Fermi National Accelerator Laboratory

FERMILAB-Pub-85/38-T
March, 1985

Corner Transfer Matrices and Lorentz Invariance on a Lattice†

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Abstract

The continuum limit of the Baxter eight-vertex lattice model is the Lorentz invariant massive Thirring/sine-Gordon field theory. Here it is shown that the Baxter model exhibits a continuous symmetry which is an exact lattice generalization of Lorentz invariance, and that the corner transfer matrix recently developed by Baxter is a lattice boost operator. The role of elliptic function parameters as lattice rapidity variables is discussed.

† Talk delivered at the International Conference on Solitons and Coherent Structures, Santa Barbara, CA, Jan. 11-16, 1985.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

I. Introduction

The formulation of quantum field theory on a lattice provides a practical framework for both analytical and numerical studies. In Euclidean space-time, lattice field theory becomes a problem in statistical mechanics. The continuum theory must be recovered from the scaling limit of the lattice theory as it approaches a critical point. The space-time symmetries of translation and Euclidean rotation or Lorentz invariance, which are violated by the introduction of the lattice, should be restored in the continuum limit. This symmetry restoration is a fundamental constraint on the scaling limit. It is interesting to study this question in the context of solvable two-dimensional lattice theories, for which a variety of powerful analytic methods have been developed. What I would like to discuss here is the behavior of a solvable two-dimensional lattice theory, the Baxter eight-vertex model,[1] whose continuum limit is the relativistically invariant massive Thirring/sine Gordon field theory.[2] I will show that this model exhibits an exact lattice generalization of Lorentz invariance. By this I mean that there exists a continuous one-parameter group of symmetry operators on the lattice which are an exact lattice analog of Lorentz boost operators. Moreover, these operators have already been constructed and studied by Baxter[3-7] under the name "corner transfer matrices" (CTM's). Although the result that the CTM is a Lorentz boost operator has not been pointed out previously, the boost operator itself, defined as a rapidity shift operator, has appeared recently in the literature on integrable models. It was introduced in the nonlinear Schrodinger model as part of a scheme for calculating thermodynamic traces,[8-9] and was also used in the formulation of the quantum Gel'fand-Levitan equation.[10] Sogo and Wadati[11] have recently discussed the role of boost operators in more general integrable models.

The idea that the 8-vertex model exhibits a lattice generalization of Lorentz invariance is already suggested by results which have been known for some time. A comparison of the Bethe ansatz solution of the massive Thirring model[12] with the treatment of the 8-vertex model by Baxter and by Johnson, Krinsky, and McCoy[13] reveals that the elliptic function parametrization of Baxter[1](specifically Baxter's v parameter) corresponds to the introduction of rapidity variables in the continuum theory.[12] In both the lattice and the continuum theory, these parametrizations are introduced in order to transform the spectral integral equations to equations with a difference kernel. This kernel is the derivative of the scattering phase shift between two Bethe's ansatz modes, and, in the continuum, the fact that it depends only on the difference of rapidities is a straightforward consequence of the Lorentz invariance of the two-body scattering amplitude. The remarkable thing is that, in the lattice theory, the elliptic function parametrization achieves the same result, i.e. the kernel depends only on the difference of rapidity parameters, suggesting that a continuous symmetry group remains operative on the lattice. Of course, on a square Euclidean lattice one would normally expect only the discrete subgroup of reflections and 90° rotations to survive. The continuous symmetry we find on the Baxter lattice can be understood by introducing a geometric interpretation of vertex weights.[14] We note that in the Baxter model, the v or u parameter which appears in the vertex weights determines the relative strength of the horizontal and vertical two-spin couplings. In the scaling limit, an asymmetric choice of couplings will give rise to a difference in relative distance scales in the space and time directions. To recover a symmetric space-time continuum, we should regard the model with asymmetric couplings as being defined on a distorted lattice whose unit cell is a rectangle or rhombus rather than a square. The v parameter defines the angle of the rhombus, and a Euclidean boost can in this way be associated with a geometrical distortion of the lattice. This makes it easier to understand how a continuous space-time symmetry can survive on the lattice.

I will begin in the next section with a brief digression on the propagation of a free scalar field on a lattice. This example illustrates in a particularly simple way the connection between lattice kinematics, hyperbolic trigonometry, and elliptic functions. After this introduction, I will return to the Baxter model and discuss the corner transfer matrix and its interpretation as a boost operator.

