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GRAVITON DOMINANCE IN QUANTUM KALUZA-KLEIN THEORY*

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Carlos R. Ordóñez

Center for Theoretical Physics
Department of Physics
The University of Texas
Austin, Texas 78712, USA

and

Mark A. Rubin

Theory Group
Department of Physics
The University of Texas
Austin, Texas 78712, USA

and

Theoretical Physics Department
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, Illinois 60510, USA†

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†present address.

ABSTRACT

We compute, at the one-loop level, the effective potential for pure gravity in a Kaluza-Klein background geometry which is the direct product of four-dimensional Minkowski spacetime M^4 with the N-sphere S^N, N odd. The computation is performed in the physical Lorentz-signature spacetime, avoiding the difficulties of "Euclideanization". We find that the contribution of each gravitational degree of freedom to the $\mathcal{O}(\hbar)$ part of the effective potential is significantly greater than that of a scalar or spinor in the same background geometry. No stable minima of the effective potential exist for $3 \leq N \leq 13$. Geometries which may be interpreted as "unstable solutions" are found for all N from 3 through 13. These results, obtained in Lorentz-signature spacetimes, differ from those obtained by "Euclideanization"; our "Euclideanized" results agree with those obtained by Chodos and Myers using a different regularization scheme.

I. Introduction

In the Kaluza-Klein approach to the unification of gauge forces with gravity [1], it is gravity (or its supersymmetric extension) which is the elementary entity; not, however, "gravity" in the usual sense of "the manifestation of the curvature of four-dimensional spacetime". Rather, the basic field in a Kaluza-Klein theory is the metric tensor of a spacetime with some number N of "extra" dimensions above and beyond the familiar four. For this (4+N)-dimensional metric tensor to give rise to an effective theory of four-dimensional gravity coupled to four-dimensional gauge fields, it is necessary that the background geometry of the (4+N)-dimensional spacetime have, at least locally, the structure of the direct product of a four-dimensional spacetime with an N-dimensional compact space, henceforth referred to as the "internal space". For the effective four-dimensional Yang-Mills coupling constants to have their observed values (i.e., of order unity) the characteristic length scale L of the internal space must be comparable to and somewhat larger than the Planck length [2]

$$L_p = G^{\frac{1}{2}} \approx 1.616 \times 10^{-33} \text{ cm} \quad . \tag{1.1}$$

where G is Newton's constant. (In Kaluza-Klein theories, gauge couplings are proportional to L_p/L [3].) We are therefore led to search for dynamically consistent "Kaluza-Klein spacetimes", i.e., spacetimes with metrics satisfying the Einstein equation, and which have the desired product structure and internal scale.

An interesting spacetime of this sort has been studied by Candelas and Weinberg [4]. In their model, the background geometry has the form $M^4 \otimes S^N$, where M^4 is four-dimensional Minkowski spacetime and S^N is the N-dimensional sphere. The fields in the model are, in addition to the (4+N)-dimensional metric tensor g_{MN}, n_b real scalar fields and n_f Dirac spinor fields. These matter fields have zero vacuum expectation values. Therefore, at the classical level (i.e., ignoring all effects proportional to Planck's constant $\hbar$), the matter fields produce no stress-energy, and the classical Einstein equation is simply

$$G_{MN} = \frac{1}{2} \bar{\Lambda}\, g_{MN} \quad , \tag{1.2}$$

where G_{MN} is the (4+N)-dimensional Einstein tensor constructed from g_{MN}, and $\bar{\Lambda}$ is the (4+N)-dimensional cosmological constant. One can verify that, except for $N = 1$ and $\bar{\Lambda} = 0$, $M^4 \otimes S^N$ is <u>not</u> a solution to (1.2) for any values of $\bar{\Lambda}$ and the radius r of S^N.

It is known, however, that one-loop quantum effects can drastically alter the nature of the Einstein equation, and its solution, in Kaluza-Klein spacetimes. For example, the original Kaluza-Klein spacetime, $M^4 \otimes S^1$, satisfies the classical Einstein equation (1.2) with $\bar{\Lambda} = 0$; one-loop effects of massless boson or fermion fields, including the gravitational field, destabilize this solution [5,6], causing the circle S^1 to contract or expand in much the same way that two plane parallel conducting plates are drawn together by virtue of the vacuum energy of the quantized electromagnetic field. It was the suggestion of Weinberg [3] that, in a Kaluza-Klein spacetime with a <u>curved</u> internal space (and thus at least two extra dimensions) quantum effects could be counter-balanced by classical curvature effects, yielding a stable equilibrium size of $\mathcal{O}(L_p)$ for the internal space without the introduction of arbitrary parameters.

In calculating the quantum corrections to (1.2) Candelas and Weinberg ignore the contribution of the degrees of freedom of the quantized gravitational field (gravitons), arguing that n_b and n_f can always be chosen sufficiently large for graviton effects to be negligible compared with quantum effects of scalars and spinors. The relevant geometrodynamical equation for the Candelas-Weinberg model is thus

$$G_{MN} = \frac{1}{2} \bar{\Lambda} \; g_{MN} - 8\pi \bar{G} \; T_{MN} \quad , \tag{1.3}$$

where T_{MN} is the effective stress energy of the quantized scalar and spinor fields, and $\bar{G}$ is the (4+N)-dimensional Newton constant. $\bar{\Lambda}$ is fine-tuned to ensure the compatibility of (1.3) with the flat M^4 subspace; the radius r of the internal space is then fixed by (1.3).

Candelas and Weinberg find that the numerical coefficients which determine the contribution of each scalar or spinor degree of freedom to T_{MN} turn out to be "unreasonably" small; e.g., $\sim 10^{-5}$ for $N = 7$, rather than ~ 1 as might be anticipated for dimensionless factors. This is significant on several counts. In the first place, quite apart from the requirement that gauge couplings be of the correct magnitude, the size of the internal space must be larger than L_p in order for the loop expansion to make sense [7]. Due to the smallness of the aforementioned coefficients, $\sim 10^5$ species of matter fields are needed to obtain an internal space which is "large enough". Secondly, the anomalous smallness of the contribution of scalars and spinors to T_{MN} leads us to wonder whether gravitons can indeed be neglected; perhaps <u>their</u> contribution to T_{MN} is "reasonable".

There is another reason why computation of the graviton contribution to T_{MN} in the context of the Candelas-Weinberg model is of interest; namely, as a "warm-up" for the study of quantum effects in Kaluza-Klein theories which incorporate supergravity [8,25]. Such theories have, potentially, the advantage of <u>predicting</u> the total number of dimensions of spacetime, as well as the split between the internal dimensions and the rest. However, in supergravity, the number of matter fields cannot be made arbitrarily large, or, indeed, varied at all; it is fixed once and for all by the requirement of supersymmetry.

In the present paper, then, we compute the $\mathcal{O}(\hbar)$ effects of gravitons (in the absence of any matter fields) in the background geometry $M^4 \otimes S^N$, using the effective-potential technique. The paper is organized as follows: In section II we diagonalize the quadratic part of the classical action, and obtain a formal expression for the effective potential. In section III this formal expression is regularized by the zeta-function method, yielding explicit numerical values for the effective potential as a function of the radius of S^N and the cosmological constant. We present our results in section IV. The application of the path integral formalism to one-loop quantum gravity in Lorentzian-signature spacetimes is discussed in appendices A and B, with particular attention paid to the problem of negative eigenvalues. We compare our results with those of Chodos and Myers [17,33] in appendix A.

II. Computation of the Effective Potential

Let us begin our analysis by specifying only that we are working in a D-dimensional spacetime with coordinates z^A, $A = 0,1,\ldots,D-1$. As the classical action for the gravitational field on this manifold we take the D-dimensional Einstein action*, with cosmological constant $\bar{\Lambda}$:

$$S[g_{AB}] = -\frac{1}{16\pi\bar{G}} \int d^Dz \sqrt{|g|}\, [R + \bar{\Lambda}] \quad . \tag{2.1}$$

g is the determinant of the D-dimensional metric tensor g_{AB}, and R is the Ricci scalar formed from g_{AB}. $\bar{G}$ is the bare D-dimensional Newton constant with dimensions of $(\text{length})^{(D-2)}$. The effective action Γ corresponding to (2.1) is, to first order in Planck's constant [9,10]

$$\Gamma[\bar{g}_{AB}] = S[\bar{g}_{AB}] + \Gamma_Q[\bar{g}_{AB}] \quad , \tag{2.2a}$$

where Γ_Q is defined by

$$e^{i\Gamma_Q[\bar{g}_{AB}]} = \int \mathcal{D}h_{AB}\, e^{iS_2[\bar{g}_{AB},h_{AB}]} \quad . \tag{2.2b}$$

$S_2[\bar{g}_{AB},h_{AB}]$ is obtained from $S[g_{AB}]$ by writing g_{AB} in (2.1) as the sum of a background metric, $\bar{g}_{AB}$, and a deviation from this background, h_{AB}:

$$g_{AB} = \bar{g}_{AB} + h_{AB} \quad . \tag{2.3}$$

$S_2[\bar{g}_{AB},h_{AB}]$ is the part of $S[\bar{g}_{AB} + h_{AB}]$ quadratic in h_{AB}:

$$S_2[\bar{g}_{AB},h_{AB}] = \frac{1}{2} \int d^Dz\, h_{AB}\, \hat{S}_2^{ABCD}(\bar{g}_{MN}) h_{CD} \quad , \tag{2.4}$$

*Our differential-geometric conventions are those of ref.[4], which are identical to those of ref.[15] except with regard to the normalization of the cosmological constant. Unless otherwise specified, Planck's constant $\hbar$ = speed of light $c = 1$.

where $\hat{S}_2^{ABCD}$ is the operator

$$\hat{S}_2^{ABCD}(\bar{g}_{MN}) = \frac{\delta^2 S[g_{MN}]}{\delta g_{AB}\delta g_{MN}}\bigg|_{g_{MN}=\bar{g}_{MN}} \quad . \tag{2.5}$$

Factoring out and discarding, in the usual manner, the infinite constant due to gauge invariance [9,10,11], (2.2b) becomes

$$e^{i\Gamma_0[\bar{g}_{AB}]} = \int \mathcal{D}h_{AB}\, e^{iS_{2f}[\bar{g}_{AB},h_{AB}]} \int \mathcal{D}\bar{V}_A \mathcal{D}V_A\, e^{iS_{gh}[\bar{g}_{AB},\bar{V}_A,V_A]} \tag{2.6}$$

where

$$S_{2f}[\bar{g}_{AB},h_{AB}] = S_2[\bar{g}_{AB},h_{AB}] + S_{g-f}[\bar{g}_{AB},h_{AB}] \quad . \tag{2.7}$$

S_{g-f} is the gauge-fixing term, also quadratic in h_{AB}, so S_{2f} is also of the form

$$S_{2f}[\bar{g}_{AB},h_{AB}] = \frac{1}{2} \int d^D z\, h_{AB}\, \hat{S}_{2f}^{ABCD}(\bar{g}_{MN})h_{CD} \quad . \tag{2.8}$$

S_{gh} is the ghost action appropriate to S_{g-f}:

$$S_{gh}[\bar{g}_{AB},\bar{V}_A,V_A] = \frac{1}{2} \int d^D z\, \bar{V}_A \hat{S}_{gh}^{AB}(\bar{g}_{AB})V_B \quad . \tag{2.9}$$

The Feynman-DeWitt-Faddeev-Popov ghosts $\bar{V}_A, V_B$ are anticommuting c-number (Grassmann) valued vector fields.

