variables" would be a better name.

where for any subscripted variable ξ_k, $k=1,2$ we shall define the 2-tuple $\boldsymbol{\xi} \equiv (\xi_1,\xi_2)$. This describes, of course, a system of two uncoupled harmonic oscillators: I_1 and I_2 are constants of the motion, and δ_1 and δ_2 increase linearly with θ at rates ν_1 and ν_2. The action variables are trivially related to horizontal and vertical emittances. Note first the equivalence of the two differential forms.

$$dx \wedge dx' = d(x/\sqrt{\beta}) \wedge d(\sqrt{\beta}\, x') = d(x/\sqrt{\beta}) \wedge d(q/\sqrt{\beta}) \tag{31}$$

We have temporarily dropped the subscript. Plotted in $(x/\sqrt{\beta},\ q/\sqrt{\beta})$ space, the orbit is a circle of radius $\sqrt{2I}$. (See Eq.(28)) Because of Eq.(31) the area of this circle is in fact the (horizontal or vertical) emittance, W, associated with the orbit, and therefore, putting the subscript back,

$$W_k = 2\pi I_k \tag{32}$$

If ν_1 and ν_2 are commensurate — that is, if they obey a *resonance condition*,

$$m_1\nu_1 + m_2\nu_2 = 0 \quad , \tag{33}$$

for some doublet of integers $(m_1,\ m_2)$ — then the orbit must be periodic (diffeomorphic to a circle). If ν_1 and ν_2 are incommensurate, then the orbit is uniformly dense on a torus.[4]

PROBLEM 2: Show that the torus is not contained in some three-dimensional subspace of four-dimensional phase space. That is, show that no rotation of the phase space coordinate system will zero one of the torus's coordinates.

Hint: Calculate the covariance matrix of the torus.

While your attention was diverted by this quick review of linear theory, another swindle has been perpetrated. The existence

---

4) Showing that the orbit is dense is easy; showing that it is uniform is harder. For a proof, see Sinai (1976).

of non-trivial lattice functions requires violating CON 2: the magnetic fields of a strong focussing accelerator must vary along the reference orbit. Therefore, the expansion for A_3 is no longer valid and we must add terms to make it more general. Rather than do that, we shall ignore this complication and treat A_3 *as though it could be switched on and off suddenly* as one passes through the edge of a magnet. Essentially, we are ignoring edge effects and closing our eyes to the fact that the vector potential we are using does not satisfy Maxwell's equations in the vicinity of magnet edges.

CON 10: Layer approximation. We are going to treat the θ dependence of A_3 as a sequence of step functions.

The final step is to write the higher order, nonlinear part of H in action-angle form. This involves nothing more than a good deal of messy algebra, which we shall try to get through as painlessly as possible. According to Eqs. (25) and (13) we have the following expression for $H^{(hot)}$.

$$
\begin{aligned}
H^{(hot)} &= -eA_3^{(hot)} \cdot R/p_3 \\
&= \frac{B_o R}{B\rho} \sum_{m',p'} c_{m'p'}(\theta)\; x_1^{m'} x_2^{p'}
\end{aligned}
\tag{34}
$$

First, write the monomials $x_1^{m'} x_2^{p'}$ in terms of (δ, I).

$$
\begin{aligned}
x_1^{m'} x_2^{p'} &= (2I_1)^{m'/2} (2I_2)^{p'/2} (-i/2)^{m'+p'} \\
&\quad \times \sum_{\substack{m,p \\ (\text{step } 2)}} (-1)^{(m'+p'-m-p)/2} \binom{m'}{(m'-m)/2} \binom{p'}{(p'-p)/2} \\
&\qquad\qquad \times W_{m\,p}^{m'p'}(\theta)\; e^{i(m\delta_1 + p\delta_2)}
\end{aligned}
\tag{35}
$$
,

$$
W_{m\,p}^{m'p'}(\theta) = \beta_1^{m'/2}(\theta)\; \beta_2^{p'/2}(\theta)\; e^{i[m\tilde{\psi}_1(\theta) + p\tilde{\psi}_2(\theta)]}
$$

The summation extends over the range,

$$m = -m', -m'+2, -m'+4, \ldots, m'-2, m',$$
$$p = -p', -p'+2, -p'+4, \ldots, p'-2, p'.$$

Then, substitute these back into Eq.(34) and interchange the order of summation.

$$H^{(hot)} = \frac{B_o R}{B\rho} \sum_{m,p} H_{mp}(\underline{I};\ \theta)\ e^{i(m\delta_1 + p\delta_2)}$$

$$H_{mp} = \sum_{k,q\geq 0} \tilde{H}_{m}^{|m|+2k}{}_{p}^{|p|+2q} \tag{36}$$

$$\tilde{H}_{m\ p}^{m'p'} = (-i)^{m+p} \binom{m'}{(m'-m)/2} \binom{p'}{(p'-p)/2} \times (I_1/2)^{m'/2} (I_2/2)^{p'/2} c_{m',p'}(\theta)\ W_{m\ p}^{m'p'}(\theta)$$

And we are finished.

SECTION III LIE TRANSFORM PERTURBATION THEORY

Having written a Hamiltonian, our task now is to solve its dynamics. Eventually we will be reduced to numerically integrating the equations of motion, but let us put off the inevitable and first try to gather information analytically. If the magnetic field nonlinearities are accidental, arising from errors in magnet construction, then the corresponding terms in the Hamiltonian will be small, and perturbation theory presents a natural approach.

For the time being it is convenient to picture the motion as taking place within the MILD scenario, although this is not a real restriction, as we shall see below. The orbits then lie on phase space surfaces that are deformed tori. What we want from perturbation theory is a method of building a transformation, order by order in the size of the perturbation, that will straighten them into normal form tori. Deprit's algorithm, the procedure that we shall describe here, was developed originally for doing such calculations in celestial mechanics; it has been used profitably in plasma physics as well.[5] (Deprit 1969; Kamel 1970) We shall first lay out the algorithm, without derivation, and then adapt it to our own problem. It is certainly not the only approach for bringing a Hamiltonian into normal form, but it does have the advantage of being completely explicit, which makes it easy to automate; for an example of a more conventional method see Ando (1983) or Ohnuma (1984). Forest (1985) has recently proved the equivalence of the normal forms computed by Deprit's algorithm and those used by MARYLIE. (Dragt 1982; Douglas 1982) If the variant of Deprit's algorithm described below is restricted (i) to first order in the transformation and second order in the Hamiltonian and (ii) by ignoring the contributions of resonances, then the resultant expressions reproduce the "distortion functions" of Collins (1984).

Deprit's equations

To set up the problem in the abstract, let us assume that the Hamiltonian of interest depends on a "small parameter", ε.

$$d\underline{z}^{\star}/d\theta = [\ \underline{z}^{\star},\ H(\ \underline{z}^{\star};\ \theta;\ \varepsilon\ )\ ] \qquad (37)$$

$$\underline{z}^{\star} \equiv \begin{pmatrix} \underline{\delta}^{\star} \\ \underline{I}^{\star} \end{pmatrix} \ ,$$

We want to find a phase space transformation, T,

$$\underline{z}^{\star} \equiv (\underline{\delta}^{\star}, \underline{I}^{\star}) \xleftarrow[T(\varepsilon,\theta)]{} \underline{z} \equiv (\underline{\delta}, \underline{I}), \qquad (38)$$

5) A more complete list of references to the papers in which Deprit's algorithm has been derived, studied, or applied can be found in the bibliography of Michelotti (1985).

such that the dynamical equations in the new coordinates, governed by a Hamiltonian K,

$$
\begin{aligned}
& d\underline{z}\,/d\theta = [\,\underline{z}\,,\,K(\,\underline{z}\,;\,\theta;\,\varepsilon\,)\,] \\
& \underline{z} \equiv \begin{pmatrix}\underline{\delta}\\ \underline{I}\end{pmatrix}\,,
\end{aligned} \tag{39}
$$

are easier to integrate. This transformation will be realized as a Lie transform generated by a function $S(\underline{z};\theta;\varepsilon)$. By this we mean formally that T must satisfy the following two conditions as a function of ε,

(a) $T(\varepsilon=0) = 1$ (the identity transformation),

(b) $dT(\varepsilon)/d\varepsilon = [\,T(\varepsilon),\,S(\varepsilon)\,] \equiv L_S(\varepsilon)T(\varepsilon)$,

where $[\,\cdot\,,\,\cdot\,]$ represents a Poisson bracket, and the "adjoint operator" L_S is defined

$L_S \equiv [\,\cdot\,,\,S\,] = -\,:S:$ (in the notation of Dragt (1982))

By inspection, the formal solution is given by

$$
\begin{aligned}
T(\varepsilon) &= 1 + \int_0^{\varepsilon} d\varepsilon'\,L_S(\varepsilon') + \int_0^{\varepsilon} d\varepsilon' \int_0^{\varepsilon'} d\varepsilon''\,L_S(\varepsilon')\,L_S(\varepsilon'') + \ldots \\
&= \varepsilon\_\text{ordered\_product}\left[\,\exp \int_0^{\varepsilon} d\varepsilon'\,L_S(\varepsilon')\,\right]
\end{aligned} \tag{40}
$$