II. Lattice Kinematics, Hyperbolic Geometry, and Elliptic Functions

The Euclidean propagator for a free scalar particle on a lattice is proportional to $(1-K)^{-1}$ where K is the hopping matrix describing the probability for the particle to hop to neighboring sites. For nearest neighbor hopping on a two-dimensional rectangular lattice we can write the hopping matrix in momentum space as

$$K = \kappa\left\{\frac{1}{a_x^2}\left(e^{ipa_x} + e^{-ipa_x}\right) + \frac{1}{a_t^2}\left(e^{i\omega a_t} + e^{-i\omega a_t}\right)\right\} \tag{2.1}$$

where κ is a constant, and a_x and a_t are the lattice spacings in the x and t directions. The continuum propagator is recovered by scaling a_x and a_t to zero while setting

$$\frac{1}{\kappa} = 2\left(\frac{1}{a_x^2} + \frac{1}{a_t^2}\right) + m^2 \tag{2.2}$$

The propagator then reduces (up to an overall constant) to the relativistic scalar Euclidean propagator $(P^2+m^2)^{-1}$ where $P^2=P_\mu P_\mu = \omega^2+p^2$ and $P_\mu = (i\omega,p)$ is a Lorentz two-vector. In Minkowski space, the relation between the energy $E=i\omega$ and the momentum p of an on-shell particle is given by the location of the propagator pole. For the continuum case, this gives

$$E^2 = p^2 + m^2 \tag{2.3}$$

The particle rapidity α can be used to parametrize the solutions of (2.3) as

$$E = m\cosh\alpha \tag{2.4a}$$

$$p = m\sinh\alpha \tag{2.4b}$$

When E and p are given by (2.4) they satisfy (2.3) for any value of α, which is simply the statement that (2.3) is a Lorentz invariant equation.

Now let us return to the lattice propagator, where the mass shell condition K=1 leads to the energy-momentum relation

$$\cosh Ea_t = \frac{a_t^2}{2\kappa} - \frac{a_t^2}{a_x^2} \cos pa_x \tag{2.5}$$

We want to parametrize this lattice dispersion relation in a manner analogous to the continuum rapidity parametrization (2.4). To motivate the correct procedure, let us define two constants K_1 and K_2^* by

$$\cosh 2K_1 \; \cosh 2K_2^* = \frac{a_t^2}{2\kappa} \tag{2.6a}$$

$$\sinh 2K_1 \; \sinh 2K_2^* = \frac{a_t^2}{a_x^2} \tag{2.6b}$$

Then (2.5) can be interpreted as an equation in hyperbolic trigonometry,

$$\cosh \varepsilon = \cosh 2K_1 \; \cosh 2K_2^* - \sinh 2K_1 \; \sinh 2K_2^* \cos q \tag{2.7}$$

where $\varepsilon = Ea_t$ and $q = pa_x$. Eq. (2.7) is the "law of cosines" for a hyperbolic triangle with two fixed sides $2K_1$ and $2K_2^*$. The energy ε and momentum q are the third side and opposite angle respectively, as shown in Fig. 1. It is a well known result of hyperbolic (or spherical) geometry that this relation can be parametrized in terms of Jacobi elliptic functions. In fact, Eq. (2.7) is identical to the relation which occurs in the two-dimensional Ising model where K_1 and K_2 are the spin-spin couplings and K_2^* is the dual of K_2 defined by

$$\sinh 2K_2^* = \frac{1}{\sinh 2K_2} \quad . \tag{2.8}$$

The elliptic function parametrization is just Onsager's "uniformization substitution,"[15] and the relevant formulas can be obtained from the Appendix of Ref. 15. Define the elliptic modulus

$$k = \frac{\sinh 2K_1}{\sinh 2K_2^*} \tag{2.9}$$

and the parameter a by

$$\sinh 2K_1^* = -i \text{ sn } ia \tag{2.10}$$

Then the values of ε and q satisfying (2.7) are given by

$$\sinh \varepsilon = -i\, k'^2 \text{ sn}(ia)/M \tag{2.11a}$$

$$\sin q = k'^2 \text{sn}(\alpha)/M \tag{2.11b}$$

where $k' = (1-k^2)^{\frac{1}{2}}$, and

$$M = \text{dn}(ia)\, \text{dn}(\alpha) + k\, \text{cn}(ia)\, \text{cn}(\alpha) \tag{2.12}$$