The actual configuration the field $\bar{g}_{AB}$ assumes is that configuration which extremizes the effective action Γ; i.e., that $\bar{g}_{AB}$ for which

$$\frac{\delta\Gamma[\bar{g}_{AB}]}{\delta\bar{g}_{AB}} = 0 \quad . \tag{2.10}$$

The physical interpretation of $\bar{g}_{AB}$ satisfying (2.10) is:

$$\bar{g}_{AB} = \langle 0_+ | \hat{g}_{AB} | 0_- \rangle \quad . \qquad (2.11)$$

That is, $\bar{g}_{AB}$ is the matrix element of the quantum metric-tensor operator between the groundstate (state of least energy) at time $t_- \to -\infty$, $|0_-\rangle$, and the groundstate at time $t_+ \to +\infty$, $|0_+\rangle$.

Using (2.8) and (2.9) in (2.6)

$$e^{i\Gamma_Q[\bar{g}_{MN}]} = \int \mathcal{D}h_{AB} \;\; \exp\left[\tfrac{i}{2}\int d^Dz \;\; h_{AB}\; \hat{S}^{ABCD}_{2f}(\bar{g}_{MN})h_{CD}\right]$$

$$\cdot \int \mathcal{D}\bar{V}_A \mathcal{D}V_A \;\; \exp\left[\tfrac{i}{2}\int d^Dz \;\; \bar{V}_A\; \hat{S}^{AB}_{gh}(\bar{g}_{MN})V_B\right] \quad . \qquad (2.12)$$

Formulas for the evaluation of path integrals of the form of those appearing in (2.12) are given in appendix A. To apply these formulas, we must know the eigenvalue spectra of the operators $\hat{S}^{ABCD}_{2f}(\bar{g}_{MN})$ and $\hat{S}^{AB}_{gh}(\bar{g}_{MN})$; indeed, since the equation of motion (2.10) for $\bar{g}_{AB}$ requires that $\Gamma[\bar{g}_{AB}]$ be stationary under <u>arbitrary</u> variations of $\bar{g}_{AB}$, we should, in principle, determine the spectra of $\hat{S}^{ABCD}_{2f}(\bar{g}_{MN})$ and $\hat{S}^{AB}_{gh}(\bar{g}_{MN})$ for arbitrary $\bar{g}_{MN}$.

In practice, we opt for the more mathematically-tractable procedure of <u>assuming</u> in advance that the background metric which extremizes Γ will belong to a <u>subclass</u> of all possible metrics. In the present paper, we shall assume that $\bar{g}_{MN}$ is the metric tensor of the one-parameter class of spacetimes of the form $M^n \otimes S^N$; the parameter which distinguishes different members of this class is just the radius of S^N.

The advantage of this "ansatz approach" is, of course, that it makes possible the computation of the necessary eigenvalues. The disadvantages are twofold:

1) Since we only obtain Γ corresponding to backgrounds in a restricted subclass, we can only evaluate the variation of Γ within this subclass. In the case at hand, Γ will be a function of r, and will determine the physical value of r via the equation which states that Γ is stationary under a small variation of r about its physical value,

$$\left.\frac{\partial \Gamma(r)}{\partial r}\right|_{r = r_{physical}} = 0 \quad . \tag{2.13a}$$

The manifold $M^n \otimes S^N$ with r given by (2.13a) is a candidate physical background. Further work is then needed to show that Γ is stationary under all other possible small variations of the background, i.e., that (2.10) is satisfied.

Aside from changes in r, there are other variations in the background metric which will keep it within the subclass of metrics $M^n \otimes S^N$; namely, those which leave S^N unchanged and take M^n into itself ("special conformal transformations"), including, e.g., uniform dilations. Γ must be stationary under these transformations as well. As shown in [4], this requirement leads to the condition

$$\left.\Gamma(r)\right|_{r = r_{physical}} = 0 \quad . \tag{2.13b}$$

2) It may turn out that <u>no</u> metric in the subclass extremizes $\bar{\Gamma}$, even with respect to the single parameter r. <u>This is precisely the case for all the values of</u> N <u>we deal with in the present work</u>. This result is discussed in section IV.

We now obtain the eigenspectra required to compute (2.12). For arbitrary $\bar{g}_{AB}$, the explicit form of the quadratic term S_{2f} defined in (2.7) is [12,13,14]

$$S_{2f}[\bar{g}_{AB},h_{AB}] = -\frac{1}{16\pi\bar{G}}\int d^Dz\sqrt{|\bar{g}|}\left\{\frac{1}{4}\tilde{h}^{AB}[(-\Box-\bar{R}-\bar{\Lambda})\bar{g}_{AM}\bar{g}_{BN} + 2\bar{g}_{AM}\bar{R}_{BN} + 2\bar{R}_{AMBN}]h^{MN}\right\} \quad , \tag{2.14a}$$

where

$$\tilde{h}_{MN} \equiv h_{MN} - \frac{1}{2}\bar{g}_{MN}\, h^A{}_A \tag{2.14b}$$

and where $\bar{g}$, $\bar{R}_{AMBN}$, $\bar{R}_{BN}$ and $\bar{R}$ are, respectively, the determinant, Riemann tensor, Ricci tensor and Ricci scalar constructed from $\bar{g}_{AB}$. (Tensor indices are lowered and raised using $\bar{g}_{AB}$ and its inverse matrix $\bar{g}^{AB}$). The gauge-fixing term used in constructing S_{2f} (see eq.(2.7)) is

$$S_{g-f}[\bar{g}_{AB},h_{AB}] = -\frac{1}{32\pi\bar{G}}\int d^Dz\sqrt{|\bar{g}|}\;\nabla^A\tilde{h}_{AM}\nabla^B\tilde{h}_B{}^M \tag{2.15}$$

The ghost action corresponding to this choice of gauge-fixing term is

$$S_{gh}[\bar{g}_{AB},\bar{V}_A,V_A] = -\frac{1}{32\pi\bar{G}}\int d^Dz\sqrt{|\bar{g}|}\;\bar{V}_A(-\bar{g}^{AB}\Box + \bar{R}^{AB})V_B \quad . \tag{2.16}$$

To determine the eigenvalues of $\hat{S}_{2f}$, we reexpress (2.14) in terms of fields linearly related to h_{AB}, so that the resulting quadratic form is a sum of squares of the new fields with no cross-terms; the coefficients of the squared fields will then be the desired eigenvalues. We first specialize the metric to the case of a product manifold $M^n \otimes B^N$, where M^n is n-dimensional Minkowski spacetime, and B^N is an N-dimensional Riemannian manifold, $n + N = D$. We denote the coordinates of M^n by x^α, $\alpha = 0,1,\ldots,n-1$, and the coordinates of B^N by y^a, $a = n, n+1,\ldots N-1$:

$$\{z^A\} = \{x^\alpha, y^a\} \quad . \tag{2.17}$$

The D-dimensional metric tensor takes the form

$$\bar{g}_{AB}(z) = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \bar{g}_{ab}(y) \end{pmatrix} \tag{2.18}$$

where $\eta_{\alpha\beta}$ is the n-dimensional Minkowski metric, and $\bar{g}_{ab}(y)$ is the metric of B^N. It is useful to define the projection operators

$$\bar{g}_{1AB} = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \bar{g}_{2AB} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{g}_{ab} \end{pmatrix} \quad . \tag{2.19}$$

Using (2.18) and (2.19), we can write h_{AB} as

$$h_{AB} = \phi_{AB} + \frac{1}{n}\,\bar{g}_{1AB}\,\phi_1 + \frac{1}{N}\,\bar{g}_{2AB}\,\phi_2 \quad . \tag{2.20}$$

ϕ_1 and ϕ_2 are the traces of h_{AB} over the external (M^n) and internal (B^N) indices, respectively, and ϕ_{AB} is the "doubly-traceless" part of h_{AB}:

$$\phi_1 = h_{AB}\,\bar{g}_1^{AB} \tag{2.21a}$$

$$\phi_2 = h_{AB}\, \bar{g}_2^{AB} \tag{2.21b}$$

$$\phi_{AB}\, \bar{g}_1^{AB} = \phi_{AB}\, \bar{g}_2^{AB} = \phi_{AB}\, \bar{g}^{AB} = 0 \tag{2.21c}$$

If the metric is of the form (2.18), the Riemann and Ricci tensors are zero unless all of their indices lie in the internal space. If the internal metric is that of an N-sphere of radius r, the components of $\bar{R}_{ambn}$ and $\bar{R}_{ab}$ are [15]

$$\bar{R}_{ambn} = r^{-2}(\bar{g}_{bm}\bar{g}_{an} - \bar{g}_{mn}\bar{g}_{ab}) \tag{2.22a}$$

$$\bar{R}_{ab} = -r^{-2}(N-1)\bar{g}_{ab} \tag{2.22b}$$

and the Ricci scalar is

$$\bar{R} = -r^{-2}\, N(N-1) \quad . \tag{2.23}$$

Using (2.15) and (2.17) - (2.23) in (2.14), and defining

$$g \equiv \det(\bar{g}_{ab}) \tag{2.24}$$

we find that

$$S_{2f} = -\frac{1}{16\pi\bar{G}} \int d^nx d^Ny \sqrt{g} \left(\tfrac{1}{4}\right)$$

$$\{\phi^{ab}[-\Box + (N(N-3)+4)r^{-2} - \bar{\Lambda}]\phi_{ab}$$

$$+ \phi^{\alpha b}(2)[-\Box + (N-1)^2 r^{-2} - \bar{\Lambda}]\phi_{\alpha b}$$

$$+ \phi^{\alpha\beta}[-\Box + N(N-1)r^{-2} - \bar{\Lambda}]\phi_{\alpha\beta}$$

$$+ \phi_1\left(\tfrac{n-2}{2n}\right)[\Box - N(N-1)r^{-2} + \bar{\Lambda}]\phi_1$$

$$+ \phi_2\left(\tfrac{N-2}{2N}\right)[\Box - (N-1)(N-4)r^{-2} + \bar{\Lambda}]\phi_2$$

$$+ \phi_1\left(\tfrac{1}{2}\right)[\Box - (N-1)(N-2)r^{-2} + \bar{\Lambda}]\phi_2$$

$$+ \phi_2\left(\tfrac{1}{2}\right)[\Box - (N-1)(N-2)r^{-2} + \bar{\Lambda}]\phi_1\} \quad . \qquad (2.25)$$

The d'Alembertian which appears in (2.25) is the d'Alembertian for the total spacetime $M^n \otimes S^N$:

$$\Box = \nabla^A\nabla_A = \nabla^\alpha\nabla_\alpha + \nabla^a\nabla_a \quad , \qquad (2.26)$$

where $\nabla^\alpha\nabla_\alpha$ is the d'Alembertian on M^n and $\nabla^a\nabla_a$ is the d'Alembertian on S^N. To diagonalize this operator we expand each of the fields ϕ_{ab}, $\phi_{\alpha b}$, $\phi_{\alpha\beta}$, ϕ_1 and ϕ_2 in terms of harmonics, as follows:

$$1)\ \phi_{ab}(x,y) = \sum_{j=1}^{\infty} \int d^nk\ C_T(j,k_\mu) \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}} H_{ab}^{(j)}(y) \qquad (2.27)$$

where $H_{ab}^{(j)}(y)$ is the j^{th} symmetric traceless tensor harmonic on the N-sphere, satisfying

$$\nabla^a \nabla_a H_{ab}^{(j)} = \Lambda_T^{(j)} H_{ab} \tag{2.28}$$

and

$$\int d^N y \sqrt{g}\, H^{(j)ab} H_{ab}^{(k)} = \delta_{jk} \quad . \tag{2.29}$$

