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Path Integral Measures for Two-Dimensional Fermion Theories*

MARK A. RUBIN
Enrico Fermi Institute
University of Chicago, Chicago, IL 60637
and
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510†

ABSTRACT

An apparent discrepancy is noted between Fujikawa's path integral analysis of anomalies and the existence of a family of distinct solutions to the Thirring model. It is proposed that this family of distinct solutions may be obtained in the path integral formalism by employing a family of distinct measures for the fermion functional integration. The new measures are constructed by means of a two-dimensional analog of the Pauli-Gürsey-Pursey transformation, and the anomalies are evaluated explicitly for those measures which are close to the usual one.

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†Present address.

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I. INTRODUCTION

Invariances of a classical field theory under continuous transformations give rise to conserved currents. In the quantized theory these invariances are expressed by the Ward-Takahashi [1] identities (WTI's). As is well known, it is possible that a current which is conserved in the classical theory is not conserved in the quantized theory on account of anomalies [2].

In a series of papers [3-5] Fujikawa has studied the origin of WTI's in the framework of the path integral formalism, and has shown that both chiral and conformal anomalies have their origin in the non-invariance of the path integral measure under the transformation associated with the classical symmetry. Fujikawa's analysis has been applied to two-dimensional fermion theories: to the Schwinger model, by Roskies and Schaposnik [6]; and to the Thirring model, by Duerksen [7]. It is in the context of these applications that the question to which the present paper addresses itself arises.

The Minkowski-spacetime action for the massless Thirring model [8], including coupling to a classical external gauge field $A_\mu(x)$, is

$$
\begin{aligned}
S &= \int d^2x \left(i\bar{\psi}\partial\!\!\!/\psi + e\, j^\mu A_\mu - \frac{\lambda}{2}\, j^\mu j_\mu \right) , \\
j^\mu &= \bar{\psi}\gamma^\mu\psi \ .
\end{aligned}
\tag{1.1}
$$

To the classical action (1.1) there corresponds a one-parameter family of inequivalent quantum theories [9]. If we call this parameter η, the anomaly equations for the vector current j^μ and the axial current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ may be written as

$$\langle\partial_\mu j^\mu\rangle = -(\eta/\pi)\cdot(e\partial_\mu A^\mu - \lambda\langle\partial_\mu j^\mu\rangle)$$

$$\langle\partial_\mu j_5^\mu\rangle = (\xi/\pi)\cdot(e\varepsilon^{\mu\nu}\partial_\mu A_\nu - \lambda\langle\partial_\mu j_5^\mu\rangle) \tag{1.2}$$

where $\xi = 1 - \eta$.

The existence of solutions possessing different WTI's seems to be at variance with Fujikawa's unambiguous regularization procedure. We propose that the resolution of this apparent paradox lies in a freedom of choosing the fermonic path integral measure. This freedom corresponds to a two-dimensional counterpart of the Pauli-Gürsey-Pursey transformation [10] familiar from four-dimensional field theories.

To begin, we summarize the application of Fujikawa's method to the Thirring model. (The details of the calculation are identical in most aspects to the calculation of the chiral anomaly in four-dimensional Q.E.D. given in ref. [3]. See appendix B and ref. [7].)

Under the chiral transformation

$$\psi \to e^{i\alpha_5\gamma_5}\psi\ , \quad \bar\psi \to \bar\psi e^{i\alpha_5\gamma_5} \tag{1.3}$$

the path integral measure μ,

$$\mu = \prod_x d\psi(x)\, d\bar\psi(x) \tag{1.4}$$

changes in the following manner:

$$\mu \to \mu\, \exp\left[-2i \int d^2x\, \mathrm{Tr}(\alpha_5\gamma_5)\right]\ , \tag{1.5}$$

where "Tr" denotes a sum over a complete set of states. The manner in which this sum is to be regulated in uniquely determined by the equation of motion for ψ, continued to Euclidean spacetime. In the case at hand, the Euclidean equation of motion is

$$\not{D}\psi = 0 \tag{1.6a}$$

where

$$D_\mu = i\partial_\mu + B_\mu \;, \quad B_\mu = eA_\mu + \lambda j_\mu . \tag{1.6b,c}$$