(We have replaced Onsager's parameter u by $2K-\alpha$ where K is the complete elliptic integral of modulus k.) With the substitutions (2.9)-(2.12), Eq. (2.7) is satisfied identically. The continuum limit of these expressions is obtained by letting the elliptic modulus $k \to 1$, whereupon

$$\begin{aligned} \text{cn, dn} &\to \text{sech} \\ \text{sn} &\to \tanh \end{aligned} \tag{2.13}$$

Then (2.11) reduces to

$$\sinh \varepsilon \sim k'^2 \sin a \cosh \alpha \tag{2.14a}$$

$$\sin q \sim k'^2 \cos a \sinh \alpha \tag{2.14b}$$

By letting

$$m = k'^2/\sqrt{a_x^2+a_t^2} \tag{2.15}$$

and choosing Onsager's parameter a to be given in terms of the lattice spacings a_x and a_t by

$$a_t/a_x = \tan a \quad , \tag{2.16}$$

we see that, in the continuum limit $k' \to 0$ with m finite, (2.11) reduces to the rapidity parametrization (2.4).

Thus, for the simple case of a free scalar field the hyperbolic triangle relation (2.7) is the lattice analog of the relativistic dispersion relation (2.3), and the elliptic function parameter α defined in (2.11)-(2.12) is the analog of a rapidity variable. Note that the double periodicity of the elliptic functions embodies both the periodicity under Euclidean rotations (imaginary rapidity shifts) as well as the Brillouin zone periodicity under real rapidity shifts. The latter period goes to infinity in the continuum limit. Similar formulas will arise in the Baxter model, where we will explicitly construct the lattice boost operator.

III. Lorentz Invariance in the Baxter Model

A. Yang-Baxter Equations

The Baxter eight-vertex model may be defined in terms of Ising-like spin variables $\sigma_i = \pm 1$ on a two-dimensional square lattice. The Boltzmann factor associated with a particular configuration is given by a product of "vertex weights" for each elementary square face of the lattice. If the four spins around a face are a, b, c, and d, as in Fig. 2, then the weight factor for this face is

$$\exp\{K_1 ac + K_2 bd + K'' abcd\} \quad , \tag{3.1}$$

i.e. there are diagonal two-spin couplings K_1 and K_2 and a four-spin coupling K''. It is easy to see that if $K''=0$, the lattice decouples into a staggered pair of Ising models. (Note that the horizontal and vertical axes of the Ising sublattices are turned at 45° relative to the Baxter lattice.) Another very useful formulation of the model is the "arrow" representation. Arrows are defined on the links of the dual lattice which separate pairs of adjacent spins, with the value of the arrow given by the product of the two spins.

Baxter's original solution of the model involved properties of the row-to-row transfer matrix T. He considered the commutator $[T,T']$ of two transfer matrices with different sets of vertex weight parameters. The commutativity condition $[T,T']=0$ led Baxter to a set of equations for the vertex parameters which have become known as the Yang-Baxter equations. Let us adopt the arrow representation and regard a local vertex L as an explicit 2×2 matrix in the horizontal arrow indices with each element being an operator which acts on a vertical arrow. Let L_n denote an L-matrix acting on site n. Thus, for example, the transfer matrix for a row of N sites is

$$T = \mathrm{Tr}\{L_1 L_2 \ldots L_N\} \tag{3.2}$$

where the matrix products and the trace are taken over horizontal indices. The Yang-Baxter equations are then

$$[L_n \otimes L_n']R = R[L_n' \otimes L_n] \tag{3.3}$$

where R is a 4×4 matrix of c-numbers. These relations are represented graphically in Fig. 3.

Investigating the solutions of Eq.(3.3) is one natural way of introducing elliptic functions into the Baxter model. Later I'll discuss Baxter's solution to an eight-vertex model on an arbitrary planar lattice and show that the Yang-Baxter route to the elliptic function parametrization is in fact closely related to the hyperbolic geometry approach discussed in the last section. For now, I will simply state the parametrization for later reference. The three couplings K_1, K_2, and K'' in Eq. (3.1) are replaced by an elliptic modulus k and two other parameters, u and λ, as follows:

$$e^{-2K_1} = k^{\frac{1}{2}}\mathrm{snh}\ u \tag{3.4a}$$

$$e^{-2K_2} = k^{\frac{1}{2}}\mathrm{snh}(\lambda-u) \tag{3.4b}$$

$$e^{-2K''} = k^{\frac{1}{2}}\mathrm{snh}\ \lambda \tag{3.4c}$$

where we have defined a hyperbolic elliptic function $\mathrm{snh}\ x = -i\ \mathrm{sn}\ ix$. (Unless otherwise indicated, elliptic functions will always be of modulus k.) For physical values of the spin couplings (i.e. real, positive Boltzmann weights) and assuming $0<k<1$, the parameters u and λ are real and satisfy the restrictions $0<u<\lambda<\tau$, where $\tau = \pi K/K'$. It is useful to regard the modulus k and the parameter λ as fixed constants and u as a variable. Note that varying u changes the relative size of