$\Lambda_T^{(j)}$ is the eigenvalue corresponding to the j^{th} symmetric traceless tensor harmonic, and δ_{jk} is the Kronecker delta.

$$2)\ \phi_{\alpha b}(x,y) = \sum_{j=1}^{\infty} \sum_{\nu=0}^{n-1} \int d^n k\, C_V(j,\nu,k_\mu) \hat{e}_\alpha^{(\nu)} \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}} V_b^{(j)}(y) \tag{2.30}$$

$\hat{e}_\alpha^{(\nu)}$ is the ν^{th} (constant) basis vector in M^n, and $V_b^{(j)}$ is an N-spherical vector harmonic.

$$\eta^{\alpha\beta} \hat{e}_\alpha^{(\nu)} e_\beta^{(\mu)} = \eta^{(\nu)(\mu)} \tag{2.31}$$

$$\nabla^a \nabla_a V_b^{(j)} = \Lambda_V^{(j)} V_b^{(j)} \quad , \tag{2.32}$$

$$\int d^N y \sqrt{g}\, V^{(j)a} V_a^{(k)} = \delta_{jk} \tag{2.33}$$

$$3)\ \phi_{\alpha\beta}(x,y) = \sum_{j=1}^{\infty} \sum_{j'=1}^{\frac{n(n+1)}{2}-1} \int d^n k\, C_s(j,j',k_\mu) \hat{e}_{\alpha\beta}^{(j')} \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}} S^{(j)}(y) \tag{2.34}$$

The $\frac{n(n+1)}{2} - 1$ constant tensors $\hat{e}_{\alpha\beta}^{(j)}$ form a basis for symmetric traceless tensors on M^n, and $S^{(j)}(y)$ is the j^{th} scalar N-spherical harmonic:

$$\eta^{\alpha\beta} \eta^{\mu\nu} \hat{e}_{\alpha\mu}^{(j')} e_{\beta\nu}^{(k')} = \delta_{j'k'} \tag{2.35}$$

$$\nabla^a \nabla_a S^{(j)}(y) = \Lambda_S^{(j)} S^{(j)}(y) \tag{2.36}$$

$$\int d^N y \sqrt{g}\, S^{(j)} S^{(k)} = \delta_{jk} \tag{2.37}$$

$$4)\quad \phi_1(x,y) = \sum_{j=1}^{\infty} \int d^n k\, C_1(j,k_\mu) \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}} S^{(j)}(y) \tag{2.38}$$

$$\phi_2(x,y) = \sum_{j=1}^{\infty} \int d^n k\, C_2(j,k_\mu) \frac{e^{ik_\mu k^\mu}}{(2\pi)^{n/2}} S^{(j)}(y) \quad . \tag{2.39}$$

Using (2.26) - (2.39), (2.25) becomes

$$S_{2f} = -\frac{1}{16\pi\bar{G}} (\tfrac{1}{4}) \int d^n k$$

$$\Big\{ \sum_{j=1}^{\infty} C_T^*(j,k_\mu) C_T(j,k_\mu)[k^\mu k_\mu - \Lambda_T^{(j)} + (N(N-3)+4)r^{-2} - \bar{\Lambda}]$$

$$+ \sum_{j=1}^{\infty} \sum_{\nu=0}^{n-1} C_V^*(j,\nu,k_\mu) C_V(j,\nu,k_\mu)(2)[k^\mu k_\mu - \Lambda_V^{(i)} + (N-1)^2 r^{-2} - \bar{\Lambda}]$$

$$+ \sum_{j=1}^{\infty} \sum_{j'=1}^{\frac{n(n+1)}{2}-1} C_S^*(j,j',k_\mu) C_S(j,j',k_\mu)[k^\mu k_\mu - \Lambda_S^{(j)} + N(N-1)r^{-2} - \bar{\Lambda}]$$

$$+ \sum_{j=1}^{\infty} \vec{C}^\dagger(j,k_\mu) \vec{M}(j,k_\mu) \vec{C}(j,k_\mu) \Big\} \quad , \tag{2.40}$$

where

$$\vec{C}(j,k_\mu) = \begin{pmatrix} C_1(j,k_\mu) \\ C_2(j,k_\mu) \end{pmatrix} \tag{2.41}$$

and $\vec{M}(j,k_\mu)$ is the 2×2 matrix

$$M(j,k_\mu)_{11} = (\tfrac{n-2}{2n})[-k^\mu k_\mu + \Lambda_S^{(j)} - N(N-1)r^{-2} + \bar{\Lambda}] \tag{2.42a}$$

$$M(j,k_\mu)_{22} = (\tfrac{N-2}{2N})[-k^\mu k_\mu + \Lambda_S^{(j)} - (N-1)(N-4)r^{-2} + \bar{\Lambda}] \tag{2.42b}$$

$$M(j,k_\mu)_{12} = M(j,k_\mu)_{21}$$

$$= (\tfrac{1}{2})[-k^\mu k_\mu + \Lambda_S^{(j)} - (N-1)(N-2)r^{-2} + \bar{\Lambda}] \tag{2.42c}$$

In a similar manner, the ghost action (2.16) can be written as

$$S_{gh} = -\frac{1}{32\pi\bar{G}} \int d^n k \left\{ \sum_{j=1}^{\infty} \bar{\sigma}_V(j,k_\mu)\sigma_V(j,k_\mu)[k^\mu k_\mu - \Lambda_V^{(j)} - (N-1)r^{-2}] \right.$$
$$\left. + \sum_{j=1}^{\infty}\sum_{\nu=0}^{n-1} \bar{\sigma}_S(j,\nu,k_\mu)\sigma_S(j,\nu,k_\mu)[k^\mu k_\mu - \Lambda_S^{(j)}] \right\} \tag{2.43}$$

where

$$\eta_a(x,y) = \sum_{j=1}^{\infty} \int d^n k\ \sigma_V(j,k_\mu)\ \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}}\ V_a^{(j)}(y) \tag{2.44}$$

$$\eta_\alpha(x,y) = \sum_{j=1}^{\infty}\sum_{\nu=0}^{n-1} \int d^n k\ \sigma_S(j,\nu,k_\mu)\hat{e}_\alpha^{(\nu)}\ \frac{e^{ik_\mu x^\mu}}{(2\pi)^{n/2}}\ S^{(j)}(y) \tag{2.45}$$

Performing the path integration, we obtain (see eqs.(2.12), (A.15), and (A.41)*)

$$e^{i\Gamma_0[\bar{g}_{MN}]} = N \prod_{k_\mu,j} [k^\mu k_\mu - \Lambda_T^{(j)} + (N(N-3)+4)r^{-2} - \bar{\Lambda}]^{-\frac{1}{2}}$$

$$\cdot \prod_{k_\mu,j,\nu} [k^\mu k_\mu - \Lambda_V^{(j)} + (N-1)^2 r^{-2} - \bar{\Lambda}]^{-\frac{1}{2}}$$

$$\cdot \prod_{k_\mu,j,j} [k^\mu k_\mu - \Lambda_S^{(j)} + N(N-1)r^{-2} - \bar{\Lambda}]^{-\frac{1}{2}}$$

$$\cdot \prod_{k_\mu,j} [m_1(j,k_\mu)m_2(j,k_\mu)]^{-\frac{1}{2}} \prod_{k_\mu,j} |k^\mu k_\mu - \Lambda_V^{(j)} - (N-1)r^{-2}|$$
$$\cdot \prod_{k_\mu,j,\nu} |k^\mu k_\mu - \Lambda_S^{(j)}| \tag{2.46}$$

*With regard to the absolute values of the ghost eigenvalues in (2.46), see footnote following eq.(A.39).

In (2.46), N is an infinite constant independent of r and $\bar{\Lambda}$, and $m_1(j,k_\mu)$ and $m_2(j,k_\mu)$ are the two eigenvalues of the matrix $\vec{M}(j,k_\mu)$ in (2.42). We note that we will not change the value of $\Gamma_Q[\bar{g}_{MN}]$ if we replace the product of eigenvalues by a product which has the same value for all j and k_μ. In what follows, we shall make the replacement

$$m_1(j,k_\mu)m_2(j,k_\mu) = (\frac{2-N-n}{nN})(k^\mu k_\mu + \xi_j)(k^\mu k_\mu + \xi_j^*) \tag{2.47}$$

where

$$\xi_j \equiv -\Lambda_S^{(j)} + N(N-1)r^{-2} - \bar{\Lambda}$$
$$- 2(N-1)r^{-2}\{1 - [1 - \frac{nN}{2(N+n-2)}]^{\frac{1}{2}}\} \quad . \tag{2.48}$$

(In all the cases we shall consider, the factor in square brackets in (2.48) is negative; we take the principal square root.)

As is customary in dealing with time-independent systems such as the one we are concerned with in the present paper, we define the quantum effective potential V_Q and the quantum effective potential density $\tilde{V}_Q$:

$$\Gamma_Q = - V_Q = - \tilde{V}_Q \cdot \int d^n x \quad . \tag{2.49}$$

Note that $\tilde{V}_Q$ is an ordinary function of r and $\bar{\Lambda}$. Making use of appendix A we see that computation of $\tilde{V}_Q$ requires the construction of the generalized zeta-function $\hat{\zeta}(s)$ (see eqs.(A.36), (A.40), (A.50) and (A.51)), and the analytic continuation of $\hat{\zeta}(s)$ to the region $s \approx 0$. To do this, we need to know in detail the values and degeneracies of $\Lambda_T^{(j)}$, $\Lambda_V^{(j)}$, and $\Lambda_S^{(j)}$. These have been obtained in refs.[16,17]; we summarize here the relevant facts:

1.) Tensors

There are three types of symmetric traceless tensor spherical harmonic eigenvalues $\Lambda_T^{(j)}$. They are (for $N \geq 3$):

a) Transverse-traceless eigenvalues

$$\Lambda_{TT}^{(\ell)} = -\frac{\ell(\ell+N-1)-2}{r^2} \quad , \quad \ell = 2,3,\ldots \tag{2.50}$$

with degeneracy

$$D_\ell(N,2) = \frac{(N+1)(N-2)(\ell+N)(\ell-1)(2\ell+N-1)(\ell+N-3)!}{2(N-1)!(\ell+1)!} \ . \tag{2.51}$$

b) Longitudinal-transverse eigenvalues

$$\Lambda_{LT}^{(\ell)} = -\frac{\ell(\ell+N-1)-(N+2)}{r^2} \quad , \quad \ell = 2,3,\ldots \tag{2.52}$$

with degeneracy

$$D_\ell(N,1) = \frac{\ell(\ell+N-1)(2\ell+N-1)(\ell+N-3)!}{(N-2)!(\ell+1)!} \ . \tag{2.53}$$

c) Longitudinal-longitudinal eigenvalues

$$\Lambda_{LL}^{(\ell)} = -\frac{\ell(\ell+N-1)-2N}{r^2} \quad , \quad \ell = 2,3,\ldots \tag{2.54}$$

with degeneracy

$$D_\ell(N,0) = \frac{(2\ell+N-1)(\ell+N-2)!}{\ell!(N-1)!} \ . \tag{2.55}$$

2.) <u>Vectors</u>

There are two types of vector spherical harmonic eigenvalues ($N \geq 3$):

a) Transverse eigenvalues

$$\Lambda_{TV}^{(\ell)} = -\frac{\ell(\ell+N-1)-1}{r^2} \quad , \quad \ell = 1,2,\ldots \tag{2.56}$$

with degeneracy $D_\ell(N,1)$ as defined in (2.53).

b) Longitudinal eigenvalues

$$\Lambda_{LV}^{(\ell)} = -\frac{\ell(\ell+N-1)-(N-1)}{r^2} \quad , \quad \ell = 1,2,\ldots \tag{2.57}$$

with degeneracy $D_\ell(N,0)$ as defined in (2.55).

