Fermi National Accelerator Laboratory

Fermilab-Pub-85/154-T

FACTORIZATION AND ZERO SLOPE LIMIT OF STRINGS

Hiroshi Itoyama

Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, Illinois 60510

ABSTRACT

Factorization properties of open strings are studied and the zero slope limit of tachyon emission is discussed from this point of view. The generating functional for the S matrix element is constructed and is shown to obey, in the zero slope limit, the same functional equation as the one of ϕ^3 field theory. Comments on the relation with recent approaches are made.

I. Introduction

String theories [1] - the old idea which arose from the endeavor to interpret the Veneziano amplitude [2] - now seem to be a leading candidate for the unified theory of fundamental interactions [3]. Recent extensive investigation tells us that anomaly free superstring theories [4, 5] have interesting phenomenological contents [6] besides their mathematical consistency.

On the other hand, it seems that there remains much to be seen in the structure of string theory itself. The present status of our understanding is reflected by the fact that most of the recent development [7] are based on the properties of strings in the local field theory (or zero slope) limit.

In the early study of string theories, it was found that four point tree amplitudes for tachyons and massless vector emission of open strings coincide with the ones in ϕ^3 theory or Yang-Mills theory respectively when the slope parameter α' approaches zero [8]. (It is actually a singular part of the amplitude in the limit $\alpha'\to 0$.) Of course, the property is not limited to the four point amplitudes: the string theory in its entity goes over to the field theory when the singular part in the limit $\alpha'\to 0$ is picked up. Such explicit demonstration requires, however, the use of other properties which are unique to strings.

In this paper, we would like to present a mathematical formulation which enables us to look at the zero slope limit through the generating functional for the S-matrix. We will see that the equation this quantity obeys agrees precisely with the field theory counterpart, thereby giving us an explicit proof of the statement made before. We will make a frequent

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

$$L(X_\mu) = -\frac{1}{2}\left(\left|\frac{\partial X}{\partial x}\right|^2 + \left|\frac{\partial X}{\partial y}\right|^2\right). \qquad (2.1)$$

The N particle Veneziano amplitude is

$$(2\pi)^D \delta^{(D)}(\sum_{i=1}^{N} k_i) V_N(k_1, k_2, \ldots k_n) =$$

$$= \int \cdots \int d\mathcal{V}_N(z_1, \ldots\ldots z_N) \prod_{i=1}^{N} |z_i - z_{i-1}|^{\alpha_0} < :\exp(i\sqrt{2\pi\alpha'} \sum_{i=1}^{N} k_i \cdot X(x_i, y_i)): >_D, \qquad (2.2)$$

where $< \cdots >_D$ means the functional average mentioned above and : : means a normal ordering, in other words, the elimination of the self contraction of $X_\mu(x_i y_i)$. $d\mathcal{V}_N(z_1 \ldots z_N)$ is integration measure defined by

$$d\mathcal{V}_N(z_1 \cdots z_N) \equiv \prod_{i=2}^{N-2} d\theta_i \frac{|z_1 - z_N||z_N - z_{N-1}||z_1 - z_{N-1}|}{\prod_{i=1}^{N} |z_i - z_{i-1}|}. \qquad (2.3)$$

The measure is invariant under the Moebius transformations which we will explain shortly.

For the sake of later discussions, it is useful to reexpress the functional average in the following rudimental amplitude [10]. We divide the circle into N segments, γ_i $(i = 1, \cdots N)$, putting N "charges" on a unit circle (Fig. 2) . Consider a set of Moebius transformations g_i such that

$$g_i ; \quad \Sigma \rightarrow \gamma_i, \qquad g_i \sigma[0] = z_i, \quad g_i \sigma[\pi] = z_{i+1} \qquad (2.4)$$

with

$$\Sigma \equiv \left\{ z = \sigma[\xi] = \frac{1}{2}(1 - e^{i\xi}), \quad 0 < \xi < \pi \right\}.$$

Then

$$<\exp(i\sqrt{2\pi\alpha'} \sum_{i=1}^{N} k_i \cdot X(x_i, y_i)): >_D \equiv \lim_{\xi \to 0} \prod_{i=1}^{N} E(g_i)^{-1} \left| \begin{matrix} k_1(\cdot), k_2(\cdot), & \cdots & k_N(\cdot) \\ g_1, \quad g_2, & \cdots & g_N \end{matrix} \right|_\Sigma \qquad (2.5)$$

where

$$k_i(\xi) = k_i \rho_\epsilon(\xi), \quad E(g_i) = \exp(\alpha' \int_0^\pi d\xi \int_0^\pi d\xi' \rho_\epsilon(\xi) \rho_\epsilon(\xi') \ln |g_i \sigma[\xi] - g_i \sigma[\xi']|)$$

$$\lim_{\epsilon \to 0} \rho_\epsilon(\xi) = k_i \delta(\xi), \qquad \alpha' k_i^2 = \alpha_0 \qquad (2.6)$$

A Moebius transformation is defined by

$$z \rightarrow z' = g[z] = \frac{az + b}{cz + d} \qquad (2.7)$$

III. Factorization Properties of N Particle Veneziano Amplitude

Like usual Feynman diagrams, the factorization of string amplitudes is done, in operator formalism, by inserting the completeness relation $\sum_{\gamma} |\gamma\rangle\langle\gamma| = \mathbb{1}$ between operators. In functional integral approach, this corresponds, schematically, to slicing the unit disk by an arc which orthogonally intersects the unit circle. For definiteness, we slice the disc as shown in Fig. 3.