The anomaly factor $\mathrm{Tr}(\alpha_5\gamma_5)$ is evaluated as follows:

$$\begin{aligned} \mathrm{Tr}(\alpha_5\gamma_5) &= \lim_{M\to\infty} \mathrm{Tr}\; \alpha_5\gamma_5 \exp\left(-\frac{\not{D}^2}{M^2}\right) \\ &= \lim_{M\to\infty} \alpha_5 \;\mathrm{tr} \int \frac{d^2k}{(2\pi)^2}\, e^{-ik\cdot x}\gamma_5 \exp\left(-\frac{\not{D}^2}{M^2}\right) e^{ik\cdot x} \end{aligned} \tag{1.7}$$

where "tr" indicates simply a sum over Dirac indices. The result is

$$\mathrm{Tr}(\alpha_5\gamma_5) = \frac{\alpha_5}{2\pi}\, \epsilon_{\mu\nu}\partial_\mu B_\nu \quad . \tag{1.8}$$

The value of an integral is unchanged by a change of integration variable, provided that any change in the integration measure is taken into account by a suitable Jacobian factor--in this case, that Jacobian factor is the exponential on the right-hand side of (1.5). We can write, for the effect of the change of variables (1.3) on the generating functional $Z = \int \mu e^{-S}$,

$$0 = \frac{\delta}{\delta\alpha_5} \ln Z$$

$$= Z^{-1} \int \left(\frac{\delta\mu}{\delta\alpha_5} - \mu \frac{\delta S}{\delta\alpha_5}\right) e^{-S}$$

$$= \left\langle -2i \frac{\delta}{\delta\alpha_5} \mathrm{Tr}(\alpha_5\gamma_5)\right\rangle - \left\langle\frac{\delta S}{\delta\alpha_5}\right\rangle \quad . \tag{1.9}$$

Using (1.1), (1.6c), (1.8) and the relation between j^μ and j^μ_5 (see appendix A) we find, back in Minkowski spacetime,

$$\langle\partial_\mu j^\mu_5\rangle = (e/\pi)\cdot\left[e\varepsilon^{\mu\nu}\partial_\mu A_\nu - \lambda\langle\partial_\mu j^\mu_5\rangle\right] \tag{1.10}$$

The same considerations applied to the gauge transformation

$$\psi \rightarrow e^{i\alpha_1}\psi \quad , \quad \bar{\psi} \rightarrow \bar{\psi}e^{-i\alpha_1} \tag{1.11}$$

easily show that

$$\langle\partial_\mu j^\mu\rangle = 0 \ . \tag{1.12}$$

Thus, we obtain only the WTI's corresponding to $\eta = 0$ in (1.2). One might conjecture that the WTI's with $\eta = 0$ result simply from regularizing the respective Jacobians of the chiral and gauge transformations in a manner different from that which was used in (1.7). Is it, then, necessary to abandon or modify in an arbitrary manner Fujikawa's simple regularization prescription, which has to date been applied with some success in a variety of disparate circumstances? We proposed that solutions to the Thirring model with $\eta \neq 0$ may be obtained by modifying, not the regularization procedure, but the measure μ which is subject to regularization. For $|\eta| \ll 1$ we shall explicitly demonstrate that this is, in fact, the case.

II. TWO-DIMENSIONAL SPINOR FORMALISM

We will find it convenient to work with Weyl spinors which, in two dimensions, are single-component objects. Using the Euclidean conventions described in appendix A, the Weyl spinors λ, ρ, $\bar{\lambda}$, $\bar{\rho}$ are related to the Dirac spinors ψ, $\bar{\psi}$ by

$$\psi = \begin{pmatrix}\lambda \\ \rho\end{pmatrix} \quad , \quad \bar{\psi} = (\bar{\rho},\bar{\lambda}) \ . \tag{2.1}$$