3.) <u>Scalars</u>

The scalar spherical harmonic eigenvalues and their degeneracies are well known [36]; the degeneracies are $D_\ell(N,0)$, and the eigenvalues are

$$\Lambda_S^{(\ell)} = -\frac{\ell(\ell+N-1)}{r^2} \quad , \quad \ell = 0,1,\ldots \tag{2.58}$$

Using (2.46) - (2.58) and appendix A, we find that

$$\hat{\zeta}(s) = \sum_{i=1}^{6} \zeta_i(s) + \zeta_{10+}(s) + \zeta_{10-}(s) - 2\sum_{i=7}^{9} \zeta_i(s) \quad , \tag{2.59}$$

where

$$\zeta_1(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=2}^{\infty} D_\ell(N,2)\,|\ell(\ell+N-1)+N(N-3)+2-\lambda|^{-\beta} \tag{2.60}$$

$$\zeta_2(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=2}^{\infty} D_\ell(N,1)|\ell(\ell+N-1)+N(N-4)+2-\lambda|^{-\beta} \tag{2.61}$$

$$\zeta_3(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=2}^{\infty} D_\ell(N,0)|\ell(\ell+N-1)+N(N-5)+4-\lambda|^{-\beta} \tag{2.62}$$

$$\zeta_4(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} (n) \sum_{\ell=1}^{\infty} D_\ell(N,1)|\ell(\ell+N-1)+N(N-2)-\lambda|^{-\beta} \tag{2.63}$$

$$\zeta_5(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} (n) \sum_{\ell=1}^{\infty} D_\ell(N,0)|\ell(\ell+N-1)+(N-1)(N-2)-\lambda|^{-\beta} \tag{2.64}$$

$$\zeta_6(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \left[\frac{n(n+1)}{2} - 1\right] \cdot \sum_{\ell=0}^{\infty} D_\ell(N,0)|\ell(\ell+N-1)+N(N-1)-\lambda|^{-\beta} \tag{2.65}$$

$$\zeta_7(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=2}^{\infty} D_\ell(N,1)|\ell(\ell+N-1)-N|^{-\beta} \tag{2.66}$$

$$\zeta_8(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=1}^{\infty} D_\ell(N,0)|\ell(\ell+N-1)-2(N-1)|^{-\beta} \tag{2.67}$$

$$\zeta_9(s) \equiv \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} (n) \sum_{\ell=1}^{\infty} D_\ell(N,0)|\ell(\ell+N-1)|^{-\beta} \tag{2.68}$$

$$\zeta_{10\pm}(s) \equiv \frac{\pm ir^{2\beta}}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{\ell=0}^{\infty} D_\ell(N,0) \cdot \left(\ell(\ell+N-1)+(N-1)(N-2) \pm 2(1-N)\left[1-\frac{nN}{2(n+N-2)}\right]^{\frac{1}{2}}\right)^{-\beta} \tag{2.69}$$

We have used the abbreviations

$$\beta \equiv s - \frac{n}{2} \tag{2.70}$$

and

$$\lambda \equiv \bar{\Lambda} r^2 \quad . \tag{2.71}$$

In all of the sums (2.60) - (2.68), any term in which the quantity within the absolute value signs is zero should be deleted [30]. (That is the reason that the $\ell = 1$ term in $\zeta_7(s)$ and the $\ell = 0$ term in $\zeta_9(s)$ do not appear.)

The third term on the right-hand side of eq.(A.36) for $\tilde{V}_0$ may be written as

$$- \frac{i\pi}{2} \hat{\zeta}^{(-)}(0) = - \frac{i}{2} \frac{\pi}{(4\pi)^{n/2}} \frac{1}{(\frac{n}{2})!} \sum_j \theta(-\lambda_j)\lambda_j^{-n/2} \quad , \tag{2.72}$$

where

$$\hat{\zeta}^{(-)}(s) \equiv \sum_{i=1}^{6} \zeta_i^{(-)}(s) \quad . \tag{2.73}$$

The quantities $\zeta_i^{(-)}(s)$, $i = 1,\ldots,6$, are defined in exactly the same manner as the $\zeta_i(s)$ in eqs.(2.60) - (2.65), except that the sums run only over those values of ℓ, if any, for which the quantity within the absolute value signs is strictly negative. This will always be a finite number of terms so s may be set equal to zero directly.

III. Analytic Continuation of the Zeta Functions

We now make another specialization in the class of background spacetimes we are investigating: we restrict our considerations to spacetimes $M^n \otimes S^N$ for which N is odd. We have already restricted the values of n to be those for which $\frac{n}{2}$ is both integral and even (see appendix A). The total number of dimensions $D = n + N$ is therefore odd.* In an odd number of dimensions there are no odd-loop anomalies [18]; this means that the coefficient of the log $\bar{\mu}$ term in (A.36) must vanish, i.e.,

$$\hat{\zeta}(0) = 0 \quad . \tag{3.1}$$

For odd $N \geq 5$ the degeneracy factors $D_\ell(N,2)$, $D_\ell(N,1)$ and $D_\ell(N,0)$ can be written as polynomials in a shifted index L:

$$D_{\ell(L)}(N(\nu),J) = \sum_{m=0}^{\nu-1} \underset{J}{A}_{\nu m} \, L^{2m+2} \quad , \quad J = 0,1,2 \quad , \tag{3.2}$$

where

$$L = \ell + \nu \quad , \tag{3.3}$$

$$\nu = \frac{N-1}{2} \quad , \tag{3.4}$$

and $\underset{J}{A}_{\nu m}$ are constants independent of L (see appendix C). For $N = 3$, the $J = 2$ and $J = 1$ degeneracies in (3.2) have an additional term independent of L (i.e., $m = -1$). For the moment we shall consider only the case of odd $N \geq 5$.

Using (3.3) and (3.4), we reexpress each of the zeta functions (2.60) - (2.69) as

*The motivation for this restriction is discussed in section IV.

$$\zeta_i(s) = \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} d_i(n) \sum_{L=L_i}^{\infty} D_{\ell(L)}(N(\nu),J_i)|L^2 - \gamma_i{}^2|^{-\beta} , \quad (3.5)$$

$$\zeta_{10\pm}(s) = \frac{\pm i}{(4\pi)^{n/2}} \frac{\Gamma(\beta)}{\Gamma(s)} \sum_{L=\nu}^{\infty} D_{\ell(L)}(N(\nu),0)(L^2 - \gamma_{10\pm}^2)^{-\beta} , \quad (3.6)$$

where $d_i(n)$, L_i, J_i, and $\gamma_i{}^2$ are given in appendix C. Following the procedure of ref.[14] and appendix D of ref.[4], we expand the quantities $|L^2 - \gamma_i{}^2|^{-\beta}$ and $(L^2 - \gamma_{10\pm})^{-\beta}$ using the binomial theorem and perform the resulting L-sums; for $\mathrm{Re}\,\beta \gg 0$ these sums converge.

We obtain, for $i = 1,\ldots,9$*

$$\begin{aligned}\zeta_i(s) = \frac{ir^{2\beta}}{(4\pi)^{n/2}} \frac{d_i(n)}{\Gamma(s)} \Bigg\{ &\sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu),J_i)|L^2 - \gamma_i{}^2|^{-\beta}\Gamma(\beta) \\ &- \sum_{L=1}^{Q_i-1} D_{\ell(L)}(N(\nu),J_i) \sum_{r=0}^{n/2} \frac{\Gamma(\beta+r)}{r!} \gamma_i^{2r} L^{-2\beta-2r} \\ &+ \sum_{m=0}^{\nu-1} A_{J_i}^{\nu m} \Bigg[\sum_{r=0}^{n/2} \frac{\Gamma(\beta+r)}{r!} \gamma_i^{2r} \zeta(2\beta+2r-2m-2) \\ &\qquad + \sum_{r=\frac{n}{2}+1}^{\infty} \frac{\Gamma(\beta+r)}{r!} \gamma_i^{2r} \zeta(2\beta+2r-2m-2,Q_i) \Bigg] \Bigg\}\end{aligned} \quad (3.7)$$

where q_i and Q_i are as defined in appendix C. $\zeta(x)$ and $\zeta(x,y)$ are the Riemann zeta function and modified Riemann zeta function, respectively. Any sum in which the upper limit is less than the lower limit is defined to be zero. Note that many of the terms in the second sum in (3.7) (i.e., $\sum_{L=1}^{Q_i-1}$) may be discarded, since, for odd $N \geq 5$,

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*For $i = 10_\pm$, $|L^2 - \gamma_i{}^2| \to (L^2 - \gamma_{\pm10}^2)$ in (3.7) and (3.11).

$$D_{\ell(L)}(N(\nu),2) = 0 \quad , \qquad L = 0,1,\ldots,\nu-2,\ \nu+1 \tag{3.8}$$

$$D_{\ell(L)}(N(\nu),1) = 0 \quad , \qquad L = 0,1,\ldots,\nu-2,\ \nu \tag{3.9}$$

$$D_{\ell(L)}(N(\nu),0) = 0 \quad , \qquad L = 0,1,\ldots,\nu-1 \tag{3.10}$$

We now Laurent-expand each factor in (3.7) about $s=0$. $[\Gamma(s)]^{-1} = s + O(s^2)$ as $s \to 0$, and the terms in the curly brackets are either $O(1/s)$ or $O(1)$ as $s \to 0$. Thus, $\zeta_i(s)$ is regular at $s=0$. Since we are only interested in $\zeta_i(0)$ and $\zeta_i'(0)$, we need only keep terms up to $O(s)$ in the Taylor expansion of $\zeta_i(s)$ about $s=0$; this can be written as $(\psi(x) \equiv \frac{d}{dx} \log \Gamma(x))$

$$\zeta_i(s) = \frac{id_i(n)}{(4\pi)^{n/2} r^n} (1 + 2s \log r) \frac{1}{(n/2)!}$$

$$\cdot \left\{ \sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu),J_i) |L^2 - \gamma_i^2|^{n/2} - \sum_{L=1}^{Q_i-1} D_{\ell(L)}(N(\nu),J_i) |L^2 - \gamma_i^2|^{n/2} \right\}$$

$$+ \frac{id_i(n)}{(4\pi)^{n/2} r^n} (s)$$

$$\cdot \left\{ \sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu),J_i) \frac{1}{(n/2)!} |L^2 - \gamma_i^2|^{n/2} [\psi(1+\tfrac{n}{2}) - \log|L^2 - \gamma_i^2|] \right.$$

$$- \sum_{L=1}^{Q_i-1} D_{\ell(L)}(N(\nu),J_i) L^n \sum_{r=0}^{n/2} \frac{(-\gamma_i^2/L^2)^r}{r!(\frac{n}{2}-r)!} [\psi(1+\tfrac{n}{2}-r) - 2\log L]$$

$$+ \sum_{m=0}^{\nu-1} A_{J_i}{}^{\nu m} \left[\sum_{r=0}^{n/2} (-1)^{m+1} \pi^{-n-5/2-2m+2r} \gamma_i^{2r} \frac{(1+\frac{n}{2}+m-r)!}{(\frac{n}{2}-r)!r!} \Gamma(\tfrac{3}{2}+\tfrac{n}{2}+m-r) \right.$$

$$\cdot \zeta(3+n+2m-2r)$$

$$\left. \left. + \sum_{r=\frac{n}{2}+1}^{\infty} \frac{(r-\frac{n}{2}-1)!}{r!} \gamma_i^{2r} \zeta(2r-n-2m-2,\ Q_i) \right] \right\} + O(s^2) \tag{3.11}$$

From (3.11) and appendix C, we discover that

$$\zeta_i(0) \neq 0 \quad , \quad i = 1,2,3,4,5,7,8 \quad . \tag{3.12}$$

This may seem distressing, in light of (3.1)...until we recall that the theorem "no odd loop anomalies in odd dimensions" is a <u>dynamical</u> statement, and has relevance only for the zeta function $\hat{\zeta}(s)$ from which the effective potential is actually constructed, not for the individual zeta functions $\zeta_i(s)$ into which we have arbitrarily decomposed $\hat{\zeta}(s)$. Indeed, when the ζ_i's are added up to give $\hat{\zeta}(s)$ using (2.59), (3.1) is satisfied for all values of r and $\bar{\Lambda}$. This fact serves as a useful check on the correctness of our computation.