All of this is implemented within a perturbation theory by expanding everything as power series in ε.

$$
H = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\,H_n \quad , \quad H_0(\underline{\delta}^{\star},\,\underline{I}^{\star};\,\theta) = \underline{\nu}\cdot\underline{I}^{\star} \tag{41}
$$

$$
K = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\,K_{0n} \tag{42}
$$

$$
S = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\,S_{n+1} \qquad ;\; S_0 \equiv 0 \tag{43}
$$

$$\underline{z}^{\star} = T(\varepsilon;\theta)\underline{z} = \sum_{n=0}^{\infty} \frac{\varepsilon^{n}}{n!} \; \underline{z}^{\star}_{0n}(\underline{z};\, \theta) \tag{44}$$

(The shifted index S_{n+1} is conventional; the double subscripts on K_{0n} and $\underline{z}^{\star}_{0n}$ will be convenient later.) The unknown functions S_n and K_{0n} are related to each other through a sequence of partial differential equations, the Deprit equations. (Deprit 1969) After defining the Lie derivative operator,

$$D \equiv \partial/\partial\theta + L_{H_0} \qquad \text{(i.e., } Df = \partial f/\partial\theta + [f, H_0]\text{)} \tag{45}$$

these equations are written as follows.

$$\begin{aligned}
DS_n + K_{0n} &= H_n + \Sigma_n^{(H)} - \Sigma_n^{(K)} \;, \\
\Sigma_n^{(H)} &= \sum_{m=1}^{n-1} \binom{n-1}{m-1} [\, H_{n-m} \,,\, S_m \,] \;, \\
\Sigma_n^{(K)} &= \sum_{m=1}^{n-1} \binom{n-1}{m-1} K_{m\;n-m} \;, \\
K_{m\;n-m} &= \sum_{j=1}^{m} \binom{m-1}{j-1} [\, S_j \,,\, K_{m-j\;n-m} \,] \;.
\end{aligned} \tag{46}$$

This system appears intimidating, but in fact it is very cleanly structured. The partial differential equations have the same form at all levels of the perturbation:

$$DS_n + K_{0n} = \mathrm{rhs}_n \;, \tag{47}$$

where each rhs_n is assembled from brackets of functions found at lower levels of the sequence. To illustrate, we shall write the first few of these explicitly.

LEVEL	EQUATIONS
n=0	$K_{00} = H_0$
n=1	$DS_1 + K_{01} = H_1$
n=2	$K_{11} = [\, S_1, K_{01} \,]$
	$DS_2 + K_{02} = H_2 + [\, H_1, S_1 \,] - K_{11}$
n=3	$K_{12} = [\, S_1, K_{02} \,]$
	$K_{21} = [\, S_2, K_{01} \,] + [\, S_1, K_{11} \,]$
	$DS_3 + K_{03} = H_3 + [\, H_2, S_1 \,] + 2[\, H_1, S_2 \,] - K_{21} - 2K_{12}$

(48)

These equations <u>define</u> <u>the</u> <u>relationship</u> between the Lie generator S and the transformed Hamiltonian K. Ideally, we would like to set all K_{0n} to zero, or at least we would prefer that K be a function of $\underline{I}$ only. Generally, this will not be possible, because any rhs_n may contain a component *not in the range of the differential operator D.* This component then must be projected out and absorbed into K_{0n} for that equation to have a solution. If we adopt the principle of keeping the new Hamiltonian as simple as possible, then these will be the <u>only</u> terms in K.

We shall see shortly that the terms absorbed by K come from "slowly varying", "long wavelength" fluctuations that give rise to resonances, while the "rapidly varying", "short wavelength" terms that simply produce distortion get relegated to the transformation $T(\epsilon)$.

Once the Lie generator is constructed and the dynamics solved using the new Hamiltonian, the full transform must be applied to the coordinate functions, $\underline{z} \equiv (\underline{\delta},\underline{I})$, in order to map them back into the original ones, $\underline{z}^\star \equiv (\underline{\delta}^\star,\underline{I}^\star)$. The functions $\underline{z}^\star_{0n}$ defined in Eq.(44) are iteratively constructed as follows.

$$
\begin{aligned}
&\underline{z}^{\star}_{00}(\ \underline{z}\ ;\ \theta\ ) = \underline{z} \\
&\underline{z}^{\star}_{0n} = \mathbb{J}\cdot\partial S_n/\partial\underline{z} - \sum_{m=1}^{n-1}\binom{n-1}{m}\underline{z}^{\star}_{m\ n-m} \quad , \quad \mathbb{J} = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \\
&\underline{z}^{\star}_{m\ n-m} = \sum_{j=1}^{m}\binom{m-1}{j-1}[\ S_j\ ,\ \underline{z}^{\star}_{m-j\ n-m}\ ]
\end{aligned}
\tag{49}
$$

Green's functions

The Lie operator defined in Eq.(45) is simply the directional derivative along an unperturbed orbit. This observation provides us with at least one way of solving Deprit's system of equations: integration along unperturbed orbits.[6] Instead of that, we shall make use of the periodicity of the Hamiltonian and construct an appropriate Green's function. Under the assumption that H_0 is as in Eq.(41), the Lie derivative operator simplifies to

$$D = \partial/\partial\theta + \underline{\nu}\cdot\partial/\partial\underline{\delta}\ , \tag{50}$$

whose 2π-periodic eigenfunctions are exponentials in angle variables multiplied by arbitrary functions of the action variables.

$$
\begin{aligned}
&D[\ f(\underline{I})\ e^{i(n\theta\ +\ \underline{m}\cdot\underline{\delta})}\ ] = i(n + \underline{m}\cdot\underline{\nu})\ f(\underline{I})\ e^{i(n\theta\ +\ \underline{m}\cdot\underline{\delta})} \\
&\underline{m}\cdot\underline{\delta} = m_1\delta_1 + m_2\delta_2 \quad , \quad \underline{m}\cdot\underline{\nu} = m_1\nu_1 + m_2\nu_2
\end{aligned}
\tag{51}
$$

Note from this that the null space of D — that is, the functions annihilated by D — consists of those exponentials for which $n + \underline{m}\cdot\underline{\nu} = 0$. But this is just a resonance condition.[7] Thus, the

6) This is actually a variant of the "method of characteristics."

7) There is some ambiguity on this point. This is certainly considered a "resonance" in papers written by accelerator physicists, but not in the writings of many nonlinear dynamicists who reserve the word for the zeroth harmonic case, n=0, as written in Eq.(33).

minimal hamiltonian K will contain only the so-called "shear" (n=0, $\underline{m}$=0) and "secular" (resonant) terms.

We shall expand functions of $\underline{\delta}$ and $\underline{I}$ in the following basis.

$$\{ \phi_{\underline{em}} \mid \underline{e} = (e_1,e_2) \ , \ \underline{m} = (m_1,m_2) \ ; \ e_1,e_2,m_1,m_2 \ : \ \text{integers} \}$$

$$\phi_{\underline{em}}(\underline{\delta}, \underline{I}) = I_1^{e_1/2} \quad I_2^{e_2/2} \quad e^{i\underline{m}\cdot\underline{\delta}} \tag{52}$$

By employing these, evaluating Poisson brackets becomes a matter of bookkeeping, not differentiation; the bracket algebra is defined by specifying it on the basis.

$$\begin{aligned} [\phi_{\underline{em}} \ , \ \phi_{\underline{e}'\underline{m}'}] = & \ \tfrac{i}{2} (e_1'm_1 - e_1m_1') \ \phi_{(e_1+e_1'-2,\ e_2+e_2'),\ \underline{m}+\underline{m}'} \\ & + \tfrac{i}{2} (e_2'm_2 - e_2m_2') \ \phi_{(e_1+e_1',\ e_2+e_2'-2),\ \underline{m}+\underline{m}'} \end{aligned} \tag{53}$$

Because the operator D is linear and does not mix the $\phi_{\underline{em}}$, we can solve Deprit's equations one component at a time.

$$\begin{aligned} D[S_{n\ \underline{em}}(\theta) \ \phi_{\underline{em}}] &= [\text{rhs}_n - K_{0n}]_{\underline{em}}(\theta) \ \phi_{\underline{em}} \\ S_{n\ \underline{em}}(\theta) &= \int_0^{2\pi} d\theta' \ G_{\underline{m}}(\theta - \theta') \ [\text{rhs}_n - K_{0n}]_{\underline{em}}(\theta') \end{aligned} \tag{54}$$

For $\underline{m}\cdot\underline{\nu} \neq$ integer — that is, off resonance — the Green's functions $G_{\underline{m}}$ satisfy the differential equation,

$$\begin{aligned} D[G_{\underline{m}}(\theta - \theta') \ \phi_{\underline{em}}] &= \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' - 2\pi n) \quad \phi_{\underline{em}} \\ &= \delta_{per}(\theta - \theta') \quad \phi_{\underline{em}} \ , \end{aligned} \tag{55}$$

whose solution is found easily:

$$G_{\underline{m}}(\tau) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{e^{in\tau}}{n + \underline{m}\cdot\underline{\nu}} = \frac{e^{-i\underline{m}\cdot\underline{\nu}\ (\ \text{mod}(\tau)\ -\ \pi\ )}}{2i \ \sin(\ \pi \ \underline{m}\cdot\underline{\nu}\ )} \tag{56}$$

where $\text{mod}(\tau) = \tau \ (\text{mod } 2\pi) \ \epsilon \ [0,2\pi)$.