What we precisely mean by slicing the disk is that the original amplitude defined on D can be expressible in terms of the product of the amplitude on D_1 and the one on D_2. This requires that each of the factors appearing in eq. (2.2) be written as such product up to the invariance it owns. Let us now look at $d\mathcal{V}_n(z_1,...,z_N)$, $\prod_{i=1}^{N} \frac{|z_i-z_{i-1}|^{\alpha_0}}{E(g_i)}$ and $\begin{bmatrix} k_1(*), & \cdots & k_N(*) \\ g_1, & \cdots & g_N \end{bmatrix}_{\Sigma}$ separately.

i) $d\mathcal{V}_N(k_1,...,k_N)$: The measure is constructed to be invariant under the Moebius transformations. It actually depends only on N-3 variables chosen and we can freely fix any of the three variables. Let us now fix z_1, z_j and z_N.

$$d\mathcal{V}_N(z_1,...z_N) = \prod_{i=2}^{j-1} d\theta_i \prod_{i=j+1}^{N-1} d\theta_i \frac{|z_1-z_N|\ |z_N-z_j|\ |z_1-z_j|}{\prod_{i=1}^{N} |z_i-z_{i-1}|}. \tag{3.1}$$

It is also possible to think the integration measure in domain I and domain II separately. (We identify z_E with z_{j+1} and z_F with z_1. See Fig. 4.) We fix three variables z_1, z_j and z_E in domain I and z_{j+1}, z_N and z_F in domain II. Then

$$d\mathcal{V}_{j+1}(z_1,...z_j, z_E) = \prod_{i=2}^{j-1} d\theta_i\, |z_1-z_j| \Big/ \prod_{i=1}^{j-1} |z_{i+1}-z_i|, \tag{3.2}$$

$$d\mathcal{V}_{N-j+1}(z_{j+1},...z_N, z_F) = \prod_{i=j+2}^{N-1} d\theta_i\ |z_{j+1}-z_N| \Big/ \prod_{i=j+1}^{N-1} |z_{i+1}-z_i|. \tag{3.3}$$

We find immediately

$$d\mathcal{V}_N(z_1,...z_N) = d\mathcal{V}_j(z_1, \cdots z_j, z_E)\, d\mathcal{V}_{N-j+1}(z_{j+1}, \cdots z_N, z_F) \frac{|z_N-z_j|}{|z_{j+1}-z_N|\ |z_{j+1}-z_j|} d\theta_{j+1}, \tag{3.4}$$

The last factor is going to be used for a string (Reggeon) propagator.

ii) $\prod_{i=1}^{N} |z_i-z_{i-1}|^{\alpha_0} / \prod_{i=1}^{N} E(g_i)$: From Eq.(2.10), we see that this factor can be written as $\lim_{\epsilon\to 0}[1/E(1)]^N \prod_{i=1}^{N} \left(\frac{|z_{i+1}-\tilde{z}_i|}{|z_i-\tilde{z}_i|}\right)^{-\alpha_0}$ with $\tilde{z}_i$ an arbitrary point on the circle. Let us choose $\tilde{z}_i = z_{i-1}$ for $i=2, \cdots j, j+2, \cdots N$, $\tilde{z}_1 = z_{j+1}$ and $\tilde{z}_{j+1} = z_1$. Then it becomes

$$\times c(\epsilon)^{N-j+1} R_{N-j+1}(p_F(\cdot), k_{j+1}\delta(\cdot), \cdots k_N \delta(\cdot))(2\pi)^D \delta^{(D)}(\sum_{i=j+1}^{N} k_i + \int_0^{\pi} d\xi \, p_F(\xi)) \quad (3.9)$$

where $c(\epsilon) = \lim_{\epsilon \to 0} \frac{1}{E(1)}$.

For later use, we relabel the indices in eq. (3.9) as shown in Figure 6. It represents the possible triangulation ($j=2 \sim N-1$) with respect to the vertex at point $j+1$.

$$(2\pi)^D \delta^{(D)}(\sum_{i=1}^{N} k_i) V_N(k_1, \cdots, k_N) g^{N-2}(\sqrt{\alpha'})^{N-4} = \int \mathcal{D}P_E \mathcal{D}P_F \mathcal{D}P_G \mathcal{D}P_H \, (2\pi)^D \delta^{(D)}(\sum_{i=j+1}^{N} k_i + \int_0^{\pi} d\xi \, P_G(\xi))$$
$$c(\epsilon)^{N-j+1} g^{N-j-1} (\sqrt{\alpha'})^{N-j-3} R_{N-j+1}(k_{j+1}\delta(\cdot), \cdots k_N \delta(\cdot), P_G(\cdot)) \frac{1}{c(\epsilon)} \sqrt{\alpha'} P(-P_G, -P_H; V_G, V_H) \sqrt{\alpha'} g \begin{bmatrix} P_H, & k_1\delta(\cdot), & P_E \\ V_H, & V_1, & V_E \end{bmatrix}$$
$$\sqrt{\alpha'} P(-P_E, -P_F; V_E, V_F) c(\epsilon)^j g^{j-2} (\sqrt{\alpha'})^{j-4} R_j(P_F(\cdot), k_2\delta(\cdot), \cdots k_j \delta(\cdot)) (2\pi)^D \delta^{(D)}(\sum_{i=2}^{j} k_i + \int_0^{\pi} P_F(\xi) d\xi). \quad \Sigma \quad (3.10)$$

IV. The Generating Functional for the S matrix Element of ϕ^3 Theory

In this section, we construct a generating functional for the tree S matrix element of $\lambda\phi^3$ field theory and derive a functional equation which it has to obey. We will see, in the next section, that the corresponding quantity in string theories obeys the same equation in the zero slope limit.