Using eqs.(2.72), (2.73), (3.5), and appendix C, the third term in eq.(A.36) for $\tilde{V}_Q$ becomes

$$- \frac{i\pi}{2} \; \hat{\zeta}^{(-)}(0) = - \frac{i}{2} \frac{\pi}{(4\pi)^{n/2}} \frac{1}{(\frac{n}{2})!} \frac{1}{r^n}$$
$$\left[\sum_{i=1}^{6} d_i(n) \sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu),J_i) \, |L^2 - \gamma_i^2|^{n/2} \right] \quad . \tag{3.13}$$

(We remind the reader that sums in which the upper limit is less than the lower limit vanish.)

If $N = 3$ (i.e., $\nu = 1$) the transverse vector and symmetric transverse-traceless tensor degeneracies are of the form

$$D_{\ell(L)}(3,J) = \sum_{m=-1}^{0} {}_J A_{1m} \, L^{2m+2} \quad , \quad J = 1,2 \tag{3.14}$$

rather than (3.2). (For $J=0$, (3.2) still applies.) The only change which this difference entails in the formula for the ζ_i's is a change in the lower limit of the sum on m in (3.7) from 0 to -1 (with ${}_0A_{1,-1} \equiv 0$). The analytic continuation to a form valid near $s=0$ also proceeds in a slightly different manner: for $m=-1$, the Laurent expansion about $s=0$ of the first summand in square brackets in (3.7) is ($C \equiv -\psi(1)$ = Euler's constant = 0.57721566490...)

$$\frac{\Gamma(\beta+r)}{r!} \gamma_i^{2r} \zeta(2\beta+2r) = \frac{\gamma_i^{\,n}}{(\frac{n}{2})!} \left[-\frac{1}{2s} + \frac{C}{2} - \log(2\pi)\right] + O(s) \quad . \tag{3.15}$$

Thus, $\zeta_i(s)$, $i=1,2,\ldots,5,7$, each have an additional contribution to the part which is nonzero at $s=0$. As in the case of odd $N \geq 5$, these nonzero parts vanish when summed to give the complete $\hat{\zeta}(s)$.

The case of $N=1$ has been dealt with previously [17] and will not be discussed here.

IV. Results and Discussion

Since the $\mathcal{O}(1)$ terms in (3.8) vanish when combined into $\hat{\zeta}(s)$, the relevant part of $\zeta_i(s)$ is simply the coefficient of the $\mathcal{O}(s)$ term. Using (2.59), (3.1), (3.8-11), (3.13), (A.35), (A.39), and (A.43), we can compute the value of the quantum effective potential density $\tilde{V}_Q$ for any values of r and $\bar{\Lambda}$. $\tilde{V}_Q$ has the functional form

$$\tilde{V}_Q = \frac{f(\lambda)}{r^n} + \frac{ih(\lambda)}{r^n} \tag{4.1a}$$

where $f(\lambda)$ and $h(\lambda)$ are real-valued functions of $\lambda = \bar{\Lambda} r^2$,

$$\frac{f(\lambda)}{r^n} = \frac{i}{2} \sum_{j=1}^{6} \zeta_j'(0) - i \sum_{j=7}^{9} \zeta_j'(0) \tag{4.1b}$$

$$\frac{ih(\lambda)}{r^n} = -\frac{i}{2} \frac{\pi}{(4\pi)^{n/2}} \frac{1}{(\frac{n}{2})!} \frac{1}{r^n} \Big[\sum_{i=1}^{6} d_i(n) \sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu), J_i) |L^2 - \gamma_i^2|^{n/2} \Big] + \frac{i}{2} \left[\zeta_{10_+}'(0) + \zeta_{10_-}'(0)\right] \quad . \tag{4.1c}$$

Define the total one-loop effective potential V and the corresponding density $\tilde{V}$ by

$$\Gamma = -V = -\tilde{V} \int d^n x \quad . \tag{4.2}$$

Using (2.1), (2.2a), (2.23), (2.49), (4.1a), (4.2) and the formula for the volume of an N-sphere [19],

$$\int d^N y \sqrt{\bar{g}} = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma(\frac{N+1}{2})} r^N \quad , \tag{4.3}$$

we obtain

$$\tilde{V} = \frac{\pi^{\frac{N-1}{2}}}{8\Gamma(\frac{N+1}{2})\bar{G}} \left[-N(N-1)r^{N-2} + \bar{\Lambda} r^N\right] + \frac{f(\bar{\Lambda} r^2)}{r^4} + \frac{ih(\bar{\Lambda} r^2)}{r^4} \quad . \tag{4.4}$$

(From this point on, we consider only the phenomenologically-interesting case $n = 4$.)

Since $\tilde{V}$ is complex, the conditions (2.13a,b) that determine the physical value of r really amount to four conditions:

$$\frac{\partial}{\partial r} \mathrm{Re}\tilde{V} \Big|_{r_{phys}} = 0 \qquad (4.5a)$$

$$\frac{\partial}{\partial r} \mathrm{Im}\tilde{V} \Big|_{r_{phys}} = 0 \qquad (4.5b)$$

$$\mathrm{Re}\tilde{V} \Big|_{r_{phys}} = 0 \qquad (4.5c)$$

$$\mathrm{Im}\tilde{V} \Big|_{r_{phys}} = 0 \qquad (4.5d)$$

In the cases we have examined it is not possible to satisfy all of these equations simultaneously for any values of r and $\bar{\Lambda}$.

In particular, (4.5b) and (4.5d) are equivalent to

$$\frac{\partial}{\partial \lambda} h(\lambda) = 0 \qquad (4.6a)$$

$$h(\lambda) = 0 \qquad (4.6b)$$

We have verified graphically that these equations fail to be simultaneously satisfied for odd N from N = 3 through N = 13 (see figures 17 through 28). Thus, for these values of N, the matrix element of the metric operator (see eq.(2.11)) is not a real metric describing the spacetime $M^4 \otimes S^N$. A fortiori, there does not exist a groundstate $|0\rangle$ such that

$$|0_-\rangle = |0_+\rangle = |0\rangle \qquad (4.7)$$

and such that the expectation value of the metric

$$\bar{g}_{AB} = \langle g_{AB} \rangle = \langle 0 | \hat{g}_{AB} | 0 \rangle \tag{4.8}$$

describes $M^4 \otimes S^N$.

Suppose we choose to view the complex effective potential $\tilde{V}$ as describing an unstable state [20].* That is, we view our spacetime as one which, in the distant past, had $\langle g_{AB} \rangle$ corresponding to $M^4 \otimes S^N$; however, there is a probability per unit time per unit 3-volume of $\langle g_{AB} \rangle$ changing (there is no way to tell from the present analysis what changes are likely). This probability density is given by

$$\rho = -2 \, \mathrm{Im} \, \tilde{V} \quad . \tag{4.9}$$

The value of $\tilde{V}$ depends on both r and $\bar{\Lambda}$; which r and $\bar{\Lambda}$ shall we use in (4.9)? The only natural (though not, perhaps, compelling) choice is those values of r and $\bar{\Lambda}$ which extremize the real part of $\tilde{V}$, i.e. the solutions to (4.5a) and (4.5c). With suitable fine-tuning of $\bar{\Lambda}$, such "unstable solutions" exist with N = 3, 5, 7, 9, 11, or 13. Some of these are described in table 1 and figs. 1 through 16.

As an example, consider $N = 7$. Fine-tune $\bar{\Lambda}$ to the

*It is not our aim here to argue that this viewpoint is correct or incorrect, but merely to point out the consequences if it is taken.

value

$$\bar{\Lambda} = 52.930\ \bar{L}_p \quad , \tag{4.10}$$

where

$$\bar{L}_p \equiv (\bar{G})^{\frac{1}{n+N-2}} \tag{4.11}$$

is the (n+N)-dimensional Planck length. Then $\mathrm{Re}\tilde{V}$ and $\frac{\partial}{\partial r}\mathrm{Re}\tilde{V}$ are both zero at

$$r = 1.1100\ \bar{L}_p \quad . \tag{4.12}$$

This is a local minimum of $\mathrm{Re}\tilde{V}$. Since $r > \bar{L}_p$, we expect on dimensional grounds that the loop expansion we have been employing is a meaningful approximation to the exact effective potential [7]. (This is not true for all the "unstable solutions" in table 1.)

G_0, the Newton constant of the effective n-dimensional gravitational field, is related - at tree level - to the bare (n+N)-dimensional Newton constant $\bar{G}$ by

$$G_0 = \bar{G}/\int d^N y \sqrt{\bar{g}}(y) \quad . \tag{4.13}$$

Using (4.3), (4.11) and (4.13), n = 4 and N = 7, we find that r is significantly larger than the tree level n-dimensional Planck length $L_p \equiv (G_0)^{\frac{1}{n-2}}$:

$$r = 9.1137\, L_p \quad . \tag{4.14}$$

However, it must be borne in mind that loop effects may cause the observed physical value of the Newton constant to differ drastically from its tree-level value [21,22,4].

At the minimum of $\mathrm{Re}\tilde{V}$ given by (4.10) and (4.12),

$$\mathrm{Im}\tilde{V} = -\,3062.9\,(\bar{L}_p)^{-4} \quad . \tag{4.15}$$

If we identify the tree-level L_p in (4.14) with its observed value, given in eq.(1.1), we obtain, using (4.9), a rate of decay of the $M^4 \otimes S^7$ state

$$\begin{aligned} \rho &= 6125.8\,\bar{L}_p^{-4} \\ &= 2.2005 \times 10^{-4} L_p^{-4} \\ &= 9.672 \times 10^{137}\mathrm{cm}^{-3}\mathrm{sec}^{-1} \quad . \end{aligned} \tag{4.16}$$

Corresponding results for other unstable solutions are given in Table 1 .* (No unstable solutions exist in regions where $\mathrm{Im}\tilde{V} > 0$, corresponding to a "negative decay probability".)