The Euclidean action for the spinors (2.1) interacting with a vector field $B_\mu(x)$ is

$$S = S_L + S_R \ , \tag{2.2a}$$

where

$$S_L = - \int d^2x \ \bar{\lambda} D_L \lambda \quad , \quad S_R = - \int d^2x \ \bar{\rho} D_R \rho \tag{2.2b,c}$$

$$D_\mu = i\partial_\mu + B_\mu \quad , \quad D_{\substack{L \\ R}} = D_1 \pm iD_2 \ . \tag{2.2d,e}$$

(Computation of the gauge and chiral anomalies using the action (2.2) will enable us to obtain the anomalies for the Thirring model as well, if we make $B_\mu(x)$ a Lagrange multiplier [7]. See appendix B and ref. [7].)

Since we shall be considering transformations mixing spinors with anti-spinors, we make still another modification in our notation. Define the "Weyl bispinors" (Majorana spinors, actually) $\underset{\sim}{\lambda}$, $\underset{\sim}{\rho}$:

$$\underset{\sim}{\lambda} = \begin{pmatrix} \lambda \\ \bar{\lambda} \end{pmatrix} \;, \quad \underset{\sim}{\rho} = \begin{pmatrix} \rho \\ \bar{\rho} \end{pmatrix} \;. \tag{2.3}$$

In terms of $\underset{\sim}{\lambda}$, $\underset{\sim}{\rho}$,

$$S_L = - \int d^2x \; \underset{\sim}{\lambda}^T \mathscr{D}_L \underset{\sim}{\lambda} \quad , \quad S_R = - \int d^2x \; \underset{\sim}{\rho}^T \mathscr{D}_R \underset{\sim}{\rho} \tag{2.4a,b}$$

where

$$\mathscr{D}_L = \frac{1}{2} \begin{pmatrix} 0 & \hat{D}_L \\ D_L & 0 \end{pmatrix} \;, \quad \mathscr{D}_R = \frac{1}{2} \begin{pmatrix} 0 & \hat{D}_R \\ D_R & 0 \end{pmatrix} \tag{2.4c,d}$$

$$\hat{D}_\mu = i\partial_\mu - B_\mu \quad , \quad \hat{D}_{\substack{L \\ R}} = \hat{D}_1 \pm i\hat{D}_2 \quad . \tag{2.4e,f}$$

(Note that, for example, $\int d^2x \; \lambda i\partial_L \bar{\lambda} = + \int d^2 \; \bar{\lambda} i \partial_L \lambda$, and $\int d^2x \; \lambda B_L \bar{\lambda} = - \int d^2x \; \bar{\lambda} B_L \lambda$, since all the spinors are anticommuting Grassman objects.) The measure (1.4) may be written as

$$\begin{aligned} \mu &= \mu_L \mu_R \\ \mu_L &= \prod_x d\underset{\sim}{\lambda}(x) \quad , \quad \mu_R = \prod_x d\underset{\sim}{\rho}(x) \;. \end{aligned} \tag{2.5}$$

If ψ, $\bar{\psi}$ are subject to the infinitesimal gauge and chiral transformation

$$\psi \to \left(1+i\alpha_1+i\alpha_5\gamma_5\right)\psi \quad , \quad \bar{\psi} \to \bar{\psi}\left(1-i\alpha_1+i\alpha_5\gamma_5\right) \ , \qquad (2.6a,b)$$

$\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ transform as

$$\underset{\sim}{\lambda} \to (1+ig_L)\underset{\sim}{\lambda} \quad , \quad \underset{\sim}{\rho} \to (1+ig_R)\underset{\sim}{\rho} \ , \qquad (2.6c,d)$$

where

$$g_L = \begin{pmatrix} \alpha_L & 0 \\ 0 & -\alpha_L \end{pmatrix} \quad , \quad g_R = \begin{pmatrix} \alpha_R & 0 \\ 0 & -\alpha_R \end{pmatrix} \qquad (2.6e,f)$$

$$\alpha_L = \alpha_1 + \alpha_5 \quad , \quad \alpha_R = \alpha_1 - \alpha_5 \ . \qquad (2.6g,h)$$