Let us start out from the generating functional for the connected Green functions denoted by W. The above mentioned quantity is then extracted from W in a similar manner to LSZ formalism [14]. Let W_{tree} be

$$W_{tree}[j] \equiv \sum_n \frac{i^n}{n!} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D} j(k_1) \ldots j(k_n) G_{tree}^{(n)}(k_1 \cdots k_n)$$
$$\equiv \sum_n W_{n,tree} = \langle out \mid in \rangle^j_{tree,c}, \quad (4.1)$$

with Lagrangian

$$L = -\frac{1}{2}\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{\lambda}{3!}\tilde{\phi}^3 + \tilde{j}\tilde{\phi}. \quad (4.2)$$

(The tilde denotes the inverse Fourier transform.) The $G_{tree}^{(n)}(k_1 \cdots k_N)$ is the n-point connected Green's function in the tree approximation. From the well-known formula

$$-i\frac{\delta W[j]}{\delta j(-k)} \equiv \phi_{cl}(k) = \underbrace{\langle out \mid \tilde{\phi}(x) \mid in \rangle_c}_{\text{Fourier transform of}}, \quad (4.3)$$

one obtains without approximation

$$(+i)(-k^2)\frac{\delta W[j]}{\delta j(-k)} = k^2 \langle out \mid \phi(k) \mid in \rangle_c^j$$

where

$$C=\int_0^{2\pi} d\theta_1 \int_{\theta_1}^{\theta_1+2\pi} d\theta_2 \int_{\theta_1}^{\theta_2} d\theta_3 \frac{1}{|z_1-z_2|\,|z_2-z_3|\,|z_3-z_1|} \tag{5.3}$$

and

$$\tilde{A}(\sqrt{2\pi\alpha'}x(z)) \equiv \int \frac{d^D k}{(2\pi)^D} :e^{i\sqrt{2\pi\alpha'}\,k\cdot X(z)}: A(k) \tag{5.4}$$

is a Fourier transform of $A(k)$ with respect to the dynamical variable X_μ.

It is straightforward to extend this to massless vector emission and eq. (5.2) becomes

$$\frac{i}{2Cg^2\alpha'^2} \langle \exp[g\sqrt{\alpha'}\,\epsilon_\mu \int d\theta \frac{\partial X^\mu(x)}{\partial\theta}\,\tilde{A}(\sqrt{2\pi\alpha'}X(x))] \rangle_0 \tag{5.5}$$

where ϵ_μ is a polarization tensor of massless vector emitted.

For closed strings, one can also proceed in a similar spirit to construct the corresponding quantity though the integration measure and combinatorics are different. The S-matrix functional for graviton and antisymmetric tensor constructed this way is expressed as

$$\frac{i}{C'g^2\alpha'^2} \Big\langle \exp g\sqrt{\alpha'} \int d^2\xi \Big\{ \frac{1}{2}\epsilon^{(S)}_{\mu\nu}\partial^a X^\mu(z)\partial^a X^\nu(z)\tilde{A}_G(\sqrt{2\pi\alpha'}X(z)) +$$

$$+ \frac{1}{2}\,\epsilon^{AS}_{\mu\nu}\epsilon_{ab}\,\partial^a X^\mu(z)\partial^b X^\nu(z)\tilde{A}_B(\sqrt{2\pi\alpha'}X(z))\Big\} \Big\rangle_{\xi_2} \tag{5.6}$$

Here, $\epsilon^{(S)}_{\mu\nu}$ and $\epsilon^{(AS)}_{\mu\nu}$ are polarization tensors for graviton and antisymmetric tensor. The rest of the notation is self-explanatory.

A few remarks are in order. Quantity (5.6) may be viewed as background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and D interacting with strings if one suppresses X dependence of $\tilde{A}$:

$$\langle \exp\int d^2\xi\, \frac{1}{2}G_{\mu\nu}\partial_a X^\mu \partial_a X^\nu + \frac{1}{2}B_{\mu\nu}\epsilon_{ab}\,\partial_a X^\mu\partial_b X^\nu + \frac{1}{2}D\,\partial_a X^\mu \partial_a X_\mu \rangle_S . \tag{5.7}$$

In fact, it agrees with the quantity first considered by Fradkin and Tseytlin [15] except for a singular coefficient. They suggested that it is an effective action. But what it agrees with in the zero slope limit is the quantity extracted from generating functional for the connected Green functions [16], as we will see below for tachyon emission.

Recently, a σ model calculation has been suggested [17] with (5.7), and interesting results are obtained for closed strings. Our approach is equally applicable to both open and closed strings and may be a way to make sense out of the quantities like (5.4) and (5.5)

With these remarks in mind, we reach

$$(2\pi)^D \delta^{(D)}(\sum_{i=1}^{N} k_i) \lim \left| g^{N-2}(\sqrt{\alpha'})^{N-4} V_N(k_{1,\dots,}k_N) \right|$$

$$= \sum_{j=2}^{N-1} \int \frac{dp'_G}{(2\pi)^D} \frac{dp'_F}{(2\pi)^D} (2\pi)^D \delta^{(D)}(\sum_{i=j+1}^{N} k_i + p'_G) \lim \left(g^{N-j-1}(\sqrt{\alpha'})^{N-j-3} V_{N-j+1}(k_{j+1}, \cdots, k_N, p'_G) \right)$$

$$\times \frac{1}{p'^2_G} (2\pi)^D \delta^{(D)}(k_1 - p'_G - p'_F) \lambda \frac{1}{p'^2_F} \lim \left| g^{j-2}(\sqrt{\alpha'})^{j-4} V_j(p'_F, k_2, \cdots, k_j) \right| (2\pi)^D \delta^{(D)}(\sum_{i=2}^{j} k_i + p'_F)$$

(Fixing an overall coefficient in the functional integral formulation is difficult. We did it by looking at lower N directly. Also $\int_0^{\pi} d\xi p_F(\xi) \equiv p'_F$, $\int_0^{\pi} p_G(\xi) \equiv p'_G$).