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*We should point out that the instability induced in $M^4 \otimes S^N$ by gravitons is of a somewhat different nature than that which may occur when only scalar and Dirac-spinor contributions to V_Q are taken into account. In the scalar-spinor case, a groundstate with $M^4 \otimes S^N$ geometry does exist. If the ratio of the number of scalar species to the number of spinor species does not satisfy certain inequalities, arbitrarily small perturbations of the background geometry will give rise to exponentially-growing deviations from $M^4 \otimes S^N$ [4,38]. However, a state which, at early times, has a background metric corresponding to a perfectly unperturbed $M^4 \otimes S^N$ will, with unit probability, be found in a state which also looks like $M^4 \otimes S^N$ at late times. This is not true in the graviton case.

However seriously one wishes to take such "unstable solutions", the following conclusions may be drawn from our results:

1) At the one-loop level, a state with $<g_{AB}>$ corresponding to $M^4 \otimes S^N$, $N = 3,5,\ldots, 13$, is not a stable groundstate for pure gravity.

2) The anomalous smallness of the contribution of scalar and spinor degrees of freedom to the effective potential on $M^4 \otimes S^N$ is not a feature shared by the contribution of gravitational degrees of freedom on this background. In table 2 we compare these contributions, in units of r^{-4}. For scalars and spinors, this quantity is independent of r; for gravitons, values are quoted both at $r = 0$ (or, equivalently $\bar{\Lambda} = 0$ and r arbitrary), and at values of r, $\bar{\Lambda}$ corresponding to "unstable solutions".

Both of these conclusions demonstrate that, in studying quantum effects in Kaluza-Klein theories, one ignores quantum-gravitational effects only at one's peril.

What implications do our results have for the construction of models with stable Kaluza-Klein background geometries and realistic particle spectra? Since a model without fermionic matter is hardly realistic, the instability of "pure-gravity compactification" is not to be regarded as catastrophic. Indeed, the quantum part of the effective potential in eleven-dimensional supergravity has been shown to vanish if the background metric is the classical solution with anti-de Sitter spacetime as the non-compact sector [23]. The corresponding analysis for supergravity in the background $M^4 \otimes S^7$ is currently in progress [24].

In addition, the existence of interesting supergravitational theories in ten dimensions [25] motivates the extension of our analysis to the case of gravity (and, ultimately, supergravity) on $M^4 \otimes S^N$ with N <u>even</u>. The motivation for restricting N to be odd is to introduce as few arbitrary parameters as possible into the effective potential. In the present case, the bare (n+N)-dimensional cosmological constant $\bar{\Lambda}$ is an arbitrary parameter which is fine-tuned in hope of obtaining flat M^4. With N even there will be an additional term in the effective action

$$\frac{1}{\bar{G}} \Lambda_{ind} \int d^{n+N}z \ \sqrt{|g|} \tag{4.18}$$

where the "induced cosmological constant" is an extra fine-tuneable parameter

$$\Lambda_{ind} = iG_0 \log \bar{\mu}\ \hat{\zeta}(0) \quad . \tag{4.19}$$

In a supergravitational theory, $\bar{\Lambda}$ is fixed rather than arbitrary; thus, the number of arbitrary parameters in the one-loop effective potential need not increase in going from N odd to N even.* Since Λ_{ind} enters the effective potential <u>only</u> through the term (4.18), without affecting the "mass eigenvalues" λ_i of the graviton modes, it should be possible to fine-tune Λ_{ind} to give vanishing Re$\tilde{V}$ at the minimum with comparative ease. Whether there exists a gravitational or supergravitational model in which all the dynamical equations (4.5a - d) are satisfied can, of course, only be determined by detailed calculation [27].

Finally, it should be kept in mind that the instability of pure gravity on spheres (or that of supergravity on spheres, should the problem arise in that case as well) could be an indication that the true groundstate has a different <u>shape</u> than S^N. Internal manifolds differing significantly from S^N--"squashed" spheres, product manifolds, etc.--are worth examining, not only for this reason, but also because their symmetry groups may correspond to those observed in Nature [39]. However, it is also of great interest to study arbitrary <u>small</u> deformations of S^N, or of other symmetric manifolds [40]. If the effective action Γ of the theory on the symmetric manifold is real, knowledge of Γ in the presence of such deformations is needed for a complete analysis of stability against these deformations, as well as for a determination of the masses, scattering cross sections,

*This point has been noticed independently by E. Myers [26].

and decay rates of ultraheavy (> $1/\bar{L}_p$) particle states ("pyrgons"). Cosmology can place strong constraints on pyrgons [41] and thus help to reduce the field of viable Kaluza-Klein theories.

Perhaps most importantly, we should expect that stable internal geometries in a Kaluza-Klein theory will possess extremely small deviations from perfect symmetry. Why so? Particle masses in Kaluza-Klein theory are determined by the spectra of operators on the internal manifold. (This is true in the quantum theory as well as in the classical theory; in the quantum theory, the relevant metric on the internal manifold is the background-field metric rather than the classical metric, and the relevant operators are second variational derivatives of the effective action rather than the classical action.) Zero mass particles correspond to zero modes of these operators, and zero modes, in turn, generally correspond to symmetries of the internal manifold. Changes in the shape of the internal manifold will tend to change the masses of all particles, including those of zero mass; in particular, deformations which destroy a given symmetry will tend to give mass to zero-mass particles. Thus, in a Kaluza-Klein context, the "hierarchy problem"--the problem of how a theory characterized by a large mass scale (in this case, the Planck mass $1/\bar{L}_p$) naturally gives rise to particles with masses smaller by many orders of magnitude (electrons, quarks, etc.)--is expressed as: Why does the internal manifold deviate by such a small, but yet nonzero amount from a perfectly-symmetric form?

If the Kaluza-Klein theory is to be of physical relevance, it must be capable of generating, without artificial fine-tuning, an extremely small dimensionless quantity, the "eccentricity" which measures the small deviation from symmetry. In this regard, it is encouraging to note that, as seen in table 2*, even the present simple model is capable of generating a "hierarchy" of several orders of magnitude between the contributions per degree of freedom of different species to the quantum part of the effective potential.

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*The quantities $f(\lambda)$ and $h(\lambda)$ in the first and second columns of table 2 do depend on $\bar{\Lambda}$, which is fine-tuned to give flat M^4; $f(0)$, $h(0)$, $C_N^{(0)}$ and $C_N^{(\frac{1}{2})}$ are all completely independent of any fine-tuned parameters, including Λ.

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APPENDIX A

Path Integrals with Negative Eigenvalues

In studying quantum gravity at the one-loop level by means of the path-integral method, a potential problem which arises in any number of dimensions is that of negative eigenvalues in the operator $\hat{S}_2$ which appears in the exponent of the integrand. The problem is particularly acute if, as is commonly done, the path integral is defined *ab initio* by "Euclideanization", in which physical Lorentz-signature spacetime is replaced by Euclidean-signature space; one is then faced with divergent integrals of the form

$$\int_{-\infty}^{\infty} da_j \, e^{-\Lambda_j a_j^2} \quad , \tag{A.1}$$

one for each mode with an $\hat{S}_2$ eigenvalue $\Lambda_j < 0$. In flat background geometries, negative eigenvalues are associated with the trace of the graviton; in curved backgrounds other modes in addition may be associated with negative eigenvalues. (This is the case in $M^n \otimes S^N$.)

To make sense of expressions such as (A.1), some authors [28,13,14] perform further analytic continuations applying just to the negative-eigenvalue modes; namely, they multiply the coefficients of these modes by factors of "i". This prescription is not only unpleasantly *ad hoc*, it is also in general ambiguous. Should one let $a_j \to ia_j$ or $a_j \to -ia_j$?-- the two choices yield regularized effective actions which differ by an amount proportional to the regularized number of negative modes. This number is, in general, a function of the background geometry, and therefore cannot simply be discarded as an irrelevant constant in the effective action.

These problems do not arise if we work in the physical Lorentz-signature spacetime. No divergent Gaussian integrals are encountered, and potential ambiguities involving factors of $\pm i$ are resolved by use of the "$-i\varepsilon$ rule" (Feynman boundary conditions). The "$-i\varepsilon$ rule", in turn, is in no sense ad hoc, but is a direct consequence of the fact that we are computing groundstate-to-groundstate amplitudes of a system in a static background (such as the present case) whose exact energy eigenspectrum is assumed to be bounded from below (see appendix B).

For notational simplicity consider a real scalar field ϕ in a D-dimensional static spacetime of Lorentzian signature. The one-loop generating functional (also referred to as the "partition function") is given by [9,10]

$$Z = e^{iS} Z_Q \quad . \tag{A.2}$$

S is the classical action, a functional of the background values of ϕ as well as (possibly) other quantities, such as an external classical source coupled to ϕ. Z_Q, the "quantum part" of the generating functional, and the object we are concerned with computing, is given by

$$Z_Q = \int \mathcal{D}\phi \exp[\tfrac{i}{2} \int d^D x \sqrt{|g|}\, \phi\, \hat{S}_2\, \phi] \quad . \tag{A.3}$$

Since we are concerned with groundstate-to-groundstate amplitudes, we perform the path integration in (A.2) over all configurations of the field ϕ, rather than keeping ϕ fixed at some initial and final times (see appendix B). Denote the eigenvalues and orthonormalized eigenfunctions of $\hat{S}_2$ by $-\Lambda_j$ and ϕ_j, respectively:

$$\hat{S}_2\phi_j = -\Lambda_j\phi_j \tag{A.4}$$

$$\int d^Dx \sqrt{|g|}\ \phi_j\phi_k = \delta_{jk} \quad . \tag{A.5}$$

(If $\hat{S}_2$ has a continuous spectrum, as it does in the cases we shall consider, the discrete notation should be thought of as shorthand for a continuum notation in which the Dirac delta function replaces δ_{jk}, integrals replace sums, etc.).

Any function of x may be expanded in terms of the ϕ_j,

$$\phi = \sum_j a_j\phi_j \quad , \tag{A.6}$$

where the a_j are x-independent coefficients. Using (A.4) - (A.6), the path integral (A.3) becomes

$$Z_Q = [\prod_j \int_{-\infty}^{\infty} \mu da_j]\exp[-\tfrac{i}{2}\sum_k \Lambda_k a_k{}^2] \tag{A.7}$$

where μ is a constant scale with the dimensions of mass. (The path integral measure in (A.7) may be taken as the definition of $\int\mathcal{D}\phi$; in any case the Jacobian [29] arising from the change of path-integration variables will be unity to one loop, since the logarithm of this Jacobian, which potentially gives rise to an anomaly, is zero in odd dimensions [18].)