K_1 and K_2 but leaves K" unchanged. With the parametrization (3.4), two transfer matrices commute for arbitrary values of u if they have the same values for k and λ. In the Yang-Baxter equations for the eight-vertex model, the R-matrix has exactly the same form as the L-matrix. The u parameter which appears in the R-matrix is determined by the difference of the two u parameters which appear in the L-matrices. Eq. (3.3) may be regarded as a fundamental statement of the integrability of the Baxter model. With the recent development of the quantum inverse method, Yang-Baxter relations have been found to be a generic property of integrable quantum systems.

B. Corner Transfer Matrices

Over the past several years, Baxter has developed a powerful new method for treating certain solvable lattice models based on the properties of an object called a corner transfer matrix. The method appears to be in some ways even more powerful than the row-to-row transfer matrix approach. For example, the CTM method provided a solution to the previously unsolved hard-hexagon model. It also produced the first calculation of the spontaneous magnetization for the eight-vertex model. Once certain elegant properties of the eigenvalue spectrum of the CTM are exposed, the calculation is remarkably simple. Even for the previously solved Ising case, it is one of the most easily understandable calculations of the spontaneous magnetization.

To introduce the corner transfer matrix, imagine calculating the partition function by choosing a spin in the middle of the lattice as the origin and fixing the spins along the vertical and horizontal axes while summing over all spins in the interior of each quadrant, as shown in Fig. 4. The final sum over the spins along the axes may be interpreted as the trace of a product of four matrices. Thus,

$$Z = \mathrm{Tr}\{ABCD\} \tag{3.5}$$

where A, B, C, and D are corner transfer matrices. Each of these represents one quadrant of the lattice with rows and columns of the CTM labeled by the configuration of spins along two edges, as shown in Fig. 5. For a finite size lattice, we must choose some boundary conditions (e.g. all spins up) along the outer edges of the lattice. However, it is a remarkable fact that the eigenvalue spectrum of the CTM remains discrete even in the limit of infinite volume. This is in contrast to the spectrum of the row-to-row transfer matrix, which develops continuous bands of eigenvalues in the thermodynamic limit. Because of this, the thermodynamic limit of the CTM is considerably less delicate than that of the row-to-row transfer matrix, and may be taken at an early stage of the investigation. For real positive values of u, we normalize the corner transfer matrix by dividing by its largest eigenvalue. This normalized CTM $\hat{A}(u)$ is then a well-defined (infinite dimensional) matrix in the infinite volume limit. A(u) may be diagonalized by a u-independent similarity transformation, and its diagonal form is given by the surprisingly simple expression

$$\hat{A}_d(u) = \begin{pmatrix} 1 & \\ & e^{-\alpha} \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & e^{-2\alpha} \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & e^{-3\alpha} \end{pmatrix} \otimes \cdots \tag{3.6}$$

where $\alpha = \pi u/2K$. Thus the eigenvalues are all of the form $e^{-n\alpha}$ where n is an integer. Note that the eigenvalues depend <u>only on α</u> and are independent of both k and λ for fixed α. In order to identify the CTM with a boost operator, we will need to define an "extended" CTM consisting of a direct product of upper-left and lower-right quadrants, as shown in Fig. 6. The upper left CTM is evaluated at -u, so the spectrum of the extended CTM corresponding to (3.6) contains both positive and negative exponents. The extended CTM carries a full row of spins into a full column of spins, and thus can be defined to act on the same Hilbert space as the row-to-row transfer matrix.