Under a Euclidean Lorentz transformation (x_1-x_2 rotation) through an angle β, $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ transform as

$$\begin{aligned} \underset{\sim}{\lambda} &\to e^{i\beta/2}\, \underset{\sim}{\lambda} \\ \underset{\sim}{\rho} &\to e^{-i\beta/2}\, \underset{\sim}{\rho} \ . \end{aligned} \qquad (2.7)$$

III. MEASURES WITH $\eta \neq 0$

The most general local linear transformation on $\underset{\sim}{\lambda}$, $\underset{\sim}{\rho}$ that commutes with Euclidean Poincare transformations is of the form

$$\underset{\sim}{\lambda}' = H_L\underset{\sim}{\lambda} \quad , \quad \underset{\sim}{\rho}' = H_R\underset{\sim}{\rho} \ , \qquad (3.1)$$

where H_L and H_R are arbitrary 2×2 matrices with spacetime-independent entries. This transformation is analogous to the four-dimensional Pauli-Gürsey-Pursey transformation; we do not, however, impose any constraints corresponding to the constraints imposed on the latter. That the transformation (3.1) are not unitary need not worry us here, since as we change the anomaly parameter η we are moving between theories with inequivalent commutation relations [11].

We now consider theories defined by generating functionals of the form (see appendix B)

$$Z' = \int \mu' \mu_2 \, e^{-(S+S_2)} \quad . \tag{3.2}$$

S is the action (2.2) constructed out of the original spinors $\underset{\sim}{\lambda}$, $\underset{\sim}{\rho}$ as in (2.4a,b); μ' is a modified measure,

$$\mu' = \mu'_L \, \mu'_R$$

$$\mu'_L = \prod_x d\underset{\sim}{\lambda}'(x) \quad , \quad \mu'_R = \prod_x d\underset{\sim}{\rho}'(x) \ , \tag{3.3}$$

given in terms of the modified spinors $\underset{\sim}{\lambda}'$, $\underset{\sim}{\rho}'$ defined in eqs. (3.1); and S_2 is any functional of external fields and/or dynamical fields appearing in the integration measure μ_2, but <u>not</u> including ψ or $\bar{\psi}$.

Under the infinitesimal gauge-plus-chiral transformation whose action on the original spinors $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ is given by (2.6), $\underset{\sim}{\lambda}'$ and $\underset{\sim}{\rho}'$ transform as

$$\underset{\sim}{\lambda}' \rightarrow (1+ig_L')\underset{\sim}{\lambda}' \quad , \quad \underset{\sim}{\rho}' \rightarrow (1+ig_R')\underset{\sim}{\rho}' \tag{3.4a,b}$$

where

$$g_L' = H_L g_L H_L^{-1} \quad , \quad g_R' = H_R g_R H_R^{-1} \ , \tag{3.4c,d}$$

and the change in the measure (3.3) is

$$\mu_L' \rightarrow \mu_L' \exp[-i \int d^2x \ \mathrm{Tr}\ g_L']$$

$$\mu_R' \rightarrow \mu_R' \exp[-i \int d^2x \ \mathrm{Tr}\ g_R'] \quad . \tag{3.5}$$

(We are dealing in this section with two distinct types of transformations, and we pause here briefly to emphasize the difference between the roles that each one plays.

At the outset, we select once and for all a pair of matrices H_L, H_R to use in (3.1). That gives us a pair of transformed spinors $\underset{\sim}{\lambda}'$, $\underset{\sim}{\rho}'$ which we use in the transformed measure μ'. Path integration with this transformed measure yields the quantum theory defined by the generating functional Z' in Eq. (3.2).

Having thus constructed a quantum theory, we then compute the WTI's for this theory in the usual way, by performing the infinitesimal change-of-integration-variables which is expressed in terms of the new spinors $\underset{\sim}{\lambda}'$, $\underset{\sim}{\rho}'$ by Eqs. (3.4).)