Let us write (A.7) as

$$Z_0 = \prod_j \mu Z_j \quad , \tag{A.8}$$

where

$$Z_j = \int_{-\infty}^{\infty} da_j \, \exp[-\tfrac{i}{2}\Lambda_j a_j^2] \quad . \tag{A.9}$$

Λ_j is a real number, since $\hat{S}_2$ is a Hermitian operator. (Z_j's with $\Lambda_j = 0$ may be discarded as physically irrelevant; see ref.[30]). We now evaluate (A.9):

$$Z_j = 2\int_0^{\infty} da_j \, \exp[-\tfrac{i}{2}\Lambda_j a_j^2] = \int_0^{\infty} du \; u^{-\frac{1}{2}}[\cos(\tfrac{1}{2}\Lambda_j u) - i\sin(\tfrac{1}{2}\Lambda_j u)]$$

$$= \int_0^{\infty} du \; u^{-\frac{1}{2}}[\cos(\tfrac{1}{2}|\Lambda_j|u) - i\delta_j \sin(\tfrac{1}{2}|\Lambda_j|u)] \quad , \tag{A.10}$$

where

$$\delta_j \equiv \frac{\Lambda_j}{|\Lambda_j|} \quad . \tag{A.11}$$

Performing the integration in (A.10) [31], we obtain

$$Z_j = e^{\frac{-i\pi\delta_j}{4}} \sqrt{2\pi}\, |\Lambda_j|^{-\frac{1}{2}} \tag{A.12}$$

or, equivalently

$$Z_j = e^{\frac{-i\pi}{4}} \sqrt{2\pi}\, \Lambda_j^{-\frac{1}{2}} \tag{A.13}$$

where we define

$$\arg \Lambda_j = -\pi \quad , \quad \Lambda_j < 0 \quad . \tag{A.14}$$

Using (A.13), (A.8) becomes

$$Z_Q = \prod_j \left(\mu e^{\frac{-i\pi}{4}} \sqrt{2\pi}\, \Lambda_j^{-\frac{1}{2}}\right) \quad . \tag{A.15}$$

Defining the quantum part of the effective potential, V_Q, by the relation

$$Z_Q = e^{-iV_Q} \quad , \tag{A.16}$$

we have

$$V_Q = i \log \bar{\mu} \sum_j - \frac{i}{2} \sum_j \log \Lambda_j \quad , \tag{A.17}$$

$$\bar{\mu} \equiv \mu e^{\frac{-i\pi}{4}} \sqrt{2\pi} \quad . \tag{A.18}$$

The sums in (A.17) are divergent and must be regularized; we employ the zeta-function method [30,32] to accomplish this. Define, for $\text{Re}\, s \gg 0$,

$$\zeta(s) \equiv \sum_j \Lambda_j^{-s} \quad . \tag{A.19}$$

For Re s sufficiently large, the sum in (A.19) will converge to an analytic function of s. Then (A.17) can be written as

$$V_Q = i \log \bar{\mu}\, \zeta(0) + \frac{i}{2} \zeta'(0) \quad , \tag{A.20}$$

where the values of $\zeta(s)$ and its derivative $\zeta'(s)$ at $s = 0$ are obtained by analytic continuation of (A.19).

Now suppose that the eigenvalues Λ_j have the same general form as the majority of the gravitational eigenvalues encountered in this paper, i.e., the sum of a continuous part and a discrete part:

$$\Lambda_j \to \Lambda_j(k_\mu) = k^\mu k_\mu + \lambda_j \quad , \tag{A.21}$$

where μ runs from zero to $n-1 < D-1$,

$$k^\mu k_\mu = -(k_0)^2 + \vec{k}\cdot\vec{k} = -(k_0)^2 + (k_1)^2 + (k_2)^2 + \cdots + (k_{n-1})^2. \tag{A.22}$$

(The eigenvalues $m_1(j,k_\mu)$, $m_2(j,k_\mu)$ which arise from the trace modes are <u>not</u> of this form; we shall deal with them separately.) λ_j is real and nonzero, but may be of either sign, depending on the particular mode. The zeta-function is equal to

$$\zeta(s) = \tilde{\zeta}(s) \cdot \int d^n x \tag{A.23}$$

where

$$\tilde{\zeta}(s) \equiv \sum_j \int \frac{d^n k}{(2\pi)^n} [-(k_0)^2 + \vec{k}\cdot\vec{k} + \lambda_j]^{-s} \quad . \tag{A.24}$$

For nonintegral s the integrand in (A.24), viewed as a function of complex k_0, has branch points at

$$k_0 = \pm\sqrt{\vec{k}\cdot\vec{k} + \lambda_j} \tag{A.25}$$

If $\vec{k}\cdot\vec{k} + \lambda_j > 0$ (whatever the sign of λ_j, this will be the case for some values of $\vec{k}$) the branch points (A.25) apparently lie directly on the contour of k_0 integration (i.e., the real-k_0 axis); "apparently", because when we compute a groundstate-to-groundstate amplitude we must view the time coordinate as the limit of a slightly complex variable,

$$t = \lim_{\varepsilon\to 0} t(1 - i\varepsilon) \quad , \quad \varepsilon > 0 \tag{A.25}$$

(See appendix B.) The momentum-space zeta function (A.24) should therefore be regarded as

$$\tilde{\zeta}(s) = \lim_{\varepsilon\to 0_+} \sum_j \int \frac{d^n k}{(2\pi)^n} \left[-(k_0)^2 + \vec{k}\cdot\vec{k} + \lambda_j - i\varepsilon\right]^{-s} \quad . \tag{A.26}$$

This "$-i\varepsilon$ rule" is, of course, quite familiar [37]. What is important to note here is that it applies whatever sign λ_j may have.*

Whatever the sign of $\vec{k}\cdot\vec{k} + \lambda_j$, the branch points in the integrand of (A.26) are seen to occur in the upper left and lower right quadrants of the complex-k_0 plane. We are thus free to rotate the contour of k_0 integration counterclockwise by any angle θ between 0 and $\pi/2$, without changing the value of the integral. (We must also have $\mathrm{Re}\, s > \frac{1}{2}$ so the contribution at $\|k_0\| \to \infty$ vanishes.) Choosing $\theta = \frac{\pi}{2}$, (A.26) becomes

$$\tilde{\zeta}(s) = \lim_{\varepsilon\to 0_+} i \sum_j \int \frac{d^n k}{(2\pi)^n} \left[k_0{}^2 + \vec{k}\cdot\vec{k} + \lambda_j - i\varepsilon\right]^{-s} \quad . \tag{A.27}$$

Performing the angular integration in Euclidean n-space,

$$\tilde{\zeta}(s) = \lim_{\varepsilon\to 0_+} i \sum_j \frac{2}{(4\pi)^{n/2}\Gamma(\frac{n}{2})} \int_0^\infty d\ell\, \ell^{n-1}(\ell^2 + \lambda_j - i\varepsilon)^{-s} \quad , \tag{A.28}$$

or

*For $k^\mu k_\mu + \lambda_e < 0$ it is consistent with (A.14).

$$\tilde{\zeta}(s) = \lim_{\varepsilon \to 0_+} \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} \sum_j (\lambda_j - i\varepsilon)^{\frac{n}{2} - s} \tag{A.29}$$

where (see reference [19], eq. 2.251.11)

$$|\arg[(\lambda_j - i\varepsilon)^{-1}]| < \pi \quad . \tag{A.30}$$

Keeping in mind (A.30), (A.29) can be written as

$$\tilde{\zeta}(s) = \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} \sum_j |\lambda_j|^{\frac{n}{2} - s} e^{i\pi(s - \frac{n}{2})\theta(-\lambda_j)} \, . \tag{A.31}$$

where $\theta(x)$ is the Heaviside function,

$$\begin{aligned} \theta(x) &= 1 \quad , \quad x > 0 \\ &= 0 \quad , \quad x < 0 \quad . \end{aligned} \tag{A.32}$$

Define yet another zeta-function (the last one!):

$$\hat{\zeta}(s) \equiv \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} \sum_j |\lambda_j|^{\frac{n}{2} - s} \quad . \tag{A.33}$$

If $\frac{n}{2}$ is even,

$$\tilde{\zeta}(0) = \hat{\zeta}(0) \tag{A.34}$$

$$\tilde{\zeta}'(0) = \hat{\zeta}'(0) - \frac{\pi}{(4\pi)^{n/2}} \frac{1}{(\frac{n}{2})!} \sum_j \theta(-\lambda_j)\lambda_j^{n/2} \quad . \tag{A.35}$$

(We can set $s = 0$ directly in the final sum in (A.35) because, in the problem we consider in this paper, the number of eigenvalues $\lambda_j < 0$ is finite.) Using (A.20), (A.23), (A.34), and (A.35), we find that

$$\tilde{V}_Q = i \log \bar{\mu}\hat{\zeta}(0) + \frac{i}{2} \hat{\zeta}'(0) - \frac{i}{2} \frac{\pi}{(4\pi)^{n/2}} \frac{1}{(\frac{n}{2})!} \sum_j \theta(-\lambda_j)\lambda_j^{n/2} \tag{A.36}$$

where $\tilde{V}_Q$ is the quantum effective potential density,

$$V_Q = \tilde{V}_Q \cdot \int d^n x \quad . \tag{A.37}$$

In a similar manner, the action for anticommuting c-number ghost fields [9,10] $\eta, \bar{\eta}$,

$$S_{gh} = \frac{1}{2} \int d^D x \, \bar{\eta} \, \hat{S}_{gh} \eta \tag{A.38}$$

gives rise to the quantum effective potential density

$$\tilde{V}_{gh} = -2 \log \mu \hat{\zeta}_{gh}(0) - i \hat{\zeta}'_{gh}(0) \tag{A.39}$$

where the eigenvalues of $\hat{\zeta}_{gh}$ are $|k^\mu k_\mu + \lambda_{ghj}|$, and*

$$\hat{\zeta}_{gh}(s) \equiv \sum_j \int \frac{d^n k}{(2\pi)^n} |k^\mu k_\mu + \lambda_{ghj}|^{-s} \tag{A.40a}$$

$$= \sum_j \int \frac{d^n k}{(2\pi)^n} [(k^\mu k_\mu)^2 + 2(k^\mu k_\mu)\lambda_{ghj} + (\lambda_{ghj})^2]^{-s/2} \tag{A.40b}$$

$$= \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(s - n/2)}{\Gamma(s)} \sum_j |\lambda_{ghj}|^{n/2 - s} \quad . \tag{A.40c}$$

Let us now consider the trace eigenvalues $m_1(j,k_\mu)$ and $m_2(j,k_\mu)$. Since these are defined to be the eigenvalues of the Hermitian matrix $\vec{M}(j,k_\mu)$ (see eqs.(2.41), (2.42)), they are both real numbers, so the analysis which leads to eq.(A.15) applies to them as much as to any other eigenvalues. The trace modes thus contribute to Z_Q in eq.(A.15) a factor which may be written as

$$Z_{TR} = \prod_{j,k_\mu} \mu^2 e^{-\frac{i\pi}{2}} (2\pi)[m_1(j,k_\mu) m_2(j,k_\mu)]^{-1/2} \tag{A.41}$$

Neither $m_1(j,k_\mu)$ nor $m_2(j,k_\mu)$ is of the form (A.21). However, their product is equal to a product of factors linear in $k^\mu k_\mu$:

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*The absolute value of the integrand appears in (A.40a) because the ghost action comes from exponentiation of a functional Jacobian, i.e., the absolute value of a functional determinant. No term like the third term on the R.H.S. of eq.(A.36) occurs in the ghost case. We are grateful to S. MacDowell for pointing this out to A. Chodos, who pointed it out to us. To obtain (A.40c), from (A.40b), use ref.[42], keeping in mind the "$-i\varepsilon$" rule.

$$Z_{TR} = \prod_{j,k_\mu} \mu^2 e^{\frac{-i\pi}{2}}(2\pi)\left(\frac{2-N-n}{nN}\right)^{-\frac{1}{2}}(k^\mu k_\mu + \xi_j)^{-\frac{1}{2}}(k^\mu k_\mu + \xi_j^*)^{-\frac{1}{2}} \tag{A.42}$$