C. Operator Form of the CTM

By studying the corner transfer matrix in a low temperature expansion,[3] Baxter was led to conjecture an exact operator form for the CTM in the thermodynamic limit. To understand this result, let us first recall the connection between the row-to-row transfer matrix and the XYZ spin chain Hamiltonian. We consider the transfer matrix near the "shift point" u=0. At u=0 the elementary vertex becomes a pair of Kronecker deltas which equate arrows through the vertex ($\delta_{\alpha\beta}\delta_{\gamma\delta}$ in Fig. 2). Thus, the row-to-row transfer matrix may be expanded around u=0 with the result depicted in Fig. 7. Note that the second term is given by the sum of nearest-neighbor two-spin operators obtained from the first u derivative of the vertex at u=0. This is just the XYZ spin chain Hamiltonian density,

$$H_{XYZ}(n,n+1) = -\frac{1}{2}\left(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z\right) + \text{const.} \qquad (3.7)$$

where the σ's are Pauli spin matrices, and the coefficients are given in terms of elliptic parameters by

$$J_x = \frac{cn(2\zeta,\ell)}{sn(2\zeta,\ell)} \tag{3.8a}$$

$$J_y = \frac{dn(2\zeta,\ell)}{sn(2\zeta,\ell)} \tag{3.8b}$$

$$J_z = \frac{1}{sn(2\zeta,\ell)} \tag{3.8c}$$

Here ℓ is related to the previously defined k by a Landen transformation, $k = (1-\ell)/(1+\ell)$, and $\zeta = (K_\ell'/\pi)\lambda$. We see that the XYZ Hamiltonian is essentially the logarithmic derivative of the transfer matrix,

$$\ln\{T(u)T^{-1}(0)\} \sim 1 - u \sum_n H_{XYZ}(n,n+1) + \ldots \tag{3.9}$$

Now let us consider a similar expansion for the corner transfer matrix. At this point it is useful to change the way we identify spins or arrows from row to row. In the row-to-row transfer matrix formalism, it was normal to identify the spins at a given position in the row, so that the n th spin in each successive row represents the time development of a single spin. To study the CTM, it is useful to identify the n th spin of a given row with the (n+1)th spin of the following row. In this formulation, the elementary vertex is a two-spin operator which acts on nearest-neighbor pairs,

$$V_n = \tfrac{1}{2}\{(a+c) + (a-c)\sigma_n^z\sigma_{n+1}^z + (b+d)\sigma_n^x\sigma_{n+1}^x + (b-c)\sigma_n^y\sigma_{n+1}^y\} \tag{3.10}$$

where a, b, c, and d are the vertex weights,

$$a = \exp(K_1+K_2+K'') \tag{3.11a}$$

$$b = \exp(-K_1-K_2+K'') \tag{3.11b}$$

$$c = \exp(-K_1+K_2-K'') \tag{3.11c}$$

$$d = \exp(K_1-K_2-K'') \tag{3.11d}$$

Note that V_n represents the same vertex previously denoted by L_n, but now regarded as a two-spin operator instead of a 2×2 matrix of one-spin operators. Now at the point u=0, the vertex becomes proportional to the identity operator, and the expansion of the CTM around u=0, shown in Fig. 8, is of the form

$$A(u) \sim 1 - uK + \ldots \tag{3.12}$$

where K is the first moment of the Hamiltonian density,

$$K = \sum_{n=-\infty}^{\infty} n\, H_{XYZ}(n,n+1) \quad . \tag{3.13}$$

We have taken the sum in (3.13) from $-\infty$ to ∞, corresponding to an extended CTM. Recall that the Lorentz boost generator in relativistic field theory is given by

$$K = \int dx\, [xH(x) - tP(x)] \tag{3.14}$$

where P(x) and H(x) are the momentum density and Hamiltonian density. Eq. (3.13) is clearly the lattice analog of (3.14) at t=0.

D. Group Property of CTM's

Recall that, in the case of the row-to-row transfer matrix, the Yang-Baxter relations led to the result that two transfer matrices with the same k and λ but different values of u commute with each other,

$$[T(u),T(u')] = 0 \tag{3.15}$$

This result implies that the eigenvectors of T(u) are independent of u. Baxter derived some important algebraic properties of the corner transfer matrix by considering a generalization of this argument to the case of an inhomogeneous transfer matrix.[6] Consider a transfer matrix T(u,v) with different vertex weights to the left and right of the origin,

$$T(u,v) = \mathrm{Tr}\left[L_{-N+1}(u)\ldots L_{-2}(u)L_{-1}(u)L_0(v)L_1(v)\ldots L_N(v)\right] \tag{3.16}$$