Using (2.4) and (3.1) we express the action (2.2) in terms of $\underset{\sim}{\lambda}'$, $\underset{\sim}{\rho}'$:

$$S_L = -\int d^2x\, \underset{\sim}{\lambda}'^T \mathscr{D}_L' \underset{\sim}{\lambda}' \quad , \quad S_R = -\int d^2x\, \underset{\sim}{\rho}'^T \mathscr{D}_R' \underset{\sim}{\rho}' \tag{3.6a,b}$$

$$\mathscr{D}_L' = \left(H_L^{-1}\right)^T \mathscr{D}_L \left(H_L^{-1}\right) \ , \ \mathscr{D}_R' = \left(H_R^{-1}\right)^T \mathscr{D}_R \left(H_R^{-1}\right) \ . \tag{3.6c,d}$$

The relevant anomaly factor for μ_L' is therefore

$$\mathrm{Tr}\ g_L' = \lim_{M\to\infty} \mathrm{tr} \int \frac{d^2k}{(2\pi)^2}\, e^{-ik\cdot x}\, g_L' \exp\left[- \frac{(\mathscr{D}_L')^+ \mathscr{D}_L'}{M^2} \right] e^{ik\cdot x} \ , \tag{3.7}$$

with a parallel expression for μ_R'. (The symbol "tr" indicates a trace over the matrix indices.) The appearance of $(\mathscr{D}_L')^+\mathscr{D}_L'$ rather than $(\mathscr{D}_L')^2$ is dictated by the non-Hermiticity of $\mathscr{D}_L$ and (in general) of $\mathscr{D}_L'$. (See appendix of Ref. 5.)

We evaluate (3.7) for transformations with small off-diagonal entries. Specifically, if we write H_L, H_R as

$$H_L = \begin{pmatrix} \hat{d}_L & -\kappa\hat{\ell}_L \\ \kappa\ell_L & d_L \end{pmatrix}, \quad H_R = \begin{pmatrix} \hat{d}_R & -\kappa\hat{\ell}_R \\ \kappa\ell_R & d_R \end{pmatrix}, \tag{3.8}$$

we shall work to second order in the small real parameter κ. The results (see appendix C) are

$$\mathrm{Tr}\ g_L = \frac{i\alpha_L}{2\pi} \left[\delta_L \partial_\mu B_\mu - (1+\delta_L)\ i\epsilon_{\mu\nu}\partial_\mu B_\nu\right] \tag{3.9a}$$

$$\mathrm{Tr}\ g_R = \frac{i\alpha_R}{2\pi} \left[\delta_R \partial_\mu B_\mu + (1+\delta_R)\ i\ \epsilon_{\mu\nu}\partial_\mu B_\nu\right] \tag{3.9b}$$

where

$$\delta_{L,R} = \frac{\kappa^2}{3}\left(\frac{\ell\hat{\ell}}{d\hat{d}} + \frac{\ell^*\hat{\ell}^*}{d^*\hat{d}^*} - \frac{\ell^*\ell}{\hat{d}^*\hat{d}} - \frac{\hat{\ell}^*\hat{\ell}}{d^*d} \right)_{L,R} . \tag{3.10}$$

Repeating the argument of eq. (1.9), we compute the Minkowskian WTI's

$$\begin{aligned} \langle \partial_\mu j^\mu \rangle &= \frac{\delta_L+\delta_R}{2\pi} \langle \partial_\mu B^\mu \rangle + \frac{\delta_L-\delta_R}{2\pi} \langle \varepsilon^{\mu\nu} \partial_\mu B_\nu \rangle \\ \langle \partial_\mu j_5^\mu \rangle &= \frac{\delta_L-\delta_R}{2\pi} \langle \partial_\mu B^\mu \rangle + \frac{2+\delta_L+\delta_R}{2\pi} \langle \varepsilon^{\mu\nu} \partial_\mu B_\nu \rangle . \end{aligned} \tag{3.11}$$