Using this expression, one finds

$$(-i) \lim \frac{\delta S[A]}{\delta A(-k)} = 2\lambda \sum_{N \geq 4} \sum_{j=2}^{N-1} \int \frac{dp}{(2\pi)^D} \frac{dp_G}{(2\pi)^D} \lim \frac{\delta S_{N-j+1}[A]}{\delta A(-p_G)} \frac{1}{p_G^2}$$

$$(2\pi)^D \delta(k - p_G - p_F) \frac{1}{p_F^2} \lim \frac{\delta S_j[A]}{\delta A(-p_F)}$$

$$= -2\lambda \int \frac{dp}{(2\pi)^D} \lim \sum_{l=2}^{\infty} \frac{\delta S_l[A]}{\delta A(-p)} \frac{1}{p^2} \frac{1}{(k-p)^2} \lim \sum_{j=2}^{\infty} \frac{\delta S_j[A]}{\delta A(-k+p)}$$

$$= 2\lambda \int \frac{dp}{(2\pi)^D} \frac{i}{-p^2} \lim \frac{\delta S[A]}{\delta A(-p)} \frac{i}{-(k-p)^2} \lim \frac{\delta S[A]}{\delta A(-k+p)}.$$

We see that Eq. (5.11) is the same as Eq. (4.7) except for a numerical factor. In particular, with $\lim S[A] = \frac{1}{8} S_0[2A]$, they become identical [18].

References:

[1] Y. Nambu, in Symmetries and Quark Models, edited by R. Chand (Gordon and Breach, New York, 1970).

L. Susskind, Nuovo Cimento, 69, 210 (1970); Phys. Rev., D1, 1182 (1970)

[2] G. Veneziano, Nuovo Cimento Ser. 10 57A, 190 (1968).

[3] J.H. Schwarz, Phys. Rep. 89, 203 (1982); M.B. Green, Surv. High Energy Phys. 3, 127 (1983).

[4] M.B. Green and J.H. Schwarz, Phys. Lett. 149B, 117 (1984).

[5] D. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54, 502 (1985; Nucl. Phys. B256 , 253 (1985); Princeton preprint 85-0694.

[6] P. Candelas, G. Horowitz, A. Strominger and E. Witten, ITP preprint, NSF-ITP-84-170.

FIGURE CAPTIONS

Figure 1: Geometry of N tachyon Amplitude

Figure 2: Division of the circle into N segments γ_i

Figure 3: Slicing the unit disk by an arc.

Figure 4: Factorization into two domains D_1 and D_2.

Figure 5: Factorization into three domains.

Figure 6: A triangulation of N tachyon Amplitude with respect to the vertex at point j+1.

Figure 7: Three diagrams in four particle amplitude.

Figure 8: Two factorizations intersecting each other.

use of the factorization property of string amplitudes. A duality property is another ingredient and plays an essential role to obtain a combinatorial property required.

To illustrate the duality property, let us take the simplest four point Veneziano amplitude:

$$V_4(k_1, k_2, k_3, k_4) = \int_0^1 dx\, x^{-\alpha_0 - \alpha' s} (1-x)^{-\alpha_0 - \alpha' t} \tag{1.1}$$

In the string calculation, we arrive at this expression, evaluating the following expression given in terms of mode expansion ($\alpha_0 = 1$) :

$$\int_0^1 dx \, <0| e^{\sqrt{2\alpha'} k_3 \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}} x^{\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n} x^{\alpha'(k_1 + k_2)^2 - 2} e^{\sqrt{2\alpha'} k_2 \cdot \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n}} |0>. \tag{1.2}$$

The constant mode gives, in the limit, the s-channel pole, whereas the t-channel pole does not arise from any single mode. It, instead, comes from the coherent summation of the nonzero modes. The zero slope limit of strings is not a naive point-like limit of one dimensional extended object.

For simplicity, we limit ourselves to tachyon emission for open strings. The extension to vector emission is straightforward. More work is necessary for the closed strings, however.

A functional integral formulation, [9, 10] developed a while ago is best suited for our purpose. Dual string amplitudes, in this formulation, are directly related to the two dimensional electrostatic problem where a set of charges is located at the boundary of a Riemann surface.

In section II, the functional integral approach is reviewed. Using the formalism, we discuss, in section III, factorization properties and the zero slope limit of N particle tree string (Veneziano) amplitude [11]. In section IV, the generating functional for the tree S-matrix element in ϕ^3 field theory is constructed. In section V, the generating functional for tree string amplitudes is given and is shown to obey the same functional equation, in the zero slope limit, as the field theory counterpart, thereby demonstrating the statement mentioned above.

