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Equations of motion for the heterotic string theory from the conformal invariance of the sigma model

Ashoke Sen[†]
Fermi National Accelerator Laboratory
P.O.Box 500, Batavia, IL 60510

ABSTRACT

In the presence of arbitrary background gauge, gravitational and anti-symmetric tensor fields, the heterotic string may be described by a two dimensional local field theory with N=1/2 supersymmetry. It is conjectured, and verified to certain approximation, that the conditions on the background fields for this model to have a vanishing β-function are identical to the equations of motion for the massless fields, as derived from the string theory. In particular, the appearance of the Chern-Simons three forms in the classical equations of motion is shown to be related to the chiral anomaly in two dimensional gauge theories.

† Address after Sept. 1, 1985: Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94305.

It was shown in a previous paper[1] that in the presence of arbitrary background gauge, gravitational and anti-symmetric tensor fields, the heterotic string may be described by a two dimensional field theory with N=1/2 supersymmetry, given by the action,

$$S_0 = \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma \, [g_{ij}(X)\{\partial_\alpha X^i \partial^\alpha X^j + i\bar{\lambda}^i \rho^\alpha (D_\alpha \lambda)^j\}$$

$$+ B_{ij}(X)\,\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j - i S_{ijk}(X)\,\bar{\lambda}^i \rho^\alpha \lambda^j \epsilon_{\alpha\beta}\partial^\beta X^k$$

$$+ i\,\delta_{st}\,\bar{\psi}^s \rho^\alpha \partial_\alpha \psi^t + \bar{\psi}^s (T^M)_{st}\,\rho^\alpha \psi^t \{A_i^M \partial_\alpha X^i$$

$$- \frac{i}{4} F^M_{i\ell}\,\bar{\lambda}^\ell \rho_\alpha \lambda^i\}] \qquad (1)$$

where g_{ij} is the background metric, B_{ij} is the background antisymmetric tensor field, and A_i^M is the background gauge field. X^i's are the eight bosonic fields, λ^i's are the eight left-handed Majorana fermions, and ψ^s's are 32 right-handed Majorana fermions, belonging to the fundamental (32) representation of SO(32), or the (16,1)+(1,16) representation of the SO(16)×SO(16) subgroup of the $E_8 \times E_8$ group, depending on which particular type of heterotic string is under consideration. ρ^α's are the two dimensional Dirac matrices, T^M's are the generators of the gauge group, and

$$S_{ijk} = \frac{1}{2}\,(B_{ij,k} + B_{jk,i} + B_{ki,j}) \qquad (2)$$

$$(D_\alpha \lambda)^j = \partial_\alpha \lambda^j + \Gamma^j{}_{k\ell}\, \lambda^k \partial_\alpha X^\ell$$

$$\equiv \partial_\alpha \lambda^j + \frac{1}{2} g^{ji} (g_{ik,\ell} + g_{i\ell,k} - g_{k\ell,i}) \lambda^k \partial_\alpha X^\ell \quad (3)$$

$$F^M_{i\ell} = A^M_{i,\ell} - A^M_{\ell,i} + f^{MNP} A^N_i A^P_\ell \quad (4)$$

Although the expression (1) of the effective action was derived in the weak field approximation, we calculated the ultraviolet divergent part of the effective action in the theory described by Eq.(1) without making any approximation. The result is,

$$\left(i \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2+i\epsilon}\right) \Big[\frac{1}{2} \tilde{R}_{ik}{}^k{}_j \,(\partial_\alpha X^i \partial^\alpha X^j - \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j) - \frac{1}{4} \bar{\psi} \rho^\alpha T^M \psi \{ D^k F^M_{k\ell} + f^{MNP} A^{Nk} F^P_{k\ell} - S^{ij}{}_\ell F^M_{ji} \} \partial_\alpha X^\ell \Big] \quad (5)$$

plus its N=1/2 supersymmetric extension. Here D denotes covariant derivative which includes the spin connection, but not the gauge connection. $\tilde{R}$ is the generalized curvature[3] defined as,

$$\tilde{R}_{ijk\ell} = R_{ijk\ell} + D_k S_{ij\ell} - D_\ell S_{ijk} + S_{mik} S^m{}_{\ell j} - S_{mi\ell} S^m{}_{kj} \quad (6)$$

where R is the Riemann tensor.