We see that use of the modified measure will yield a quantum theory which is not invariant under improper Lorentz transformations, unless we restrict ourselves to those measures for which

$$\delta_L = \delta_R . \tag{3.12}$$

Imposing (3.12), and defining

$$\eta = 1 - \xi = -\delta_L = -\delta_R , \tag{3.13}$$

we obtain from (3.11)

$$\begin{aligned} \langle \partial_\mu j^\mu \rangle &= - \frac{\eta}{\pi} \langle \partial_\mu B^\mu \rangle \\ \langle \partial_\mu j_5^\mu \rangle &= \frac{\xi}{\pi} \langle \varepsilon^{\mu\nu} \partial_\mu B_\nu \rangle . \end{aligned} \tag{3.14}$$

Choosing B_μ and S_2 in (3.2) appropriate to the massless Thirring model (see appendix B), we obtain the desired Eqs. (1.2) for the case of

$|\eta| \ll 1$.

The present results provide further confirmation of the correctness of Fujikawa's view of anomalies as the consequence of non-invariance of the path integral measure under a symmetry of the classical action. The family of two-dimensional measures may be of use in string theories, since string theories may be viewed as theories of fields living in two spacetime dimensions [12]. Work on explicit evaluation of the anomalies for general values of η is currently in progress.

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APPENDIX A: NOTATION, CONVENTIONS, AND USEFUL FORMULAS

<u>Minkowski spacetime</u>

metric and alternating tensors;

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \varepsilon^{10} = +1 \qquad \varepsilon^{\mu\nu}\varepsilon_{\nu\tau} = g^{\mu}{}_{\tau}$$

gamma matrices:

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\gamma_5 = \gamma^0\gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^\mu\gamma_5 = \varepsilon^{\mu\nu}\gamma_\nu \qquad \gamma^\mu\gamma^\nu = g^{\mu\nu} - \gamma^5\varepsilon^{\mu\nu}$$

current and pseudocurrent:

$$j^\mu = \bar\psi\gamma^\mu\psi \qquad j_5^\mu = \bar\psi\gamma^\mu\gamma_5\psi$$

$$j_5^\mu = \varepsilon^{\mu\nu}j_\nu$$

generating functional and action for Thirring model:

$$Z = \int \prod_x d\psi(x)\, d\bar{\psi}(x)\, e^{iS}$$

$$S = \int d^2x \left(i\bar{\psi}\partial\!\!\!/\psi + ej^\mu A_\mu - \frac{\lambda}{2} j^\mu j_\mu \right)$$

Euclidean spacetime

metric and alternating tensors:

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \varepsilon_{12} = +1 \qquad \varepsilon_{\mu\nu}\varepsilon_{\nu\tau} = -\delta_{\mu\tau}$$

gamma matrices:

$$\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma^5 = -i\gamma_1\gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_\mu\gamma^5 = -i\varepsilon_{\mu\nu}\gamma_\nu \qquad \gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\gamma^5\varepsilon_{\mu\nu}$$

current and pseudocurrent:

$$j_\mu = \bar{\psi}\gamma_\mu\psi \qquad j_\mu^5 = \bar{\psi}\gamma_\mu\gamma^5\psi$$

$$j_\mu^5 = -i\, \varepsilon_{\mu\nu} j_\nu$$

generating functional and action for Thirring model:

$$Z = \int \prod_{x} d\psi(x)\, d\bar{\psi}(x)\, e^{-S}$$

$$S = - \int d^2x \left(i\bar{\psi}\not{\partial}\psi + e\, j_\mu A_\mu + \frac{\lambda}{2} j_\mu j_\mu \right)$$