II. Functional Integral Formulation of String Amplitudes

In this section, the functional integral approach [9, 10] is reviewed briefly. In this approach, the N-particle Veneziano amplitude is expressed as a functional average of $\exp(i\sqrt{2\pi\alpha'} \sum_{i=1}^{N} k_i \cdot X(x_i, y_i))$ over $X_\mu(x, y)$ on a unit disk, where $z_i = x_i + iy_i = e^{i\theta_i}$ is a point on the boundary of the disk, and k_i is the momentum of the i th tachyon. (Fig. 1) The Lagrangian used is that of a free string conformally transformed to a unit disk [11];

with a, b, c and d complex number satisfying $ad-bc=1$. A set of these points z_i, $i = 1, 2, 3$ and their image $g(z_i)$, $i=1, 2, 3$ specify g_i uniquely. One can therefore write [13]

$$g = \begin{vmatrix} z_1, z_2, z_3 \\ z_1', z_2', z_3' \end{vmatrix}. \tag{2.8}$$

In this notation,

$$g_i = \begin{vmatrix} 1, & 1, & 0 \\ z_i, & z_{i+1}, & \tilde{z}_i \end{vmatrix} \begin{vmatrix} 0, & 1, & \infty \\ 1, & -1, & 0 \end{vmatrix}, \tag{2.9}$$

where $\tilde{z}_i$ is an arbitrary point on the unit circle. Using this, one can show

$$E(g_i) = \left|\frac{\partial(g_i\sigma[\xi])}{\partial\sigma[\xi]}\right|^{\alpha_0}_{\xi=0} = \left|\frac{|z_{i+1}-\tilde{z}_i||z_{i+1}-z_i|}{|z_i-\tilde{z}_i|}\right|^{\alpha_0} E(1). \tag{2.10}$$

It is possible to define a more general amplitude with extended momentum distribution $p_l(\xi)$ and an arbitrary reference curve to parameterize the boundary of simply connected domain D:

$$\begin{vmatrix} p_1(\cdot), p_2(\cdot), \ldots, p_m(\cdot) \\ g_1, \quad g_2, \quad \ldots \quad g_m \end{vmatrix}_\Sigma = \langle \exp(i\sqrt{2\pi\alpha'}\sum_l \int_0^\pi d\xi p_l(\xi)\cdot X(g_l\sigma[\xi])\rangle_D \,. \tag{2.11}$$

It has the following properties:

(i) Cyclic symmetry

$$\begin{vmatrix} p_1 \ldots p_m \\ g_1 \ldots g_m \end{vmatrix}_\Sigma = \begin{vmatrix} p_2 \ldots p_m p_1 \\ g_2 \ldots g_m g_1 \end{vmatrix}_\Sigma$$

(ii) Conformal invariance

$$\begin{vmatrix} p_1 \cdots p_m \\ g_1 \cdots g_m \end{vmatrix}_\Sigma = \begin{vmatrix} p_1 \ldots p_m \\ gg_1 \cdots gg_m \end{vmatrix}_\Sigma$$

(iii) Reference curve invariance

$$\begin{vmatrix} p_1 \cdots p_m \\ g_1 \cdots g_m \end{vmatrix}_\Sigma = \begin{vmatrix} p_1 \ldots p_m \\ g_1\Lambda^{-1} \cdots g_m\Lambda^{-1} \end{vmatrix}_{\Lambda\Sigma}$$

The corresponding amplitude is

$$(2\pi)^D \delta^{(D)}(\sum_i \int_0^\pi d\xi\, p_i(\xi))\;\; R_N(p_1(\cdot), \ldots, p_N(\cdot))$$

$$\equiv \int d\psi_N(z_1, \ldots z_N) \prod_{i=1}^N \left|\frac{|z_{i+1}-\tilde{z}_i|}{|z_i-\tilde{z}_i|}\right|^{-\alpha_0} \begin{vmatrix} p_1(\cdot), \ldots, p_m(\cdot) \\ g_1, \quad \ldots \quad g_m \end{vmatrix}_\Sigma \tag{2.12}$$

$$\prod_{i=1}^{j}\left(\frac{|z_{i+1}-z_{i-1}|}{|z_i-z_{i-1}|}\right)^{-\alpha_0}_{z_0=z_{j+1}}\prod_{i=j+1}^{N}\left(\frac{|z_{i+1}-z_{i-1}|}{|z_i-z_{i-1}|}\right)^{-\alpha_0}_{z_j=z_1}$$

$$=\prod_{i=1}^{j,E}\left(\frac{|z_{i+1}-\tilde{z}_i|}{|z_i-\tilde{z}_i|}\right)^{-\alpha_0}_{D_I}\prod_{i=j+1}^{N,F}\left(\frac{|z_{i+1}-\tilde{z}_i|}{|z_i-\tilde{z}_i|}\right)^{-\alpha_0}_{D_{II}}\left(\frac{|z_E-z_j|}{|z_1-z_j|}\right)^{-\alpha_0}\left(\frac{|z_F-z_N|}{|z_{j+1}-z_N|}\right)^{-\alpha_0} \tag{3.5}$$

iii) $\begin{vmatrix} k_1(\cdot), k_2(\cdot), \cdots k_N(\cdot) \\ g_1, \quad g_2, \quad \cdots \quad g_N \end{vmatrix}$: It can be factorized by using the slicing rule of ref. [10]. (The slicing rule can be readily given from the functional analog of $1=\sum_\lambda |\lambda\rangle\langle\lambda|$.)

$$\begin{vmatrix} k_1(\cdot),k_2(\cdot),\ldots k_N(\cdot) \\ g_1, \quad g_2, \ \cdots \ g_N \end{vmatrix}_\Sigma =$$

$$=\int Dp_E\, Dp_F \begin{vmatrix} k_1(\cdot),k_2(\cdot),\cdots k_j, p_E \\ \tilde{g}_1, \quad g_2 \quad \cdots g_j, V_E \end{vmatrix}_\Sigma \begin{vmatrix} -p_E, -p_F \\ V_E, \ V_F \end{vmatrix}_\Sigma \begin{vmatrix} p_F, p_{j+1}, \cdots k_N \\ V_F, \tilde{g}_{j+1}, \cdots g_N \end{vmatrix}_\Sigma \tag{3.6}$$

The tilde in $\tilde{g}_1$ and $\tilde{g}_{j+1}$ mean $\tilde{z}_1=z_E$ and $\tilde{z}_{j+1}=z_F$ respectively, and V_E and V_F parameterize respectively the segment γ_E and γ_F.