Near a resonance, the situation is only a little more complicated. Suppose that for some integer n_o we have $n_o + \underline{m}\cdot\underline{\nu} = 0$. Then, because D annihilates the function $e^{in_o\theta}\,\phi_{\underline{em}}$, we must have

$$\int_0^{2\pi} d\theta\ [\ rhs_n - K_{0n}\ ]_{\underline{em}}\ (\theta)\ e^{-in_o\theta} = 0\ , \quad \text{for all } \underline{e}. \tag{57}$$

As we mentioned before, this is accomplished by filtering any offending terms into the definition of K_{0n}. The Green's function can be modified as well by subtracting the contribution coming from the resonance term under consideration.

$$G_{\underline{m}}(\tau) = \frac{e^{-i\underline{m}\cdot\underline{\nu}(\ \mathrm{mod}(\tau) - \pi\ )}}{2i\ \sin(\ \pi\ \underline{m}\cdot\underline{\nu}\ )} - \frac{1}{2\pi i}\ \frac{e^{in_o\tau}}{n_o + \underline{m}\cdot\underline{\nu}} \tag{58}$$

This assures a finite result as the resonance is approached. In practice, one should filter resonances into K whenever $n_o + \underline{m}\cdot\underline{\nu}$ is "sufficiently close" to zero. The judgment required to make that decision is as yet non-algorithmic.

Even in the absence of non-trivial resonances, one must perform a subtraction for $\underline{m} = \underline{0}$, $n_o = 0$. Applying Eq.(58) in this limit we get the result

$$G_{\underline{0}}(\tau) = \frac{1}{2}\ [\ 1 - \frac{1}{\pi}\ \mathrm{mod}(\tau)\ ]\ . \tag{59}$$

Correspondingly,

$$K_{0n} = \langle\ rhs_n\ \rangle \tag{60}$$

where $\langle \cdots \rangle$ represents an average over all angles.

PROBLEM 3: Find $G_0(\tau)$ by applying the fundamental theorem of calculus to periodic functions with zero average.

Example: quadrupole field errors

As an example of how all this works, we shall consider a perturbation due to quadrupole field errors. In the notation of Sec.II, the Hamiltonian is written,

$$H = \underline{\nu}\cdot\underline{I} + \varepsilon H_1 \quad ,$$
$$H_1 = (B_oR/B\rho)\,[\,\tfrac{1}{2}\, b_1(\theta)\;(\,x_1^2 - x_2^2\,) - a_1(\theta)\; x_1x_2\,] \quad . \tag{61}$$

PROBLEM 4: What is the value of ε?

Hint: Write Eq.(61) without ε and then replace b_1 with $[b_1]b_1$, I with [I]I, β with $[\beta]\beta$, and so forth. A quantity $[\zeta]$ represents the *scale* of the corresponding variable ζ; the entities ζ are then dimensionless and O(1). The dimensionless parameter ε is then a combination of these scales. If $[B_oR/B\rho] \simeq 1$, then $\varepsilon \simeq [b_1][\beta]$ for this problem.

(Note: I am dropping the $\underline{\delta}^*$, $\underline{I}^*$ notation.) We shall calculate the new Hamiltonian out to second order.

$$K \simeq \underline{\nu}\cdot\underline{I} + \varepsilon K_{01} + \tfrac{1}{2}\varepsilon^2 K_{02} \tag{62}$$

The first step consists of expanding H_1 in our set of basis functions, $\{\,\phi_{em}\,\}$.

$$\begin{aligned} H_1 &\equiv \sum_{\underline{e},\underline{m}} H_{1;\underline{em}}(\theta)\;\phi_{\underline{em}} \\ &= \tfrac{1}{2}\, b_1 \Big\{\, \text{latt}_{(20)(00)}\;\phi_{(20)(00)} \;-\; \text{latt}_{(02)(00)}\;\phi_{(02)(00)} \\ &\qquad - \tfrac{1}{2}\;[\,\text{latt}_{(20)(20)}\;\phi_{(20)(20)} \;+\; \text{c.c.}\,] \\ &\qquad + \tfrac{1}{2}\;[\,\text{latt}_{(02)(02)}\;\phi_{(02)(02)} \;+\; \text{c.c.}\,] \;\Big\} \\ &\quad + \tfrac{1}{2}\, a_1 \Big\{\; [\,\text{latt}_{(11)(11)}\;\phi_{(11)(11)} \;+\; \text{c.c.}\,] \\ &\qquad - [\,\text{latt}_{(11)(1\,-1)}\;\phi_{(11)(1\,-1)} \;+\; \text{c.c.}\,] \;\Big\} \end{aligned} \tag{63}$$

$$\mathrm{latt}_{\underline{em}}(\theta) \equiv (B_o R/B\rho)\ \beta_1(\theta)^{e_1/2}\ \beta_2(\theta)^{e_2/2}\ e^{i\underline{m}\cdot\underline{\tilde{\psi}}(\theta)} \tag{64}$$

Using Eq.(60) we solve immediately for K_{01}, ***in the absence of resonances.***

$$\begin{aligned}\epsilon K_{01} = \langle \epsilon H_1 \rangle = \langle\ \epsilon b_1(\theta)/2 \cdot [\ &\mathrm{latt}_{(20)(00)}\ \Phi_{(20)(00)} \\ &- \mathrm{latt}_{(02)(00)}\ \Phi_{(02)(00)}\ \ ]\ \rangle \\ \equiv \Delta\nu_1^{(1)}\ I_1\ -\ \Delta\nu_2^{(1)}\ I_2\quad &,\ \text{where}\end{aligned} \tag{65}$$

$$\Delta\nu_{1,2}^{(1)} = \frac{1}{2}\int \left[\ (B_o R/B\rho)\ \frac{d\theta}{2\pi}\ \right]\ \epsilon b_1(\theta)\ \beta_{1,2}(\theta) \tag{66}$$

Proceeding to second order, we have from Eq.(48),

$$\begin{aligned}K_{02} &= \langle\ [H_1\ ,\ S_1]\ -\ K_{11}\ \rangle \\ &= \int \frac{d\theta}{2\pi}\ [\ [H_1\ ,\ S_1]\ -\ K_{11}\ ]\big|_{\underline{m}=\underline{0}}\end{aligned} \tag{67}$$

where the "$\underline{m}=\underline{0}$" notation means "project out all components along basis vectors $\phi_{\underline{e}0}$. But now we have a simplification.

$$\begin{aligned}K_{11}\ \big|_{\underline{m}=\underline{0}} &= [S_1\ ,\ K_{01}]\big|_{\underline{m}=\underline{0}} \\ &= \sum_{\underline{e}} K_{01;\underline{e0}}\ [S_1\ ,\ \phi_{\underline{e0}}]\big|_{\underline{m}=\underline{0}} \\ &= \sum_{\underline{e},\underline{e}'} K_{01;\underline{e0}}\ S_{1;\underline{e}'\underline{0}}\ [\phi_{\underline{e}'\underline{0}}\ ,\ \phi_{\underline{e0}}] \\ &= 0\end{aligned} \tag{68}$$

So, we only have to work with the other term.

$$\begin{aligned}K_{02} &= \langle\ [H_1\ ,\ S_1]\ \rangle \\ &= \int \frac{d\theta}{2\pi}\ [H_1\ ,\ S_1]\big|_{\underline{m}=\underline{0}}\end{aligned}$$

$$= \sum_{\substack{\underline{e},\underline{m} \\ \underline{e}',\underline{m}'}} \left[\int \frac{d\theta}{2\pi} \; H_{1;\underline{em}}(\theta) \; S_{1;\underline{e}'\underline{m}'}(\theta) \right] [\phi_{\underline{em}} \, , \, \phi_{\underline{e}'\underline{m}'}]\Big|_{\underline{m}+\underline{m}'=\underline{0}} \tag{69}$$

$$\equiv \sum_{\underline{e},\underline{m}} (-ig_{\underline{em}}) \, [\phi_{\underline{em}} \, , \, \phi_{\underline{e}\,-\underline{m}}]$$

$$= \sum_{\substack{\underline{e},\underline{m} \\ (m_1>0) \text{ or } (m_1=0 \text{ and } m_2>0)}} \mathrm{Re}(g_{\underline{em}}) \, [-2i[\phi_{\underline{em}} \, , \, \phi_{\underline{e}\,-\underline{m}}]] \quad , \text{ where}$$

$$g_{\underline{em}} = \iint \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \; H_{1;\underline{em}}(\theta) \; 2\pi i G_{-\underline{m}}(\theta-\theta') \; H_{1;\underline{e}\,-\underline{m}}(\theta') \tag{70}$$

In going from the second line to the third the meaning of the symbol $\underline{m}$ gets changed, but there should be no confusion; line five follows from line four by using anti-commutativity of the bracket and the identity $-ig_{e\,-m} = (-ig_{em})^{*}$. The relevant brackets are evaluated below.