Euclideanization procedure

active transformations: (M = Minkowski spacetime

E = Euclidean spacetime)

$$x_M^0 \to -ix_{2_E} \qquad A_{0_M} \to iA_{2_E}$$

$$\bar{\psi}_M \to -i\bar{\psi}_E$$

substitutions:

$$x_M^1 = x_{1_E} \qquad A_{1_M} = A_{1_E}$$

$$\psi_M = \psi_E$$

$$\gamma_M^0 = \gamma_{2_E} \qquad \gamma_M^1 = i\gamma_{1_E}$$

APPENDIX B: INTRODUCTION OF A LAGRANGE MULTIPLIER

From appendix A (in Minkowski spacetime):

$$Z = \int \prod_x d\psi(x)d\bar{\psi}(x) \exp\left[i \int d^2x \left(i\bar{\psi}\not{\partial}\psi + ej^\mu A_\mu - \frac{\lambda}{2} j^\mu j_\mu\right) \right]$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$. We may rewrite this as

$$Z = \int \prod_x d\psi(x)d\bar{\psi}(x)dk^\mu(x) \prod_{x,\mu} \delta\left(k^\mu(x) - \bar{\psi}(x)\gamma^\mu\psi(x)\right)$$

$$\cdot \exp\left[i \int d^2x \left(i\bar{\psi}\not{\partial}\psi + ek^\mu A_\mu - \frac{\lambda}{2} k^\mu k_\mu\right)\right]$$

$$= \int \prod_x d\psi(x)d\bar{\psi}(x)dk^\mu(x) \frac{dh^\mu(x)}{2\pi} \exp\left[i \int d^2x\, h^\mu(k_\mu - \bar{\psi}\gamma_\mu\psi)\right]$$

$$\cdot \exp\left[i \int d^2x \left(i\bar{\psi}\not{\partial}\psi + ek^\mu A_\mu - \frac{\lambda}{2} k^\mu k_\mu \right) \right]$$

$$= \int \prod_x d\psi(x)d\bar{\psi}(x)dk^\mu(x) \frac{dh^\mu(x)}{2\pi} \exp\left[i \int d^2x\, (i\bar{\psi}\not{\partial}\psi - j^\mu h_\mu)\right.$$

$$\cdot \exp\left[i \int d^2x \left(ek^\mu A_\mu + k^\mu k_\mu - \frac{\lambda}{2} k^\mu k_\mu \right) \right].$$

Upon continuation to Euclidean spacetime, this is of the form (3.2).

APPENDIX C: COMPUTATION OF Tr g'_L

From the definitions (2.4), (3.6), we find

$$\mathscr{D}'_L = \frac{1}{2r^2}\begin{pmatrix} -2\Gamma i\partial_L & si\partial_L - rB_L \\ si\partial_L + rB_L & 2\hat{\Gamma} i\partial_L \end{pmatrix}$$

where $\Gamma = \kappa d_L \ell_L$, $\hat{\Gamma} = \kappa \hat{d}_L \hat{\ell}_L$, $r = d_L \hat{d}_L + \kappa^2 \ell_L \hat{\ell}_L$, and $s = d_L \hat{d}_L - \kappa^2 \ell_L \hat{\ell}_L$. To evaluate (3.7), we first perform the rescaling $k_\mu \to Mk_\mu$, so

$$\mathrm{Tr}\; g'_L = \lim_{M\to\infty} \mathrm{tr}\, M^2 \int \frac{d^2k}{(2\pi)^2}\, e^{-iMk\cdot x}\, g'_L \exp\left[- \frac{(\mathscr{D}'_L)^{+}\mathscr{D}'_L}{M^2} \right] e^{iMk\cdot x}$$

Using $\partial_\mu e^{iMk\cdot x} = e^{iMk\cdot x}(\partial_\mu + iMk_\mu)$, we move $e^{iMk\cdot x}$ through to the left. This yields

$$\mathrm{Tr}\; g'_L = \lim_{M\to\infty} \mathrm{tr}\, M^2 \int \frac{d^2k}{(2\pi)^2}\, g'_L \exp\left[- \frac{\Lambda_0 k_\mu k_\mu}{4(r^* r)^2} \right] e^{-Q}$$