It was pointed out that the criterion for the vanishing of the one loop divergence is identical to the equations of motion in the weak field approximation[1,4]. In this paper, we propose that this correspondence holds beyond the weak field approximation, and the classical equations of motion of the full fledged string theory are identical to the criteria for the vanishing of the β-function in the model described by (1). We produce evidence in support of this conjecture from the explicit one and two loop results for the vanishing of the β-functions in this model. First, we shall make the following important observations, which will be useful in our analysis.

i) The action (1) transforms as,

$$S \to S/\Lambda \tag{7}$$

under,

$$\begin{aligned} g_{ij} &\to g_{ij}/\Lambda \\ B_{ij} &\to B_{ij}/\Lambda \\ \delta_{st} &\to \delta_{st}/\Lambda \\ (T^M)_{st} &\to (T^M)_{st}/\Lambda \end{aligned} \tag{8}$$

while all other quantities remain unchanged. Thus, in

ℓ-loop order, the effective action must transform as,

$$S_{eff}^{(\ell)} \rightarrow \Lambda^{\ell-1} S_{eff}^{(\ell)} \tag{9}$$

under the transformation (8), provided we write down all factors of δ_{st} explicitly even though it is just the Kronecker δ symbol. $S_{eff}^{(\ell)}$ is said to have conformal weight[5] $\ell-1$.

ii) If we set the background B field to zero, and the spin connection ω constructed from the Christoffel symbol Γ to be equal to the gauge connection A^M, the action (1) reduces to that of an N=1 supersymmetric sigma model[1]. If the background is Ricci flat, such models are known to be finite to all orders in the perturbation theory[6]. This fact may be used to obtain non-trivial constraints on the structure of the ultra-violet divergent terms in our model.

iii) Under a field redefinition,

$$\psi \rightarrow U(x)\,\psi \equiv \exp\,(i\,T^M \theta^M(x))\,\psi \tag{10}$$

the action (1) is mapped into a similar expression with A_i^M replaced by A'^M_i, given by,

$$A'^M_\lambda\, T^M = i\, U(x)\, \partial_\lambda U^{-1}(x) + U(x)\, A_\lambda^M T^M\, U^{-1}(x) \tag{11}$$

Hence the theories described by the gauge potentials A^M and A'^M are identical at the classical level. However, since A couples to chiral fermions, the gauge symmetry given by (11) is anomalous. As a result, we do not expect the β-functions of the theory to be invariant under this symmetry. As we shall see, this indeed happens at the two loop level, and is responsible for the appearance of the Chern-Simons term (which is not gauge invariant by itself) in the equations of motion.

iv) Since we are working in the light cone gauge, we have effectively assumed from the beginning that the x^0 and the x^1 directions are flat, and none of the dynamical fields acquire any VEV in these directions. This causes us to loose some information from the equations of motion. This is best illustrated by an example. Consider, for example, Einstein's equation in the presence of a background electromagnetic field $F_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \right) \tag{12}$$

Since the 0 and 1 directions are flat, and F does not acquire any VEV in these directions, the component of Eq.(12) in the (0,0) or (1,1) direction is,

$$R = -4\pi G \; F_{\rho\sigma} F^{\rho\sigma} \tag{13}$$

Eq. (12) may then be written as,

$$R_{\mu\nu} = (-8\pi G)\ F_{\mu\rho} F_{\nu}{}^{\rho} \tag{14}$$

As we shall see, it is exactly equations of type (14), without the $g_{\mu\nu}$ terms, that we shall obtain by vanishing of the β-function in the model described by Eq. (1).

v) In our analysis, we effectively assume the dilaton field to be constant throughout space, in which case it may be absorbed into the gauge coupling constant[7]. As a result it will never appear explicitly in our analysis.

We may now proceed to analyze the equations for the vanishing of the β-function at the one loop level. They may be derived easily from Eq. (5) and are[1],

$$R_{i\ell} + S_{ikm} S_{\ell}{}^{km} = 0 \tag{15}$$

$$D^{j} S_{ij\ell} = 0 \tag{16}$$

$$D^{k} F^{M}_{k\ell} + f^{MNP} A^{Nk} F^{P}_{k\ell} - S^{ij}{}_{\ell} F^{M}_{ji} = 0 \tag{17}$$

where $R_{i\ell}$ is the Ricci tensor.

After suitable normalization of the various fields, these equations may be shown to be identical to the equations of motion derived from the string theory, (or the limiting field theory)[8] if we ignore all terms with conformal weight larger than zero for Eqs.(15) and (16), and all terms with conformal weight larger than one for Eq.(17). Note that the last term on the left hand side of Eq.(17) appears from the term in the effective action involving the Chern-Simons term for the gauge field, although it appears in the equations of motion in a gauge covariant fashion.

Next, we turn to the two loop calculation. In order to simplify our analysis, we shall only evaluate the contribution to the renormalization of the operators involving the X^i fields. Furthermore, we shall calculate this contribution only in the presence of background gauge fields. As a result, we loose all terms involving B_{ij} and Γ_{ijk}. Terms involving only the gravitationalfields, however, may be recovered up to terms proportional to the Ricci tensor, by demanding that the two loop counterterm must vanish when we set B=0, the spin connection ω=A, and R_{ij}=0. On the other hand, any term involving the Ricci tensor, or its derivatives, may be ignored in writing down the criteria for vanishing of the β-function, if we ignore all terms with conformal weight larger than 1. This may be seen from Eq.(15), which says that if B=0, R_{ij} must be equal to terms with conformal weight>0 at the zero of the β-function. When substituted into the two loop effective action, this gives terms with conformal weight >1. Similarly, any term in the two loop effective action proportional to the left hand side of Eq.(17) may be set to zero in this approximation.