$\underline{e}$	$\underline{m}$	$[\phi_{\underline{em}} \, , \, \phi_{\underline{e}\,-\underline{m}}]$
2 0	2 0	$4i \; \phi_{(20)(00)}$
0 2	0 2	$4i \; \phi_{(02)(00)}$
1 1	1 1	$i \; \phi_{(02)(00)} + i \; \phi_{(20)(00)}$
1 1	1 -1	$i \; \phi_{(02)(00)} - i \; \phi_{(20)(00)}$

(71)

Writing the final answer is now just a matter of putting all of these pieces together. We can do this conveniently in terms of the auxiliary function,

$$\begin{aligned} c_{\underline{e}}(\theta) &= b_1(\theta) \; , \quad \underline{e} = (20) \text{ or } (02) \\ &= a_1(\theta) \; , \quad \underline{e} = (11) \end{aligned} \tag{72}$$

and the symbols,

$$f_{\underline{em}} = -\frac{\pi/2}{\sin \pi\underline{m}\cdot\underline{\nu}} \iint (B_oR/B\rho)^2 \frac{d\theta}{2\pi}\frac{d\theta'}{2\pi} \times c_{\underline{e}}(\theta)c_{\underline{e}}(\theta')$$

$$\times \left[\beta_1(\theta)\beta_1(\theta') \right]^{e_1/2} \left[\beta_2(\theta)\beta_2(\theta') \right]^{e_2/2}$$

$$\times \cos \underline{m}\cdot[\underline{\tilde{\psi}}(\theta) - \underline{\tilde{\psi}}(\theta') + \underline{\nu}(\mathrm{mod}(\theta-\theta') - \pi)] \tag{73}$$

as follows:

$$K_{02} = [\, f_{(20)(20)} + f_{(11)(11)} - f_{(11)(1\,-1)} \,]\, I_1 + [\, f_{(02)(02)} + f_{(11)(11)} + f_{(11)(1\,-1)} \,]\, I_2 \quad . \tag{74}$$

Because K represents a decoupled system of oscillators, the tunes obtained from it must be the eigenfrequencies of the original coupled system.

Example: sextupole fields

As another example, consider a Hamiltonian with a small admixture of normal sextupoles.

$$H = \underline{\nu}\cdot\underline{I} + \epsilon \frac{1}{3}\frac{B_oR}{|B\rho|}\, b_2(\theta)(x_1^3 - 3x_1x_2^2\,), \tag{75}$$

which is written in terms of our set of basis functions as follows.

$$H_1 = -i/6\surd 2 \sum_{\{\underline{e},\underline{m}\}} c_{\underline{em}}\, \mathrm{latt}_{\underline{em}}(\theta)\, \phi_{\underline{em}} \ ,$$

$$\mathrm{latt}_{\underline{em}}(\theta) = (B_oR/B\rho)\, b_2(\theta)\, \beta_1(\theta)^{e_1/2}\, \beta_2(\theta)^{e_2/2}\, e^{i\underline{m}\cdot\underline{\tilde{\psi}}(\theta)} \ ,$$

$$c_{(30)(30)} = -1 \ , \quad c_{(12)(10)} = -6 \ , \tag{76}$$

$$c_{(30)(10)} = c_{(12)(12)} = c_{(12)(1\,-2)} = 3 \ .$$

In the absence of resonances, $K_{01} = K_{11} = 0$, and once again the lowest order non-trivial term in the new Hamiltonian is $K_{02} = \langle [H_1, S_1] \rangle$. We shall leave it as an exercise to the reader to evaluate this and show that the new Hamiltonian written to second order is given by

$$\begin{aligned} K(\underline{I}) = \underline{\nu}\cdot\underline{I} - \varepsilon^2 \{ & I_1^2 [\tfrac{1}{8} f(3,0;3,0;3,0) \\ & \qquad + \tfrac{3}{8} f(3,0;3,0;1,0)] \\ & + I_2^2 [\tfrac{1}{2} f(1,2;1,2;1,0) \\ & \qquad + \tfrac{1}{8} f(1,2;1,2;1,2) \\ & \qquad + \tfrac{1}{8} f(1,2;1,2;1,-2)] \\ & + I_1 I_2 [- f(3,0;1,2;1,0) \\ & \qquad + \tfrac{1}{2} f(1,2;1,2;1,2) \\ & \qquad - \tfrac{1}{2} f(1,2;1,2;1,-2)] \} \\ & + O(\varepsilon^4) \end{aligned} \tag{77}$$

where the coefficients that appear are the following quadratic functionals of b_2:

$$\begin{aligned} f(\underline{e};\underline{e}';\underline{m}) &= \int \frac{d\theta\, d\theta'}{(2\pi)^2} \operatorname{Re}\Big[\operatorname{latt}_{\underline{e}\,-\underline{m}}(\theta)\; 2\pi i\, G_{\underline{m}}(\theta-\theta')\; \operatorname{latt}_{\underline{e}',\,\underline{m}}(\theta') \Big] \\ &= \frac{\pi}{\sin(\pi\, \underline{m}\cdot\underline{\nu})} \iint d\xi(\theta,\theta')\, b_2(\theta)\, b_2(\theta') \\ &\qquad \beta_1(\theta)^{e_1/2}\, \beta_1(\theta')^{e_1'/2}\, \beta_2(\theta)^{e_2/2}\, \beta_2(\theta')^{e_2'/2} \\ &\qquad \cos[\, |\underline{m}\cdot(\underline{\psi}(\theta) - \underline{\psi}(\theta'))| - \pi\, \underline{m}\cdot\underline{\nu} \,] \end{aligned} \tag{78}$$

$$d\xi(\theta,\theta') = (B_o R/B\rho)^2 \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}, \qquad \theta,\theta' \in [0,2\pi)$$

Let me emphasize again that the validity of this result depends on being far from prevailing resonances. The only ones that can appear through second order in sextupole strength are: $3\nu_1$, $\nu_1 \pm 2\nu_2$, ν_1 (first order); $4\nu_1$, $4\nu_2$, $2\nu_1$, $2\nu_2$, and $2\nu_1 \pm 2\nu_2$

(second order). "Near" any of these the projected resonance term must be subtracted from the Green's function and absorbed into the Hamiltonian.

PROBLEM 5: Why does the resonance $6\nu_1$ *not* appear in second order?

Carrying out calculations to higher orders requires far too many algebraic manipulations to consider doing by hand. Fortunately, the explicit nature of Deprit's algorithm makes it especially suitable for symbolic algebra coding. A MACSYMA program written to process, as described in this section, periodic polynomial perturbations has been used to extend this computation to fourth order. The resulting expressions take up too much space to be included here.

SECTION IV THE SINGLE RESONANCE TERM HAMILTONIAN

Using Deprit's algorithm, or any other appropriate method, we can transform the Hamiltonian into one containing only "shear" (independent of angles) and "secular" (resonance) terms. If only the shear terms are present, we are finished: action variables are constants, and angle variables increase linearly with θ. If there is but one, isolated resonance,

$$K = K_s(\underline{I}) + \left(K_r(\underline{I})\, e^{i(\underline{m}\cdot\underline{\delta} + n\theta)} + \text{c.c.} \right) \tag{79}$$

then it is possible to do a little more analysis before resorting to numerical procedures.

PROBLEM 6: If one starts from a multipole Hamiltonian, as in Section II, would it ever be possible for K_s to contain fractional powers of the action?

Two invariants

The Hamiltonian of Eq.(79) possesses two invariants of motion, which have been imaginatively named "first invariant" and "second invariant."

The first invariant comes from the following simple observation.

$$d\underline{I}/d\theta = -\ \partial K/\partial \underline{\delta} = -i\underline{m}\ K_r e^{i(\underline{m}\cdot\underline{\delta} + n\theta)}\ +\ c.c. \tag{80}$$

Projected into $\underline{I}$-space, the orbit is therefore constrained to lie on a straight line in the direction $\underline{m}$. For two degrees of freedom, there will be some constant, Δ, such that

$$I_1/m_1 - I_2/m_2 = \Delta \tag{81}$$

Because the actions necessarily lie in the quadrant $I_1, I_2 \geq 0$, if m_1 and m_2 have <u>opposite</u> signs, the motion must be confined to the region

$$\begin{aligned} 0 &\leq I_1 \leq m_1\Delta\ , \\ 0 &\leq I_2 \leq -m_2\Delta\ . \end{aligned} \tag{82}$$

But if they have the <u>same</u> sign there is no *a priori* upper bound for I_1 and I_2, and they may, in principle, grow indefinitely.