From i), ii) and iii), one finds

$$(2\pi)^D\delta^{(D)}(\sum_i k_i)V_N(k_1,\ldots,k_N) =$$

$$=\int Dp_E\, Dp_F \lim_{\epsilon\to 0}[\frac{1}{E(1)}]^N(2\pi)^D\delta^{(0)}(\sum_{i=1}^{j}k_i + \int_0^\pi d\xi\, p_E(\xi))R_{j+1}(k_1\delta(\cdot),\cdots k_j\delta(\cdot),p_E(\cdot))$$

$$P(-p_E,-p_F;V_E,V_F)R_{N-j+1}(p_F,k_{j+1}\delta(\cdot)\cdots,k_N\delta(\cdot))(2\pi)^D\delta^{(D)}(\sum_{i=j+1}^{N}k_i + \int_0^\pi d\xi\, p_F(\xi)) \tag{3.7}$$

where

$$P(-p_E,-p_F;V_E,V_F)\equiv\int_0^\pi d\theta_E\frac{|z_N-z_j|}{|z_{j+1}-z_N|\ |z_{j+1}-z_j|}(\frac{|z_E-z_j|}{|z_1-z_j|})^{-\alpha_0}(\frac{|z_F-z_N|}{|z_{j+1}-z_N|})^{-\alpha_0}\begin{vmatrix} -p_E, -p_F \\ V_E, V_F \end{vmatrix}_\Sigma$$

$$=\int_0^1 dx\ x^{\alpha_0-1}(1-x)^{\alpha_0-1}\begin{vmatrix} -p_E & -p_F \\ 1 & V_E^{-1}V_F \end{vmatrix}_\Sigma \tag{3.8}$$

is in fact a string propagator. The $x\equiv\frac{(z_E-z_N)}{(z_E-z_F)}\ \frac{(z_j-z_F)}{(z_j-z_N)}$ is a Moebius invariant cross ratio.

We repeat this procedure once more and obtain the factorization as indicated in Figure 5.

$$(2\pi)^D\delta^{(D)}(\sum_i k_i)V_N(k_1\cdots k_N)$$

$$=\int Dp_E\, Dp_F\, Dp_G\, Dp_H(2\pi)^D\delta^{(D)}(\sum_{i=1}^{j-1}k_i+\int_0^\pi d\xi\, p_G(\xi))c(\epsilon)^j R_j(k_1\delta(\cdot),\cdots,p_G(\cdot))\times$$

$$\times\frac{1}{c(\epsilon)}P(-p_G,-p_H;V_G,V_H)\begin{vmatrix} p_H, k_j\delta(\cdot) p_E \\ V_H, \ \tilde{V}_j, \ V_E \end{vmatrix}_\Sigma P(-p_E,-p_F;V_E,V_F)\times$$

$$= j(k) + \frac{\lambda}{2}\int \frac{d^D p}{(2\pi)^D} < out \mid \phi(p)\phi(k-p) \mid in >_c^j \quad . \tag{4.4}$$

In the tree approximation, the second term in the right hand side is also written as $\frac{\lambda}{2}\int \frac{d^D p}{(2\pi)^D}\phi_{cl}(p)\phi_{cl}(k-p)$. In fact, W_{tree} consists of 1-particle reducible diagrams only. The process of amputation given by the left-hand side decomposes any diagram into two disconnected parts. So, we obtain

$$(+i)(-k^2)\frac{\delta W_{tree}}{\delta j(-k)} = j(k) + \frac{\lambda}{2}\int \frac{d^D p}{(2\pi)^D}(-i)\frac{\delta W_{tree}}{\delta j(-p)}(-i)\frac{\delta W_{tree}}{\delta j(-k+p)} . \tag{4.5}$$

(Of course, one reaches the above formula directly from Feynman diagrams, working out the combinational problem.) The generating functional for the tree S matrix element is defined from W_{tree} by

$$S_0 \equiv \lim_{\substack{j \to 0, k^2 \to 0 \\ j(k)/k^2 \equiv A(k) = \text{finite}}} W_{tree}[j(k) = k^2 A(k)] \tag{4.6}$$

It obeys

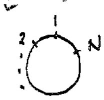
$$-i\frac{\delta S_0}{\delta A(-k)} = \frac{\lambda}{2}\int \frac{d^D p}{(2\pi)^D}\frac{+i}{-p^2}\frac{\delta S_0}{\delta A(-p)}\frac{+i}{-(-k+p)^2}\frac{\delta S_0}{\delta A(-k+p)} . \tag{4.7}$$