In order to further simplify our analysis, we shall assume that the background gauge field is Abelian in nature. This will cause us to loose all the cubic and quartic terms involving the gauge fields from the effective action. We may now calculate the two loop contribution to the effective action using the background field method[9]. In doing this calculation we always first calculate the fermionic loop integral, We use the exact result[10] for the fermionic determinant in the presence of arbitrary background vector field a_α, coupling to a right handed Weyl fermion. The contribution to the effective action for the a_α field from this determinant is given by,

$$(8\pi)^{-1}\, a_\alpha \left\{ g^{\alpha\beta} - (g^{\alpha\alpha'} + \epsilon^{\alpha\alpha'}) \frac{\partial_{\alpha'}\partial_{\beta'}}{\partial^2} (g^{\beta'\beta} - \epsilon^{\beta'\beta}) \right\} a_\beta \tag{18}$$

[Although the coefficient of the $g^{\alpha\beta}$ term in (18) is ambiguous[10], it may be fixed by demanding that if a_α couples to a right, as well as a left handed Weyl fermion, the total contribution to the effective action should be gauge invariant].

Since (18) does not have any ultraviolet divergences, we always get single poles in ε, and the evaluation of the graphs is straightforward. The total contribution to the effective action at the two loop order is given by,

$$\left[i \int \frac{d^2\ell}{(2\pi)^2 (\ell^2 + i\epsilon)} \right] \frac{1}{16} \left[\{ F^M_{ki}\, F^{M}{}_{\ell}{}^{i}\, \partial_\alpha X^k \partial^\alpha X^\ell \right.$$

$$+\frac{1}{2} D^{i} (A^M_i F^M_{\ell k} + A^M_\ell F^M_{ki} + A^M_k F^M_{i\ell}) \epsilon^{\alpha\beta} \partial_\alpha X^\ell \partial_\beta X^k \}$$

$$- \left\{ \begin{matrix} A \to \omega \\ F \to R \end{matrix} \right\} + O(B) + O(A\omega) + O(A^3) + O(\omega^3) \tag{19}$$

Note the appearance of the Abelian Chern-Simons term in the above expression, which appears due to chiral anomaly in two dimensions. We have chosen the normalization as $tr(T^M T^N) = \delta^{MN}$.

We may now compute the two loop β-function in this model using the method of Friedan[11], and write down the condition for the vanishing of the β-function. Eqs. (15) and (16) are replaced by,

$$R_{i\ell} + S_{ikm} S_\ell{}^{km} + \frac{1}{4} (F^M_{ki} F^{Mk}{}_\ell - R_{imnp} R_\ell{}^{mnp}) = 0 \tag{20}$$

$$D^i [S_{i\ell k} + \frac{1}{8} \{ (A^M_i F^M_{\ell k} + A^M_\ell F^M_{ki} + A^M_k F^M_{i\ell})$$

$$- \left(\begin{matrix} A \to \omega \\ F \to R \end{matrix} \right) \}] = 0 \tag{21}$$

with, of course, the error terms indicated in Eq. (19), as well as terms with conformal weight >1. Appearance of the R^2 term in Eq. (19) was predicted in Ref. 12.

Writing down the explicit factors of κ and g absorbed in S and F respectively, using the relation $g^2 \sim \kappa^2/\alpha'$, and the fact that we have set $\alpha'=1/2$ in our analysis, and properly normalizing F and S, we may show that Eqs. (20) and (21) are identical to the equations of motion for the graviton and the antisymmetric tensor fields, as derived from string theory.

Thus our analysis indicates that there is an exact correspondence between the equations of motion of the string theory, and the condition for the vanishing of the β-function in the model described by Eq.(1). Before concluding, we wish to make the following remarks.

i) In our analysis, we have not considered the equations of motion for the dilaton field. Since there are only three independent dimension two operators in the model described by Eq.(1), that are not related to each other by supersymmetry transformation, we cannot expect to get the equations of motion for the dilaton field by looking at the conformal invariance of the model. There is, however, another non-trivial constraint which must be satisfied by the models of this kind[13], namely, that the correlation functions of the ++ and the -- components of the energy momentum tensor must remain unrenormalized from their free field values. This constraint may reproduce the equations of motion for the dilaton field.

ii) The current approach may also provide a way to tackle loop corrections in string theories. The higher loop amplitudes in string theories correspond to formulating the string theory on world surfaces of non-trivial topology. However, the β-function of the sigma model usually depends on the local properties of the manifold. Hence the constraints imposed on the background fields by demanding the vanishing of the β-function are expected to remain valid even when we include higher loop corrections in the string theory.

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