The second invariant is best motivated by transforming to a rotating frame, with the objective of eliminating the explicit θ-dependence in Eq.(79). There is a continuum of canonical[8]

---

8) What are the etymologies of the religious terminology that appear in classical mechanics? "Canonical" refers to canon law, the set of rules governing the modes of worship for members of a Christian church. It is in opposition to secular, or civil law. The word "secular" refers to something pertaining to the temporal rather than the eternal. And yet, in the context of classical dynamics the secular terms — the "slowly varying" terms — are those viewed as having the <u>least</u> transient character, precisely the opposite of the word's true meaning.

transformations which will do this: for any real number ρ_1 ($\rho_2 \equiv 1-\rho_1$) we can define new dynamical variables,

$$
\begin{aligned}
&(\delta_k , I_k) \longrightarrow (\bar{\delta}_k , J_k) \quad , k = 1,2 , \\
&\underline{J} = \underline{I} \quad , \\
&\bar{\delta}_k = \delta_k + \rho_k n\theta/m_k \quad ,
\end{aligned}
\tag{83}
$$

according to the generating function,

$$
\begin{aligned}
F(\underline{\delta} , \underline{J} ; \theta) = \; &(\delta_1 + \rho_1 n\theta/m_1) J_1 \\
&+ (\delta_2 + \rho_2 n\theta/m_2) J_2 \quad .
\end{aligned}
\tag{84}
$$

The Hamiltonian is changed by substituting $\underline{m}\cdot\bar{\underline{\delta}}$ for $\underline{m}\cdot\underline{\delta} + n\theta$ and by adding a tune-shift term to K_s:

$$K_s \longrightarrow \text{new } K_s = K_s + (\rho_1 n/m_1)I_1 + (\rho_2 n/m_2)I_2 \tag{85}$$

(Since $\underline{J}=\underline{I}$, we will not bother to change that particular symbol.) One way of interpreting this is to say that the zero-amplitude tunes have shifted.

$$(\nu_1, \nu_2) \longrightarrow (\nu_1' , \nu_2') = (\nu_1 + \rho_1 n/m_1 , \nu_2 + \rho_2 n/m_2) \tag{86}$$

If we are on resonance — that is, if $\underline{m}\cdot\underline{\nu} + n = 0$ — then $\underline{m}\cdot\underline{\nu}' = 0$, so that the new zero-amplitude tunes are commensurate.

PROBLEM 7: Show that the condition $\underline{m}\cdot\underline{\nu} + n = 0$ with $n \neq 0$ implies that either ν_1 and ν_2 are both rational or they are incommensurate.

The new Hamiltonian contains no explicit θ-dependence and therefore is a constant of the motion.[9]

9) The fact that the "second invariant" is actually the Hamiltonian means that it probably should be called the "first invariant," and vice versa; in fact, Schoch (1958) does label them as such. In identifying them as we have I am following the lead of Guignard (1978).

$$\tilde{K}(\bar{\underline{\delta}}, \underline{I}) = \text{new } K_s(\underline{I}) + \left(K_r(\underline{I})\, e^{i\underline{m}\cdot\bar{\underline{\delta}}} + \text{c.c.} \right) \equiv \tilde{K}_s(\underline{I}) + \tilde{K}_r(\underline{I}) \cos(\, \underline{m}\cdot\bar{\underline{\delta}} + \xi(\underline{I})\,) \tag{87}$$

The invariance of $\tilde{K}$ and bounds on the cosine function can be used to restrict the allowed dynamical region in $\underline{I}$-space further.

$$|\cos(\, \underline{m}\cdot\bar{\underline{\delta}} + \xi\,)| \le 1 \longrightarrow |\tilde{K} - \tilde{K}_s(\underline{I})| \le |\tilde{K}_r(\underline{I})| \tag{88}$$

It is possible for this inequality to stabilize an orbit even when m_1 and m_2 have the same sign.

PROBLEM 8: Is this a strong condition? That is, will some orbit fill the region allowed by the inequality? Do all orbits fill their allowed regions?

PROBLEM 9: Because ρ_1 can be any real number, Eq.(87) potentially describes not a single invariant but an infinite family of invariants. Show that the difference between any two members of this family is only a multiple of the second invariant.

PROBLEM 10: Devise a canonical transformation $(\bar{\underline{\delta}},\underline{I}) \longrightarrow (\underline{\phi},\underline{J})$ such that $\phi_1=\underline{m}\cdot\bar{\underline{\delta}}$. Show that the first invariant — the particular combination of actions written in Eq.(81) — is proportional to J_2, the momentum canonically conjugate to the ignorable coordinate ϕ_2.

Hint: Use the generating function

$$F(\bar{\underline{\delta}},\underline{J}) = (\, J_1 \; J_2 \,) \begin{pmatrix} m_1 & m_2 \\ m_2 & -m_1 \end{pmatrix} \begin{pmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \end{pmatrix}$$

Fixed points and the resonance width

The Hamiltonian in Eq.(87) possesses fixed points, obtained by setting $d\underline{I} = d\underline{\bar{\delta}} = \underline{0}$. To keep the argument smooth, we shall ignore the possibility that $\tilde{K}_r(\underline{I})$ might vanish somewhere. The fixed point equations are then written as follows.

$$d\underline{I}/d\theta = -\partial\tilde{K}/\partial\underline{\bar{\delta}} = \underline{0}$$

$$\Rightarrow \quad \underline{m}\cdot\underline{\bar{\delta}} + \xi(\underline{I}) = p\pi, \quad \text{for some integer } p \tag{89}$$

$$d\underline{\delta}/d\theta = \partial\tilde{K}/\partial\underline{I} = \underline{0}$$

$$\Rightarrow \quad \nabla\tilde{K}_s(\underline{I}) + (-1)^p\nabla\tilde{K}_r(\underline{I}) = \underline{0} \tag{90}$$

Eqs.(90) are first to be solved for real, positive $\underline{I}_o$, after which Eq.(89) defines the correspondingly allowed values of $\underline{\bar{\delta}}_o$.[10] Note that these equations describe two fixed curves — one for even, the other for odd values of p. We shall also refer to these curves as the *resonant orbits*.

Resonant orbits are not necessarily periodic; for that we must pay some attention to the values assigned to ρ_1 and ρ_2 in Eq.(83). Since $\underline{I}$ is itself a constant of the motion at a fixed point, we see from Eq.(28) that the corresponding orbit will be periodic if and only if there are an integer M, the *periodicity*, and a doublet of integers $\underline{N} \equiv (N_1, N_2)$, the *winding numbers*, such that

$$\underline{\delta}|_{\theta=2\pi M} - \underline{\delta}|_{\theta=0} = 2\pi\underline{N} \; . \tag{91}$$

Reversing the transformation in Eq.(83) and remembering that $\underline{\bar{\delta}}$ is constant provide the conditions

$$\rho_1 nM + m_1 N_1 = 0 \quad , \quad \rho_2 nM + m_2 N_2 = 0 \; . \tag{92}$$

10) It is possible, presumably, to extend dynamical systems to complex valued phase spaces. There may even be some advantage in doing this. Think, for example, of all that was gained by introducing complex variables into analysis.

Adding Eqs.(92) together yields the periodicity,

$$M = - \underline{m} \cdot \underline{N}/n \quad , \tag{93}$$

which, when fed back, produces the result

$$\rho_k = m_k N_k / \underline{m} \cdot \underline{N} \quad , \quad k=1,2. \tag{94}$$

To generate an "allowed" rotating frames, therefore, we can proceed as follows: (1) choose winding numbers N_1 and N_2 such that $n | \underline{m} \cdot \underline{N}$; (2) use Eq.(94) to find ρ_1 and ρ_2; (3) Eq.(93) then gives the periodicity of the orbit, if it exists.

Even when the resonant orbits are not periodic, they still possess an experimental signature: the condition $d\underline{I}/d\theta=0$ means that they must follow an envelope equivalent to linear motion.

Let us temporarily simplify the problem in order to introduce a few key geometric concepts. Consider a one degree of freedom calculation. To be definite, suppose that K_s represents a purely linear Hamiltonian, so that in the rotating frame,

$$K = (\nu + n/m)I + [\, K_r(I) e^{im\bar{\delta}} \; + \; \text{c.c.} \,] \quad , \tag{95}$$

where now $I=I_1$, $\bar{\delta}=\bar{\delta}_1$, and $I_2 \equiv 0$. Again, to be definite, let us say that $K_r(I)$ arises from the appropriate harmonic of the leading term ($m_1=m=m'$, $m_2=p=p'=0$) in Eq.(36).

$$\begin{aligned} K_r(I) &\simeq \frac{1}{m} \left(\frac{1}{2} \, I \right)^{m/2} \zeta \quad , \\ \zeta &= (-i)^m \int (B_o R/B\rho) \, \frac{d\theta}{2\pi} \;\, b_{m-1}(\theta) \;\, \beta^{m/2}(\theta) \;\, e^{i[m\tilde{\psi}(\theta) \, - \, n\theta]} \end{aligned} \tag{96}$$

(The term must come from a normal, not skew, multipole, since this is a one degree-of-freedom problem.) Note in passing that if we are concerned with the contribution from field errors in dipoles, then $B_o=B$, and the integral becomes an average over all dipoles.

$$\int \frac{R d\theta}{2\pi\rho} \ (\cdots) \ = \ \langle \ (\cdots) \ \rangle_{\text{dipoles}} \ \simeq \ \frac{1}{N} \sum_{\text{dipoles}} (\cdots) \tag{97}$$

We now express the transverse Hamiltonian in terms of these quantities.