Using the same argument that lead to the result that eigenvectors of $T(u)$ are independent of u, we find that the eigenstates of $T(u,v)$ depend only on $u-v$. Now consider a more-or-less arbitrarily chosen state $|\omega\rangle$ along the lower boundary of the lattice and apply the transfer matrix $T(u,v)$ many times, giving the configuration depicted in Fig. 9. Assuming that there is a mass gap between the ground state and the excitation spectrum, and supposing that $|\omega\rangle$ has some overlap with the ground state, we may write

$$\Lambda_0^{-N}[T(u,v)]^N|\omega\rangle \quad \rightarrow \quad \text{const.}\times|\Omega\rangle \tag{3.17}$$

where $|\Omega\rangle$ is the ground state, and Λ_0 is the ground state (largest) eigenvalue of T. Thus, the configuration of spins along the top row in Fig. 9 is proportional to the ground state $|\Omega\rangle$. This is an eigenstate of $T(u,v)$ and hence depends only on $u-v$. But note that, if boundary conditions can be ignored, Fig. 9 may also be interpreted as the product of two corner transfer matrices. We conclude that the product of a lower-left and a lower-right CTM depends only on the difference of their arguments,

$$A(v)B(u) = X(v-u) \ , \tag{3.18}$$

where X is an operator to be determined. Next we note that the change of variables $u \rightarrow \lambda-u$ converts a lower-left CTM into a lower-right CTM, and hence,

$$B(u) = A(\lambda-u) \ . \tag{3.19}$$

Thus,

$$A(v)A(u) = X(v+u-\lambda) \ , \tag{3.20}$$

i.e. the product of two CTM's depends only on the sum of the arguments.

This equation is very restrictive.[6] If we define the normalized CTM $\hat{A}(u)$ by dividing by its largest eigenvalue a_0, $\hat{A}(u) = A(u)/a_0(u)$, then the property (3.20) implies that

$$\hat{A}(u)\hat{A}(v) = \hat{A}(v)\hat{A}(u) = \hat{A}(u+v) \tag{3.21}$$

Combining this with the small u expansion (3.12), this implies that

$$\hat{A}(u) = \exp\{-uK\} \tag{3.22}$$

where K is the lattice boost generator (3.13) (with possibly a different constant term in H_{XYZ}). Thus, up to some normalization factor, the CTM A(u) induces a Euclidean rotation of angle u, or equivalently, the analytically continued CTM $A(i\alpha)$ for real α is the Lorentz boost operator in Minkowski space.

E. Rapidity Shift Property of the CTM

It is interesting to derive the transformation property of the row-to-row transfer matrix under the action of the boost operator. This was studied by Sogo and Wadati[11] who obtained the result by direct calculation. I will give a derivation here which obtains the relevant commutation relations by a simple application of the Yang-Baxter equations. Consider the YBE shown in Fig. 10, where two of the vertices $L(v)$ and $L(v+\varepsilon)$ differ in rapidity by an infinitesimal amount ε. This implies that the third vertex must have rapidity ε. Note that this application of the YBE is somewhat different than that which arises in the quantum inverse formalism. The vertices are being commuted horizontally rather than vertically, and the "R-matrix" is not a matrix but a two-spin operator. By expanding in powers of ε,

$$L_n(v+\varepsilon) \sim L_n(v) + \varepsilon\, L_n'(v) + O(\varepsilon^2) \tag{3.23a}$$

$$V_n(\varepsilon) \sim 1 + \varepsilon\, H_{XYZ}(n,n+1) + O(\varepsilon^2) \tag{3.23b}$$

we find the following commutation relation,

$$\left[H_{XYZ}(n,n+1), L_n(v)L_{n+1}(v)\right] = L_n'(v)L_{n+1}(v) - L_n(v)L_{n+1}'(v) \tag{3.24}$$

Using (3.2) and (3.13) we obtain the commutator of the boost generator with the row-to-row transfer matrix,

$$[K,T(v)] = \frac{\partial}{\partial v}T(v) \tag{3.25}$$

From this it follows that the corner transfer matrix is a shift operator

$$A^{-1}(u)\ T(v)\ A(u) = T(u+v) \tag{3.26}$$

The relation between commuting transfer matrices with different values of v is clarified by this result. They are transfer matrices for the same lattice field theory evaluated in different Lorentz frames.