where Q is the 2×2 matrix

$$Q = \frac{1}{4(r^*r)^2} \cdot \left(\begin{array}{l|l}
\Lambda_2 k_\mu k_\mu & 2\Omega^* k_\mu k_\mu \\
+ \frac{1}{M}\left[-2i\Lambda k_\mu \partial_\mu\right. & + \frac{1}{M}\left[-4i\Omega^* k_\mu \partial_\mu\right. \\
\left.- (r^* s t_{\mu\nu} + s^* r t^*_{\mu\nu}) B_\mu k_\nu\right] & \left.-2(\hat{\Gamma} r^* t_{\mu\nu} + \Gamma^* r t^*_{\mu\nu}) B_\mu k_\nu\right] \\
+ \frac{1}{M^2}\left[-\Lambda \partial_\mu \partial_\mu\right. & + \frac{1}{M^2}\left[-2\Omega^* \partial_\mu \partial_\mu\right. \\
+ i(r^* s t_{\mu\nu} + s^* r t^*_{\mu\nu}) B_\mu \partial_\nu & +2i(\hat{\Gamma} r^* t_{\mu\nu} + \Gamma^* r t^*_{\mu\nu}) B_\mu \partial_\nu \\
\left.+ i s^* r t_{\mu\nu} \partial_\mu B_\nu + r^* r B_\mu B_\mu\right] & \left.+2i\Gamma^* r t_{\mu\nu} \partial_\mu B_\nu\right] \\
\hline
2\Omega k_\mu k_\mu & \hat{\Lambda}_2 k_\mu k_\mu \\
+ \frac{1}{M}\left[-4i\Omega k_\mu \partial_\mu\right. & + \frac{1}{M}\left[-2i\hat{\Lambda} k_\mu \partial_\mu\right. \\
\left.-2(\Gamma r^* t_{\mu\nu} + \hat{\Gamma}^* r t^*_{\mu\nu}) B_\mu k_\nu\right] & \left.+(r^* s t_{\mu\nu} + s^* r t^*_{\mu\nu}) B_\mu k_\nu\right] \\
+ \frac{1}{M^2}\left[-2\Omega \partial_\mu \partial_\mu\right. & + \frac{1}{M^2}\left[-\hat{\Lambda} \partial_\mu \partial_\mu\right. \\
+2i(\Gamma r^* t_{\mu\nu} + \hat{\Gamma}^* r t^*_{\mu\nu}) B_\mu \partial_\nu & -i(r^* s t_{\mu\nu} + s^* r t^*_{\mu\nu}) B_\mu \partial_\nu \\
\left.+2i\hat{\Gamma}^* r t_{\mu\nu} \partial_\mu B_\nu\right] & \left.-i s^* r t_{\mu\nu} \partial_\mu B_\nu + r^* r B_\mu B_\mu\right]
\end{array}\right)$$

We have defined $t_{\mu\nu} = \delta_{\mu\nu} + i\epsilon_{\mu\nu}$ and, to order κ^2,

$$\Lambda_0 = d_L^* d_L \hat{d}_L^* \hat{d}_L$$

$$\Lambda_2 = \kappa^2 \left(4 d_L^* d_L \ell_L^* \ell_L - d_L^* \hat{d}_L^* \ell_L \hat{\ell}_L - d_L \hat{d}_L \ell_L^* \hat{\ell}_L^* \right)$$

$$\hat{\Lambda}_2 = \kappa^2 \left(4 \hat{d}_L^* \hat{d}_L \hat{\ell}_L^* \hat{\ell}_L - d_L^* \hat{d}_L^* \ell_L \hat{\ell}_L - d_L \hat{d}_L \ell_L^* \hat{\ell}_L^* \right)$$

$$\Lambda = \Lambda_0 + \Lambda_2 \ , \ \hat{\Lambda} = \Lambda_0 + \hat{\Lambda}_2$$

$$\Omega = \hat{\Gamma}^* s - \Gamma s^*$$

Every term in Q contains either a factor of κ or $1/M$. Since we are working to order κ^2, and since any term with more than two powers of $1/M$ will vanish in the limit $M \to \infty$, the expansion of e^{-Q} yields a finite number of terms. Upon performing the expansion, k_μ-integration, and matrix trace, we obtain the result (3.9a). The corresponding computation of Tr g_R' gives (3.9b).