$$\begin{aligned} K &\simeq (\nu + n/m)I + \frac{2g}{m} I^{m/2} \cos(m\bar{\delta} + \xi) \ , \\ g &= (1/2)^{m/2} \ |\zeta| \ , \\ \xi &= \arg(\zeta) \end{aligned} \tag{98}$$

Solutions of the fixed point equations are written below.

$$\begin{aligned} \bar{\delta}_o &= (p\pi - \xi)/m \ , \quad p = \text{odd (even) integer} \\ &\qquad\qquad\qquad\quad \text{for } m\nu > -n \ (\ m\nu < -n \ ) \ ; \\ I_o &= |(\nu + n/m)/g|^{2/(m-2)} \end{aligned} \tag{99}$$

Notice that the reduced dimensionality of the problem has collapsed the fixed curve into a discrete set containing m fixed points, equally displaced in angle by $2\pi/m$, and all at amplitude $\sqrt{2I_o}$ in $(x/\sqrt{\beta},\ q/\sqrt{\beta})$ space. They automatically represent a periodic orbit — and that is a <u>very</u> nice feature of one degree of freedom systems — corresponding to $\rho_1=1$, $\rho_2=0$, $M=m$, and $N=n$ (assuming n and m are coprime). Since the Hamiltonian, and therefore the phase-space flow, is invariant under the symmetry transformation $\bar{\delta} \longrightarrow \bar{\delta} + 2\pi/m$, all the fixed points must be either stable, *elliptic*, or unstable, *hyperbolic*. Since the former is topologically impossible, the latter must be the case. Fig.(1) illustrates the situation. The asymptotic directions of the flow can be obtained by examining the behavior of Eq.(98). We see that $\cos(m\bar{\delta} + \xi)$ must vanish as I approaches infinity if K is to remain constant. (The easiest way to see this is to divide through by I.) Thus, the asymptotic directions are

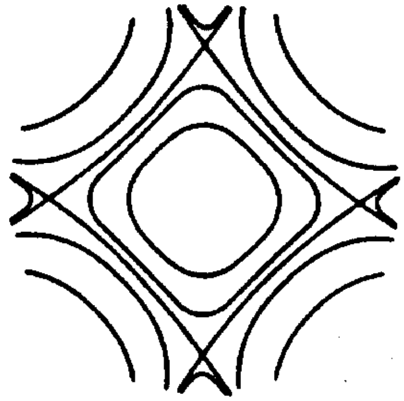

Fig. 1 Sketch of phase space flow for the Hamiltonian of Eq.(98) with m=4. (taken from Guignard (1978))

$$\bar{\delta}_{\infty} = \frac{1}{m} \left[\frac{\pi}{2} (2p'+1) - \xi \right] \quad , \text{ for integer } p'. \tag{100}$$

Had K_s contained a term of $O(I^{\mu})$ with $\mu > m/2$ it would have been impossible for I to become indefinitely large while keeping K constant. Therefore, no orbit would diverge, and the flow would have been as in Fig.(2): there would have been twice as many

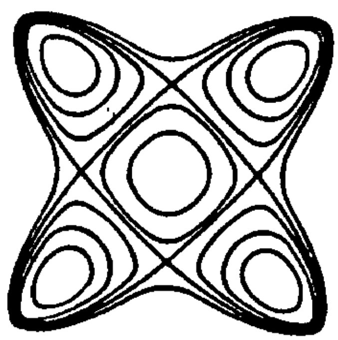

Fig. 2 Adding a higher order shear term to K_s can close the separatrix. (taken from Guignard (1978))

solutions to the fixed point equations, half of them elliptic, the other half hyperbolic. The fact that these pictures look so simple is due to the basic forms given to K_s and K_r in Eq.(98); for an example of more complicated patterns, see Michelotti (1983).

PROBLEM 11: Even with the extra term the Hamiltonian would possess the same symmetry as before. What happened to our previous argument for demonstrating that all the fixed points must have the same stability?

So, we have a workable strategy for analyzing the system in one degree of freedom: the fixed points of the transformed Hamiltonian lead to a unique, small set of coperiodic orbits which completely characterizes the flow. Loosely speaking, one of these resonant orbits is "locally stable" if neighboring orbits remain close to it for all θ; it is "locally unstable" if this is not the case. Almost all neighboring orbits of an unstable resonant orbit will diverge away from it for $\theta \rightarrow \pm\infty$. However, two special families of orbits converge on it as θ either increases or decreases: the set of orbits that approach a resonant orbit as $\theta \rightarrow +\infty$ is called its *stable manifold*, while those that approach it as $\theta \rightarrow -\infty$ comprise its *unstable manifold*.[11] When the stable manifold of one resonant orbit is the unstable manifold of another, as was the case in Figs.(1) and (2), or when stable and unstable manifolds of a single orbit coincide, the union of all these sets of orbits forms a *separatrix*, so called because it partitions, or "separates," phase space into a number of disconnected, open regions. Any two orbits in the same region can be connected by passing through a series of dynamically equivalent orbits.

If the origin of a "single resonance term" system, such as in Eq.(79), is stable, then the "region of stability", or *dynamic aperture*, is identified with that region which is bounded by the

11) These terms are unpopular with some nonlinear dynamicists who prefer to call these orbits the "inset" and "outset" instead.

separatrix and contains the origin.[12] For one degree of freedom, and viewed from the rotating frame, it is roughly a regular m-gon with the fixed points at the vertices. We can use this geometric approximation to estimate its phase space area, W, which, by Eq.(31), is the maximum possible emittance of a stable beam.

$$
\begin{aligned}
W &\simeq m \times [\ \sqrt{2I_o}\ \cos(\pi/m)\ ] \times [\ \sqrt{2I_o}\ \sin(\pi/m)\ ] \\
&= |(\nu + n/m)/g|^{2/(m-2)}\ m \sin(2\pi/m)
\end{aligned}
\tag{101}
$$

The *resonance width* is defined as $\Delta\nu \equiv 2|\nu + n/m|$: it is the size of the tune interval within which no beam of emittance W can fit into the central stable region. Solving Eq.(101) for $\Delta\nu$ yields the desired result.

$$
\Delta\nu = 2g \left(\frac{W}{m \sin(2\pi/m)} \right)^{\frac{m}{2} - 1} \tag{102}
$$

PROBLEM 12: Show that the dynamic aperture, viewed in the rotating frame, of the third integer resonance (i.e., m=3) is an equilateral triangle.

Complications arise when we try extending this picture to two degrees of freedom. We shall examine one possible scenario, certainly not the only one, using the variables $\underline{\phi}=(\phi_1,\phi_2)$, $\underline{J}=(J_1,J_2)$ introduced in Problem #10. Take a section of four-dimensional phase space along an invariant hypersurface: J_2=constant. The flow within this surface is sketched in Fig. 3 using the coordinates (ϕ_1,ϕ_2,J_1). This is only a conceptual

---

12) It is difficult to determine whether the term "dynamic aperture," as it is commonly used, refers to the region of stability or to its phase space volume. The choice made here is not universal. Its meaning also varies when more complicated, and thus more realistic, dynamics are considered. Therefore, be wary: the term is vague.

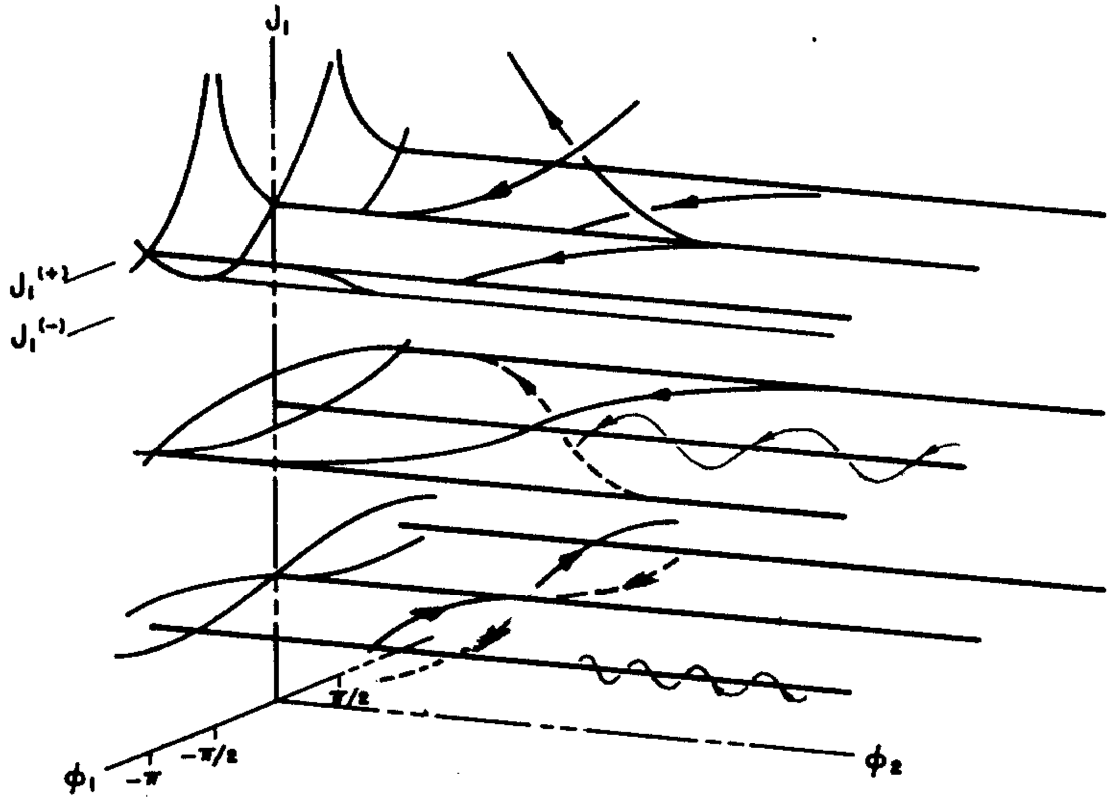

Fig. 3 Conceptual sketch of the flow described by a two degree of freedom resonance Hamiltonian on a hypersurface determined by the first invariant.