F. Lattice Poincaré Algebra

In continuum two-dimensional field theory, the generators of the Poincaré group consist of P, H, and K, the generators of space and time translations and Lorentz boosts respectively. Their algebra closes:

$$[H,P] = 0 \tag{3.27a}$$

$$[K,P] = iH \tag{3.27b}$$

$$[K,H] = iP \tag{3.27c}$$

The lattice analog of the Poincaré algebra for the Baxter model involves the entire infinite set of conserved operators which are generated by expanding the row-to-row transfer matrix in powers of u,[16]

$$\ln T(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} C_n \tag{3.28}$$

or

$$C_n = (d^n/du^n) \log T(u) \big|_{u=0} \tag{3.29}$$

The first two of these are $C_0=P$ and $C_1=H$. The conserved operators all commute with each other,

$$[C_n, C_m] = 0 \tag{3.30a}$$

while, from the shift property (3.25) it follows that the boost generator K acts as a ladder operator on the infinite sequence of conserved operators,

$$[K, C_n] = iC_{n+1} \qquad n=0,1,2... \tag{3.30b}$$

The infinite dimensional algebra (3.30) is the lattice analog of the Poincaré algebra (3.27). Notice that the operators C_n contain terms of different orders in the lattice spacing. The leading term is always proportional to P if n is even or to H if n is odd. This is how (3.30) reduces to the Poincaré algebra (3.27) in the continuum limit.

G. Baxter model on an Arbitrary Planar Lattice and Geometrical Interpretation of the Vertex Weights

Several years ago Baxter[14] solved an interesting generalization of the eight-vertex model on a lattice consisting of arbitrary straight lines in the two-dimensional plane, with no three lines intersecting at the same point. An Ising spin is placed on each face of the lattice, or alternatively an arrow on each line segment connecting two vertices. The vertex weight for site j is given in terms of the spins around the vertex as before, but now we allow the weight parameters to vary from site to site. For site j we have a factor

$$\exp\{K_j ac + K_j' bd + K_j'' abcd\} \quad . \tag{3.31}$$

Baxter showed that the model could be solved under the following conditions:

(1) The 4-spin coupling $K''_j = K''$ is site-independent.

(2) A certain combination of the two-spin couplings K_j and K'_j,

$$\Delta = -\sinh 2K_j \sinh 2K'_j - \tanh 2K'' \cosh 2K_j \cosh 2K'_j \tag{3.32}$$

is site independent.

(3) The remaining freedom in K_j and K'_j is determined by an elliptic function parameter which is proportional to the angle between the two lines which form the vertex.

Thus, for a lattice composed of n lines, the model contains, in addition to K'' and Δ, (n-1) additional angle or rapidity parameters which determine the orientation of each line in the plane (up to some overall rotation of the lattice). Baxter arrived at these restrictions by examining the general conditions under which three vertices could satisfy the Yang-Baxter equations. Note that the definition of Δ, Eq. (3.32) may be rewritten as a hyperbolic triangle equation,

$$\cosh\omega = \cosh 2K_j \cosh 2K'_j - \sinh 2K_j \sinh 2K'_j \cosh\Omega \tag{3.33}$$

where

$$\cosh\Omega = \coth 2K'' \tag{3.34}$$

$$\cosh\omega = -\Delta \coth 2K'' \tag{3.35}$$

Eq. (3.33) describes a hyperbolic triangle with fixed side ω and fixed opposite angle $i\Omega$ and variable sides $2K_j$ and $2K'_j$. The solutions can be parametrized by elliptic functions, as before. We introduce elliptic parameters α_j, β_j, and λ,

$$\sinh 2K_j = (k \,\mathrm{snh}\, \alpha_j)^{-1} \qquad (3.36a)$$

$$\sinh 2K'_j = (k \,\mathrm{snh}\, \beta_j)^{-1} \qquad (3.36b)$$

$$\sinh 2K'' = (-k \,\mathrm{snh}\, \lambda)^{-1} \qquad (3.36c)$$

The hyperbolic triangle relation (3.33) then requires that

$$\alpha_j + \beta_j = \lambda \qquad (3.37)$$

The Yang-Baxter equations relating three vertices j=1, 2, and 3, lead to the simple requirement

$$\beta_1 + \beta_2 + \beta_3 = \lambda \qquad (3.38)$$

These conditions may be interpreted geometrically if we associate with each elliptic parameter α_j, β_j, or λ, an angle θ_j given by

$$\theta_j = (\pi/\lambda)\times\text{elliptic parameter} \qquad (3.39)$$

The requirements for solvability are then satisfied if the two-spin couplings at each vertex are chosen so that $(\pi/\lambda)\alpha_j$ and $(\pi/\lambda)\beta_j$ are the obtuse and acute angles at that vertex. The hyperbolic triangle requirement (3.37) simply states that the sum of these two angles is equal to π, while the Yang-Baxter relation (3.38) reduces to the statement that the sum of the interior angles of a planar triangle is equal to π.