V. The Generating Functional for String Amplitudes and the Zero Slope Limit

Let us first count the number of N point string diagrams, which we denote by $D(N)$. In $N=4$, we have $D(4)=3$ diagrams corresponding to the different possibilities of putting labels (see Fig. 7). Each diagram represents s-t channel, s-u channel and t-u channel exchange respectively and the S-matrix element is given by the summation of these three diagrams. Clearly $D(N)=(N-1)D(N-1)$, and $D(N)=\frac{(N-1)!}{2}$. The $D(N)$ N point string diagrams are same modulo the labeling of momenta. This tells us the right definition of the generating functional.

$$S[A] \equiv \sum_{N \geq 4} g^{N-2}(\sqrt{\alpha'})^{N-4} S_N[A] =$$

$$i\sum_N \frac{1}{N!}\frac{(N-1)!}{2} g^{N-2}(\sqrt{\alpha'})^{N-4}\int \frac{dk_1}{(2\pi)^D}\cdots\frac{dk_N}{(2\pi)^D} A(k_1)\cdots A(k_N)\,(2\pi)^D\delta(\sum_i k_i)\,V_N(k_1, k_2, \cdots, k_N)$$

$$\equiv i\sum_{N>=4} g^{N-2}(\sqrt{\alpha'})^{N-4}\frac{1}{2N}$$ 

$$\tag{5.1}$$

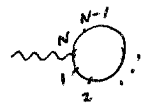
Note that, in $\alpha_0=1$, $S[A]$ with suitable definition of $S_i[A]$ $(i=1,2,3)$ can formally be written as [9]

$$\frac{i}{2Cg^2\alpha'^2} < \exp[g\sqrt{\alpha'}\int d\theta \tilde{A}(2\pi\alpha' X(z))] >_D , \tag{5.2}$$

which are too singular to be tractable in the conventional σ model calculations.

Now coming back to Eq.(5.1), we would like to evaluate the zero slope limit of the quantity $\frac{\delta S[A]}{\delta A(-k)}$

$$\frac{\delta S[A]}{\delta A(-k)} = i \sum_{N>=4} g^{N-2}(\sqrt{\alpha'})^{N-4}\frac{1}{2} \qquad (5.8)$$



In the previous section, we discussed the N particle tree string amplitudes and derived its factorized expression (Eq. (3.10)): The N particle amplitude is written by the product of the j particle amplitude, string propagator, three Reggeon vertex, string propagator and the $N-j+1$ particle amplitude $(2 \leqslant j \leqslant N-1)$. Correspondingly, there are $N-2$ such factorizations of $\frac{\delta S_N}{\delta A}$. Let us take any two of such factorizations and draw solid lines for the one triangulation with respect to the particular vertex and dotted lines for the other. (Fig. 8) We see that there is always an intersection between a dotted line and a solid line. This means that these two towers of resonances in two different channels are not represented separately.

In the zero slope limit, however, the contribution from the lowest poles comes from the singularity of the intergrand in various domains of the integration variables. The two different contributions from the above two channels, in the limit, correspond to the singularities in two distinct domains. It is, therefore, legitimate and mandatory to add $(N-2)$ possibilities in the zero slope limit.

When we take the limit of eq.(3.10),i.e. $\alpha' \to 0, g \to 0$ and $g^2\alpha' = \lambda = finite$, the $N-j+1$ amplitude and the j amplitude with appropriate powers of g, α' and $C(\epsilon)$ are finite. The three string vertex is also finite. But the propagator has $1/\alpha'$ singularity due to the edge singularity of the integrand at $x=0$ and 1:

$$P(-p_E, p_F; V_E, V_F) \approx (2\pi)^D \delta^{(D)}\left(-\alpha'\int_0^\pi d\xi' p_E(\xi') - \alpha'\int_0^\pi d\xi' p_F(\xi')\right) \qquad (5.9)$$

$$\int_0^1 dx\ x^{-1+\alpha'(\int_0^\pi d\xi' p_E(\xi'))\cdot(\int_0^\pi d\xi' p_F(\xi'))}(1-x)^{-1+\alpha'(\int_0^\pi p_E(\xi')d\xi')\cdot(\int_0^\pi d\xi' p_F(\xi'))}$$

$$= (2\pi)^D \delta^{(D)}\left(\alpha'\int_0^\pi d\xi' p_E(\xi') + \alpha'\int_0^\pi d\xi' p_E(\xi')\right) B\left(-\alpha'\left[\int_0^\pi d\xi p_E(\xi')\right]^2, -\alpha'\left[\int_0^\pi d\xi p_E(\xi)\right]^2\right)$$

$$= (2\pi)^D \delta^{(D)}\left|\alpha'\int_0^\pi d\xi'(\xi') + \alpha'\int_0^\pi d\xi' p_F(\xi')\right| \frac{-2}{\alpha'} \frac{1}{\left[\int_0^\pi d\xi p_E(\xi)\right]^2} + O(1)$$

where we have set $\alpha_0=0$ to project out the zero mass grand state. Two powers of $1/\alpha'$, combined with $(\sqrt{\alpha'})^3 g$, have the finite limit.

[7] See, for example, "Symposium on Anomalies, Geometry and Topology". Edited by W.A. Bardeen and A. White, World Scientific, 1985.

[8] J. Scherk, Nucl. Phys. B31 ,222 (1971); A. Neveu and J. Scherk, Nucl. Phys. B36 ,155 (1972); J.L. Gervais and A. Neveu, Nucl. Phys. B46 , 381 (1972).

[9] C.S. Hsue, B. Sakita and M.A. Virasoro, Phys. Rev. D2 , 2851 (1970).

[10] J.L. Gervais and B. Sakita, Phys. Rev. D4 ,2291 (1971).