drawing and is not meant to represent any particular resonance. To keep the picture simple we have assumed that $\xi(\underline{I})=0$, for all $\underline{I}$. Because ϕ_2 is an ignorable coordinate, surfaces of constant $\tilde{K}$ run parallel to that axis. The range of ϕ_1 has been truncated to $[-\pi, \pi)$; the surfaces are periodic beyond this range. Three families of resonant orbits — the fixed curves of Eqs.(89) and (90) — are shown along with their separatrices: two produce islands; the third borders on a region of unbounded orbits. The value of ρ_1 has been chosen to correspond with the flow on the lowest resonant orbits, which makes them into lines of fixed points. If this number is one of the rationals defined by Eq.(94), then these orbits are truly periodic with periodicity given by Eq.(93). Choosing a different value for ρ_1 does not alter the surfaces but merely adds a constant to $d\phi_2/d\theta$, as you discovered in Problem #9. The hyperbolic (unstable) resonant orbits are limit cycles for the orbits embedded in their associated separatrices; around the elliptic (stable) resonant orbits are invariant tori (cylinders closed by the periodic boundary conditions on ϕ_2) created by the orbits that wind around them. The regions between the separatrices are *fibrated*, as the mathematicians say, by other tori (not shown) which span the full range of both ϕ_1 and ϕ_2. All orbits starting with $J_1 < J_1^{(-)}$ are bounded; all those starting with $J_1 > J_1^{(+)}$ are unbounded; those in between will be either bounded or unbounded depending on the initial phase.

This picture shifts as we change J_2: the separatrices move around; new ones may be created or old ones destroyed via catastrophes; global bifurcations may change the flow's topology. We can conservatively associate the four-dimensional region underneath the lowest separatrix — or it's volume, depending on how the term is to be defined — with the dynamic aperture of the resonance.[13] That volume could certainly be computed, although a numerical integration would most likely be necessary. If this separatrix is plotted in (ϕ_1, J_1, J_2) coordinates, the volume of the dynamic aperture equals the three-dimensional volume between it and the plane $J_1=0$ multiplied by $2\pi(|m_1| + |m_2|)$, which takes the range of ϕ_2 into account.

Were we to continue the analysis, the next step would be to construct Poincare maps on the two-dimensional invariant manifolds. In principle these could be as complicated as any torus mapping, possessing their own hyperbolic periodic orbits, sepa-

---

13) If you want to be more liberal, associate the dynamic aperture with the top separatrix.

ratrices, or even chaotic regions. None of this possible complexity would affect the dynamic aperture of this Hamiltonian: the two invariants prevent diffusion.

It is disturbing to leave our examination of coupling resonances in such an untidy state, but little more of a generic nature can be said, and we should touch upon modulational diffusion before finishing. We shall leave open the problem of extending the "resonance width" concept to two degrees of freedom; for an interesting discussion of this see Ohnuma (1980), whose references on the subject include Sturrock (1958), Lysenko (1973), and Guignard (1978).

SECTION V OVERLAPPING RESONANCES, MODULATIONAL DIFFUSION, AND THE COURANT-CHIRIKOV CRITERION

When a dynamical system is "dominated" (whatever that means) by more than one resonance term, there in general are no global invariants to help reduce the dimensionality of the problem, and we must rely on some method of numerically integrating the equations of motion to make any progress whatsoever. There are few theorems and not much more in the way of heuristics to help light the way. This is not the place to attempt a survey of this field of very active research. Instead, we shall focus on one idea: the *Chirikov criterion*, which is a test for judging whether motion will be regular or chaotic. As expressed by Chirikov himself,

> "A plausible condition for the occurrence of the stochastic instability seems to be the approach of resonances down to the distance of the order of a resonance size. Such an approach was naturally called the *resonance overlap*. To be precise the overlap of resonances begins when their separatrices touch each other. The possibility for a system to move from one resonance to another under the above

condition is quite obvious....The overlap of two, or a few, resonances results only in a *confined instability*.... But that's quite a different thing if there are many resonances. Then a trajectory of motion may go within a set of overlapping resonances far away from the initial position." (Chirikov 1979)

Courant applied this ansatz to the problem of *modulational diffusion* — such as might arise in the coupling of longitudinal (synchrotron) and transverse (betatron) oscillations — connected with the ν=113/5 resonance in ISABELLE. (Courant 1980) The basic idea is that a slow modulation of some parameter of a dynamical system will induce a family of "satellite" resonances around any natural resonance of the system. If these satellites overlap, undesirable stochasticity may result. We shall generalize Courant's argument below; for examples of other recent work see Chirikov *et al.* (1985), Lichtenberg & Lieberman (1983, ch.6.2d), or Tennyson (1982, sec.2.2).

WARNING: The "swindle" potential of this section is enormous. This is not to say that the arguments are wrong, but do not be lulled into accepting them easily or applying them indiscriminately.

We begin by writing an approximate, one degree-of-freedom, transverse Hamiltonian in which (1) the essential effect of synchrotron oscillations is represented as a slow tune modulation, induced by chromaticity, and (2) the transverse dynamics are dominated by a single resonance and the zeroth harmonic octupole term. It is assumed that we have already transformed into the rotating phase space.

$$H = (\nu_0 + k/m + \nu_1 \cos\nu_s\theta)\cdot I + \frac{\alpha}{2} I^2 + \kappa I^{m/2} \cos m\bar{\delta} \quad + \text{"everything else"} \tag{103}$$

Here, ν_0 is the tune, $\nu_s \ll \nu_0$ (why?) is the synchrotron oscillation frequency, and ν_1 is the amplitude of tune modulation. The effect of "everything else" are assumed negligible. At the minimum, this means that the rest of the Hamiltonian at most distorts orbits, without altering their topology, in the phase space region of interest. It is therefore certainly necessary, though not sufficient, that the only important resonance be the one singled out: $m\nu_0 + k = 0$. Notice also that the tune modulation is treated as a single, pure sinusoid. That is certainly true near the center of a longitudinal bucket, but closer to the edges it may more questionable. (One can argue that this only produces a very small effect, but the whole calculation is about small effects.)

Our first step is to absorb the explicit θ dependence into the cosine function, a reversal of the mapping carried out in Eq.(83). This is done using the following canonical transformation.

$$\begin{aligned} &W(\bar{\delta}, J) = (\bar{\delta} - \frac{\nu_1}{\nu_s} \sin \nu_s\theta) J \quad , \\ &I = J \ , \qquad \bar{\delta} = \phi + \frac{\nu_1}{\nu_s} \sin \nu_s\theta \end{aligned} \tag{104}$$

The new canonical variables are $(\phi,J) = (\phi,I)$; the new Hamiltonian is written as follows.

$$H \longrightarrow (\nu_0 + k/m)I + \frac{\alpha}{2} I^2 + \kappa I^{m/2} \cos [m(\phi + \frac{\nu_1}{\nu_s} \sin \nu_s\theta)] \tag{105}$$

Now comes the purely mathematical step of expanding the cosine in a Bessel-Fourier series,

$$\cos(u + z\sin\gamma) = \sum_{m=-\infty}^{\infty} J_m(z) \cos(u+m\gamma) \quad , \tag{106}$$

with the identifications $u := m\phi$, $z := m\nu_1/\nu_s$, and $\gamma := \nu_s\theta$.

$$H = (\nu_0+k/m)I + \frac{\alpha}{2} I^2 + \kappa I^{m/2} \sum_n J_n(m\nu_1/\nu_s) \cos m(\phi + n\nu_s\theta/m) \tag{107}$$

Voilà — behold the satellite resonances. To progress to the next stage we shall treat them one at a time. This is marginally valid only if they are widely separated, and as that is Chirikov's criterion, the argument at least is self consistent. Formally, we simply erase the summation symbol and transform back to an appropriately rotating phase space via the usual canonical transformation.

$$\bar{\phi} \equiv \phi + \frac{n}{m} \nu_s \theta \tag{108}$$

Viewed from the rotating frame, the tune has been shifted once again.

$$\begin{gathered} H \longrightarrow \delta\nu\, I + \frac{\alpha}{2} I^2 + \kappa' I^{m/2} \cos m\bar{\phi} \\ + \text{lots more stuff that's being ignored} \quad , \\ \delta\nu = \nu_0 + k/m + n\nu_s/m \quad , \\ \kappa' = \kappa\, J_n(\, m\nu_1/\nu_s\,) \end{gathered} \tag{109}$$

The resonance orbit is found as in the previous section by solving the two equations

$$\begin{gathered} \cos m\bar{\phi} = \pm 1 \quad , \\ \delta\nu + \alpha I + \frac{m}{2} \kappa' I^{m/2 - 1} \cos m\bar{\phi} = 0 \quad . \end{gathered} \tag{110}$$

Now we are ready to estimate the island width, δI.[14] We shall assume that κ is small and work only to first order in κ. Let H_u (H_s) represent the value of the Hamiltonian at the unstable (stable) fixed point. H_u is also its value on the separatrix, the island boundary. If κ is small then the undulations of the

14) Some accelerator physicsts call this quantity the "resonance width," but we've already used this term to mean a tune interval in the previous section.