The interpretation of elliptic parameters as angles in the plane is the same one we arrived at by looking at the free scalar propagator in Section II, specifically Eq. (2.16). In particular, a homogeneous eight-vertex model (i.e. site-independent vertex weights) may be thought of as being defined on a rhombic lattice as shown in Fig. 11. From (3.39) the angle of the rhombus θ is determined by the weight parameter u in Eq. (3.4). Note that since the two-spin couplings are across the diagonals of an elementary rhombus, this is equivalent to a rectangular Ising sublattice with $a_t/a_x = \tan(\theta/2)$.

IV. Discussion

The existence of an exact lattice analog of Lorentz symmetry in the eight-vertex model is quite remarkable. But what is perhaps more intriguing is the fact that the transformation operators of this symmetry, the corner transfer matrices, have proven to be such a powerful calculational tool. For example, the spontaneous magnetization of the model follows almost immediately from the diagonal representation of the CTM, Eq. (3.6), along with the observation that, in this representation, the central spin on the lattice is also diagonal, and given by

$$\sigma_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \cdots \tag{4.1}$$

From this one easily obtains an infinite product expression for $\langle\sigma_0\rangle$,

$$\langle\sigma_0\rangle = \frac{\mathrm{Tr}\{\sigma_0 ABCD\}}{\mathrm{Tr}\{ABCD\}} = \frac{1-x^2}{1+x^2}\,\frac{1-x^6}{1+x^6}\,\frac{1-x^{10}}{1+x^{10}}\cdots \tag{4.2}$$

where $x = \exp(-\lambda)$. One of the most important unsolved problems in this model, as well as in most other integrable two-dimensional models, is that of calculating correlation functions. In the CTM formalism, the spin-spin correlation function is given by

$$\langle\sigma_0\sigma_i\rangle = \frac{\mathrm{Tr}\{\sigma_0\sigma_i ABCD\}}{\mathrm{Tr}\{ABCD\}} \tag{4.3}$$

Although the central spin σ_0 has a simple diagonal form in the representation in which the CTM's are diagonal, the spin σ_i does not. In order to evaluate (4.3) we need to know the eigenstates as well as the eigenvalues of the CTM. The identification of the CTM as a Lorentz transformation suggests that these eigenstates might be constructed from the known eigenstates of the Hamiltonian H_{XYZ} by Fourier transforming over the rapidity variables which label the latter states. The discreteness of the CTM eigenvalue spectrum would then follow from the Brillouin zone periodicity in rapidity space. The use of the CTM formalism may provide a fresh approach to the longstanding problem of calculating correlation functions.

Some of the results we have obtained here in the eight-vertex model may have more general implications. It is easy to define a corner transfer matrix for higher dimensional lattice theories, but it is not yet clear whether it is a useful concept. The interpretation of the CTM as a Euclidean boost operator leads to a particular way of defining an extended CTM, e.g. in n dimensions it would be an object swept out by the rotation of an (n-1)-dimensional hyperplane. However, it is far from obvious that this operator will have the elegant properties which it exhibits in the eight-vertex model. For the eight-vertex model, it was found that the values of lattice couplings or hopping constants have a direct geometrical significance in terms of how the lattice is embedded in physical space-time. This geometrical interpretation is exposed by the parametrization of the couplings in terms of elliptic functions. The discussion of free field theory in Section II suggests that the elliptic function parametrization and its geometrical interpretation may be a more general feature of lattice kinematics. On the other hand, the arguments appear to be closely tied to the integrability of the system. It remains an open question whether any of these ideas will be useful in more realistic theories, for example, in studies of lattice gauge theory with asymmetric couplings.

Figure Captions

1. The hyperbolic triangle associated with Eq. (2.7).
2. An elementary vertex of the eight-vertex model.
3. Graphical representation of the Yang-Baxter equations.
4. CTM calculation of the partition function. Circled spins are summed over last.
5. The corner transfer matrix $A_{\sigma\sigma'}$.
6. An extended corner transfer matrix.
7. Expansion of the row-to-row transfer matrix around u=0.
8. Expansion of the corner transfer matrix around u=0.
9. Configuration obtained by multiplying many inhomogeneous row-to-row transfer matrices or two corner transfer matrices, giving Eq. (3.18).
10. Yang-Baxter equations leading to the commutation relation (3.24).
11. Geometric interpretation of the vertex weight parameter u. θ is given in terms of u by Eq. (3.39).

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