[11] Z. Koba and H. Nielsen, Nucl. Phys. B12 , 517 (1969).

[12] Starting from the Polyakov approach, one can deduce, in D = 26, this Lagrangian after fixing to the conformal gauge.

[13] C. Lovelace, Phys. Lett. 32B , 490 (1970).

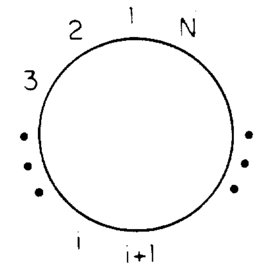
[14] See, for instance, J. Bjorken and S. Drell, "Relativistic Quantum Fields"

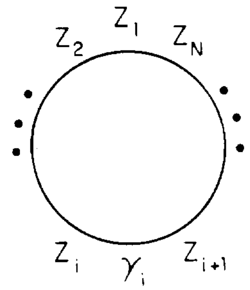
[15] E.S. Fradkin and A.A. Tseytlin, Lebedev Institute preprint N 261 (1984); Phys. Lett. 158B , 315 (1985).

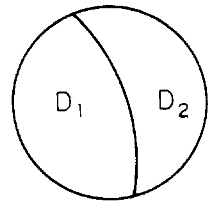
[16] See also, K. Kikkawa and M. Yamasaki, Osaka preprint 1985.

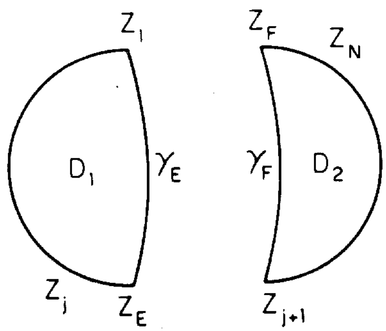
[17] C.G. Callan, E.J. Martinec, M.J. Perry and D. Friedan, Princeton preprint 1985; A. Sen, Fermilab preprint, Pub. - 85/77-T, (1985).

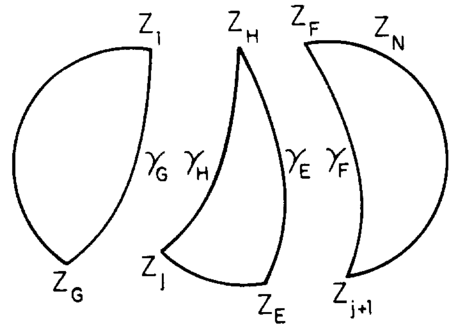
[18] The counting of Feynman diagrams suggests the choice 2A in the argument.

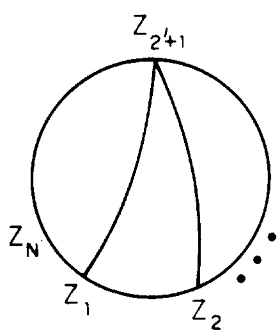

1
2
N
3
i
i+1

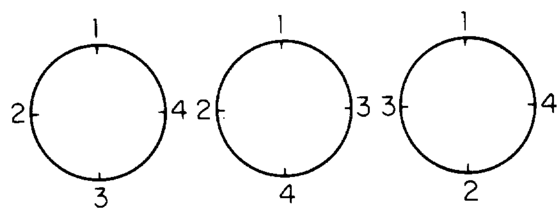

Z_1
Z_2
Z_N
Z_i
γ_i
Z_{i+1}

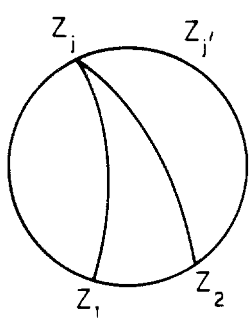

D_1
D_2


Z_1
Z_F
Z_N
D_1
γ_E
γ_F
D_2
Z_j
Z_E
Z_{j+1}


Z_1
Z_H
Z_F
Z_N
γ_G
γ_H
γ_E
γ_F
Z_G
Z_j
Z_E
Z_{j+1}


$Z_{2'+1}$
Z_N
Z_1
Z_2


1
2
4
3
1
2
3
4
1
3
4
2


Z_j
$Z_{j'}$
Z_1
Z_2

FERMILAB-Pub-85/154-T (1)

To: Supplement of Progress of Theoretical Physics

Factorization and Zero Slope Limit of Strin[g]

Hiroshi Itoyama
Fermi National Accelerator Laboratory, P.O. Box 500
Batavia, Illinois 60510.

B. Sakita
Physics Department, City College,
City University of New York,
New York, N.Y. 10031.

Abstract

Factorization properties of Nambu strings are studied and the zero slope limit of tachyon emission is discussed from this point of view. The generati[ng] functional for the S matrix element is constructe[d] and is shown to obey, in the zero slope limit, the same functional equation as the one of ϕ^3 field theory. Comments on the relation with recent approaches are made.

FERMILAB-Pub-80/154

For Supplement of Progress of Theoretical Physics

Factorization and Zero Slope Limit of Strings

Hiroshi Itoyama
Fermi National Accelerator Laboratory, P.O. Box 500
Batavia, Illinois 60510.

B. Sakita
Physics Department, City College,
City University of New York,
New York, N.Y. 10031.

Abstract

Factorization properties of Nambu strings are studied and the zero slope limit of tachyon emission is discussed from this point of view. The generating functional for the S matrix element is constructed and is shown to obey, in the zero slope limit, the same functional equation as the one of ϕ^3 field theory. Comments on the relation with recent approaches are made.