Hamiltonian surface are so shallow that it's quadratic approximation should be valid all the way to the separatrix. Accordingly we expand

$$H_u - H_s \simeq \frac{1}{2}\,(\partial^2H/\partial I^2)_s\,(\delta I/2)^2 \quad , \tag{111}$$

Now expand the left hand side of this using Eqs. (109) and (110).

$$\begin{aligned} H_u - H_s = {} & (1 - m/2)\kappa'(I_u^{m/2}\cos m\bar{\phi}_u \; - \; I_s^{m/2}\cos m\bar{\phi}_s\,) \\ & - \frac{\alpha}{2}\,(\,I_u^2 - I_s^2\,) \end{aligned} \tag{112}$$

Calculating to lowest order in κ:

$$\begin{aligned} & I_u \simeq I_s \equiv I_o \simeq -\delta\nu/\alpha \\ & (\partial^2H/\partial I^2)_s = \alpha \\ & -\frac{\alpha}{2}\,(\,I_u^2 - I_s^2\,) = \frac{m}{2}\,\kappa'\,I_o^{m/2}\,(\,\cos m\bar{\phi}_u - \cos m\bar{\phi}_s\,) \end{aligned} \tag{113}$$

Substitute these into Eq.(111) to get the final expression for the island width.

$$\begin{aligned} \frac{\alpha}{2}\,(\tfrac{1}{2}\,\delta I)^2 & \simeq \kappa' I_o^{m/2}\,(\,\cos m\bar{\phi}_u - \cos m\bar{\phi}_s\,) \\ & = 2\,\mathrm{sign}(\alpha)\,|\kappa'|\,I_o^{m/2} \quad , \\ \delta I & = 4\,|\kappa'/\alpha|^{1/2}\,I_o^{m/4} \end{aligned} \tag{114}$$

We are now ready to apply Chirikov's criterion: stability requires that the island widths be much less than the spacing between the resonance bands. The latter quantity is approximately given by the following expression.

$$\Delta I_o = -\Delta(\delta\nu)/\alpha \; = \; |\nu_s/m\alpha| \tag{115}$$

The non-overlap condition is realized by demanding that

$$\delta I < \Delta I_o \tag{116}$$

which in turn implies that

$$\nu_s > 4m\ |\alpha\kappa'|^{1/2}\ I_o^{m/4} \tag{117}$$

Or, putting all the pieces together, we get the intimidating result:

$$\delta I < \Delta I_o \ \Rightarrow\ \forall n:\ \nu_s > 4m\ |\alpha\kappa J_n(m\nu_1/\nu_s)|^{1/2} \left| \frac{1}{\alpha}(\nu_0 + \frac{k}{m} + \frac{n\nu_s}{m}) \right|^{m/4} \tag{118}$$

This is the condition that must be satisfied if the satellite resonances are not to overlap, thus avoiding diffusion across the resonance islands.

Believe it, or not!

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It is a pleasure to acknowledge the discussions I have had with Dr.'s Don Edwards and Sho Ohnuma while writing this lecture note. I would also like to thank Mr. Enrique Henestroza for carefully reading the manuscript for errors.

POSTSCRIPTUM There are two major themes that I tried to incorporate into this lecture: (1) no formalism, no Hamiltonian, no equation should be accepted without regard to its CONtext, and (2) nonlinear phenomena can be arranged very roughly in a hierarchy ranging from the simple (e.g., amplitude dependent tunes, distortion of tori) to the complicated (e.g., diffusion). Regrettably, the discussion of two-degree-of-freedom resonances in Section IV was cumbersome. Had I time to do it again, I would not bother to introduce the parameter ρ_1 but simply write the transformation of Problem #10 in terms of $\underline{m}\cdot\underline{\delta}$, not $\underline{m}\cdot\bar{\underline{\delta}}$.

BIBLIOGRAPHY

Ando,A. 1983. "Emittance distortion with sextupoles." In Proceedings of the 12th International Conference on High-Energy Accelerators (Aug 11-16, 1983). Fermi National Accelerator Laboratory.

Chirikov,B.V. 1979. "A universal instability of many-dimensional oscillator systems." Physics Reports 52(5),263-379.

————, M.A.Lieberman, D.L.Shepelyansky, and F.M.Vivaldi. 1985. "A theory of modulational diffusion." Physica 14D, 289-304.

Cole,F.T. 1961. *Notes on accelerator theory*. MURA Technical Note TN-259. (Dec 12 1961).

Collins,T. 1984. *Distortion functions*. Fermilab-84/114. (Oct 23 1984).

————. 1983. "Concepts in the design of circular accelerators." In Physics of High Energy Particle Accelerators (SLAC Summer School, 1982). A.I.P. Conference Proceedings No. 105. New York: American Institute of Physics.

Courant,E.D. 1982. "Introduction to accelerator theory." In Physics of High Energy Particle Accelerators (Fermilab Summer School, 1981). A.I.P. Conference Proceedings No. 87. New York: American Institute of Physics.

————. 1980. *Periodic passage through nonlinear resonances during acceleration in Isabelle*. Brookhaven National Laboratory: Tech. Note. 163, ISABELLE Project. (Feb 12 1980).

———— and H.S.Snyder. 1958. "Theory of the alternating-gradient synchrotron." Annals of Physics 3(1),1.

Coxeter,H.S.M. 1969. Introduction to Geometry. John Wiley & Sons, NY.

Deprit,A. 1969. "Canonical transformations depending on a small parameter." Celestial Mechanics 1,12.

Douglas,D.R. 1982. Lie algebraic methods for particle accelerator theory. Dissertation submitted to the University of Maryland, College Park.

Dragt,A.J. 1982. "Lectures on nonlinear orbit dynamics." In Physics of High Energy Particle Accelerators. *See* Courant 1982.

Edwards,D. 1985. "An introduction to circular accelerators." In Physics of High Energy Particle Accelerators (BNL/SUNY Summer School, 1983). A.I.P. Conference Proceedings No. 127. New York: American Institute of Physics.

Forest,E. 1985. *Equivalence of Michelotti's normal form and the map normal form as used by the MARYLIE code*. U.R.A. Design Center, Lawrence Berkeley Laboratory: SSC-Report SSC-30.

Guignard,G. 1978. *A general treatment of resonances in accelerators*. European Organization for Nuclear Research, CERN 78-11. (Nov 10 1978).

Hagedorn,R. 1957. *Stability and amplitude ranges of two dimensional non-linear oscillations with periodical Hamiltonian*. European Organization for Nuclear Research: CERN 57-1. (Mar 7 1957).

Kamel,A.A. 1970. "Perturbation method in the theory of nonlinear oscillations." Celestial Mechanics 3,90-106.

Kerst,D.W. and R.Serber. 1941. "Electron orbits in the induction accelerator." Phys.Rev.60,53.

Lichtenberg,A.J. and M.A.Lieberman. 1983. Regular and Stochastic Motion. New York: Springer-Verlag.

Lysenko,W.P. 1973. "Nonlinear betatron oscillations." Particle Accelerators 5(1),1-21.

Michelotti,L. 1985. "Moser-like transformations using the Lie transform." Particle Accelerators 16,233-252.

———. 1983. *Catastrope and Maxwell surfaces of the half-integer resonance excited by quadrupoles and octupoles*. Fermi National Accelerator Laboratory: FN-393. (Nov 1983).

Ohnuma,S. 1984. *Effects of correction sextupoles in synchrotrons*. Fermi National Accelerator Laboratory: FN-401. (Jun 10 1984).

———. 1980. *A study of the width of nonlinear resonance*. Fermi National Accelerator Laboratory: TM-988. (Sep 15 1980).

Schoch,A. 1958. *Theory of linear and non-linear perturbations of betatron oscillations in alternating gradient synchrotrons*. European Organization for Nuclear Research: CERN 57-23. (Feb 1 1958).

Sinai,Ya.G. 1976. Introduction to Ergodic Theory. Princeton University Press.

Snowdon,S.C. 1969. *Transformations useful in linear betatron theory*. Fermi National Accelerator Laboratory: FN-185. (May 8 1969).

Sturrock,P.A. 1958. "Nonlinear effects in alternating-gradient synchrotrons." Annals of Physics: 3,113-189.

Tennyson,J.L. 1982. "The dynamics of the beam-beam interaction." In **Physics of High Energy Particle Accelerators**. *See* Courant 1982.