(ξ_j is defined in eq.(2.48).) The contribution of the trace modes to the effective potential density is therefore

$$\tilde{V}_{TR} = i[\log \bar{\mu} - \frac{1}{4}\log(\frac{2-N-n}{nN})][\tilde{\zeta}_{10_+}(0) + \tilde{\zeta}_{10_-}(0)] + \frac{i}{2}[\tilde{\zeta}'_{10_+}(0) + \tilde{\zeta}'_{10_-}(0)] \tag{A.43}$$

where

$$\tilde{\zeta}_{10+}(s) \equiv \sum_j \int \frac{d^n k}{(2\pi)^n}[-(k_0)^2 + \vec{k}\cdot\vec{k} + \xi_j^*]^{-s} \tag{A.44}$$

and

$$\tilde{\zeta}_{10-}(s) \equiv \sum_j \int \frac{d^n k}{(2\pi)^n}[-(k_0)^2 + \vec{k}\cdot\vec{k} + \xi_j]^{-s} \quad . \tag{A.45}$$

These zeta-functions are similar to (A.24), except that ξ_j, ξ_j^* have nonzero imaginary parts for any finite value of r. Whether or not one takes care to follow the "$-i\varepsilon$ rule", the k_0-integrand in $\tilde{\zeta}_{10_+}(s)$ has branch points in the upper left and lower right quadrants of the complex-k_0 plane, while $\tilde{\zeta}_{10_-}(s)$ has branch points in the upper right and lower left quadrants. As before, we can simplify the k_0-integrals by analytic continuation -- i.e., by moving the contour of k_0-integration in such a way that the value of the integral is unchanged. However, the allowed motions of the contour are now different for $\tilde{\zeta}_{10_+}(s)$ and $\tilde{\zeta}_{10_-}(s)$ due to the different locations of the branch points. The most convenient choices are

$$\tilde{\zeta}_{10_+}(s) : \quad k_0 \to ik_0 \tag{A.46}$$

$$\tilde{\zeta}_{10_-}(s) : \quad k_0 \to -ik_0 \tag{A.47}$$

So

$$\tilde{\zeta}_{10_+}(s) = i \sum_j \int \frac{d^n k}{(2\pi)^n} [(k_0)^2 + \vec{k}^2 + \xi_j^*]^{-s} \quad (A.48)$$

$$\tilde{\zeta}_{10_-}(s) = -i \sum_j \int \frac{d^n k}{(2\pi)^n} [(k_0)^2 + \vec{k}^2 + \xi_j]^{-s} \quad (A.49)$$

or, upon performing the k_μ-integration,

$$\tilde{\zeta}_{10+}(s) = \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(s - n/2)}{\Gamma(s)} \sum_j (\xi_j^*)^{\frac{n}{2} - s} \quad (A.50)$$

$$\tilde{\zeta}_{10_-}(s) = \frac{-i}{(4\pi)^{n/2}} \frac{\Gamma(s - n/2)}{\Gamma(s)} \sum_j (\xi_j)^{\frac{n}{2} - s} \quad . \quad (A.51)$$

Since the ξ_j's are complex numbers whose phases depend continuously on r and $\bar{\Lambda}$, it turns out to be convenient to use (A.50), (A.51) directly in the expression (A.43) for the effective potential density, rather than express (A.50), (A.51) in a form analogous to (A.31) - (A.35).

We see that, at least at the one-loop level and with the background geometry $M^n \otimes S^N$, it is possible to calculate the effective potential without ad hoc Euclideanization and its attendant difficulties. One may ask: In what way, if any, does the effective potential calculated in this manner differ from the effective potential calculated with Euclideanization?

A Euclideanized calculation of the effective potential for gravitons on $M^4 \otimes S^N$ has been performed by Chodos and Myers [17,33]. Their Euclideanization is equivalent to replacing, in all zeta-functions, k_0 by $+ik_0$. As we have seen, this is - with one exception! - precisely what one does if one stays in Lorentzian spacetime and performs mathematically valid analytic continuations of contours in momentum integrals, keeping in mind that one is computing a vacuum-to-vacuum amplitude ("$-i\varepsilon$ rule"). The exception is the trace mode zeta function $\zeta_{10_-}(s)$. Analytic continuation from Minkowski spacetime yields the factor of "-i" in eq.(A.51); working in Euclideanized space would have the net effect of replacing the "-i" in (A.51) by "+i". Thus, the Euclidean version of the calculation <u>can</u> yield results which differ from those obtained by a Lorentzian calculation.

In our notation, the result of Chodos and Myers for the quantum part of the effective potential density may be written as

$$\tilde{V}_{CM} = \frac{f_{CM}(\lambda)}{r^n} + \frac{i\, h_{CM}(\lambda)}{r^n} \qquad \text{(A.52a)}$$

where

$$\frac{f_{CM}(\lambda)}{r^n} = \frac{i}{2}\left[\sum_{j=1}^{6}\zeta_j'(0) + \zeta_{10_+}(0) - \zeta_{10_-}(0)\right] - i\sum_{j=7}^{9}\zeta_j'(0) \qquad \text{(A.52b)}$$

$$\frac{ih_{CM}(\lambda)}{r^n} = -\frac{i}{2}\,\frac{\pi}{(4\pi)^{n/2}}\,\frac{1}{(\frac{n}{2})!}\,\frac{1}{r^n}$$

$$\cdot\left[\sum_{i=1}^{6} d_i(n)\sum_{L=L_i}^{q_i} D_{\ell(L)}(N(\nu),J_i)\,|L^2-\gamma_i^2|^{n/2}\right] \qquad \text{(A.52c)}$$

in contrast with the Lorentzian result $\tilde{V}_0$ (eqs.(4.1a-c)). The physics of $\tilde{V}_{CM}$ differs both qualitatively and quantitatively from that of $\tilde{V}_0$. For example, for odd N from 3 through 11, $\tilde{V}_{CM}$ has no "unstable solutions" [33], whereas $\tilde{V}_0$ does (see section IV). For $N = 13$ both have unstable solutions, but with different radii.

In addition to the question of Euclideanization, the calculation of Chodos and Myers differs from ours in a purely technical aspect: the two calculations use very different methods to analytically continue the individual zeta-function sums to the physical region $s \approx 0$. If both methods are mathematically correct, then we should obtain Chodos and Myer's results for the total effective potential $\tilde{V}$ simply by replacing (4.1a-c) with (A.52a-c). We have verified that this is, in fact, the case. For example, compare fig. 41 with fig. 4 of ref.[33].

In the absence of any relevant experimental data in the realm of quantum gravity*, there seems to be no way of determining whether it is the Lorentzian or Euclidean procedure which yields physically correct results in cases where they disagree. Certainly, one may supplement one's theory with the postulate that quantum effects be calculated by Euclideanization, and proceed from there to obtain results which are mathematically correct.** These results must still be interpreted in Lorentzian spacetime, since all available data show quite clearly that our world is, not Euclidean, but Lorentzian. Given this fact, and given that the Lorentzian procedure avoids the ad hoc aspects of the Euclidean procedure, it seems to us that the Lorentzian procedure is to be preferred.

*We are aware of only one quantum-gravitational experiment which has been performed to date [34].

**If spinor fields are included, topological obstructions may even render the Euclidean theory mathematically inconsistent [43]. We would like to thank R. Pisarski for pointing this out to us.

APPENDIX B

Groundstate-to-Groundstate Amplitude

We review here the path integral formalism for the groundstate-to-groundstate amplitude of a system with a time-independent Hamiltonian whose spectrum contains a unique state of lowest energy.

For notational simplicity consider a system with a single degree of freedom, corresponding to the Schrödinger-picture operator $\hat{q}$. The eigenvalues of $\hat{q}$ are denoted by either q or Q. We add to the time-independent Hamiltonian $\hat{H}$ a time-dependent term $-J(t)\hat{q}$, where J(t) is a time-dependent c-number source which vanishes during all but a finite interval of time:

$$J(t) \neq 0 \quad \text{only if} \quad t_1 < t < t_2 \quad . \tag{B.1}$$

Let $|q,t>$ be the eigenstate of $\hat{q}(t)$ in the Heisenberg picture, i.e.

$$\hat{q}(t)|q,t> = q|q,t> \quad , \tag{B.2}$$

and let $|E_i>$ be the i^{th} (time-independent) Heisenberg-picture eigenstate of the unmodified Hamiltonian $\hat{H}$,

$$\hat{H}|E_i> = E_i|E_i> \quad . \tag{B.3}$$

For notational simplicity we shall think of the states $|E_i>$ as discrete; it would make no difference if we were to think of them as a continuum. What _is_ important is that we shall assume that there exists a unique state of lowest energy $|E_0>$,

$$E_0 < E_i \quad \forall i \neq 0 \quad . \tag{B.4}$$

Define the generating functional in the presence of an external source, Z[J], as

$$Z[J] \equiv \lim_{\substack{T\to-\infty \\ T'\to+\infty}} \int dQ'dQ \; <Q',T'|Q,T>^J \quad , \tag{B.5}$$

where

$$T < t_1 < t_2 < T' \quad , \tag{B.6}$$

the integrals are over the entire eigenspectrum of $\hat{q}$, and the superscript "J" is a reminder that the amplitude $<Q',T'|Q,T>$ must be computed using the modified Hamiltonian $\hat{H} - J(t)\hat{q}$. Inserting complete sets of states, we can write (B.5) as

$$Z[J] = \lim_{\substack{T\to-\infty \\ T'\to+\infty}} \int dQ'dQdq_2dq_1 \; <Q',T'|q_2,t_2><q_2,t_2|q_1,t_1>^J<q_1,t_1|Q,J> \tag{B.7}$$

where (B.1) allows us to drop the "J" superscripts from the initial and final amplitudes.

$$Z[J] = \lim_{\substack{T\to-\infty \\ T'\to+\infty}} \sum_{i,j} \int dQ'dQdq_2dq_1$$

$$\langle Q',T'|E_i\rangle\langle E_i|q_2,t_2\rangle\langle q_2,t_2|q_1,t_1\rangle^J\langle q_1,t_1|E_j\rangle\langle E_j|Q,T\rangle$$

$$= \lim_{\substack{T\to-\infty \\ T'\to+\infty}} \sum_{i,j} \int dQ'dQdq_2dq_1$$

$$\langle Q'|E_i\rangle e^{-iE_iT'}\langle E_i|q_2,t_2\rangle\langle q_2,t_2|q_1,t_1\rangle^J\langle q_1,t_1|E_j\rangle e^{iE_jT}\langle E_j|Q\rangle \quad (B.8)$$

where $|Q\rangle$ is the Schrödinger-picture eigenstate of $\hat{q}$.

Now, evaluate (B.8) as the real-time limit of a system whose time coordinate has been rotated through an angle $-\varepsilon$ in the complex-time plane ($\varepsilon \gtrsim 0$):

$$\lim_{\varepsilon\to 0_+} Z[J] = \lim_{\varepsilon\to 0_+} \left\{ \lim_{\substack{T\to-(1-i\varepsilon)\infty \\ T'\to+(1-i\varepsilon)\infty}} \sum_{i,j} \int dQ'dQdq_2dq_1 \right.$$

$$\left. \langle Q'|E_i\rangle e^{-iE_iT'}\langle E_i|q_2,t_2\rangle\langle q_2,t_2|q_1,t_1\rangle^J\langle q_1,t_1|E_j\rangle e^{iE_jT}\langle E_j|Q\rangle \right\}$$

(B.9)

In the limit $T\to -(1-i\varepsilon)\infty$, $T'\to +(1-i\varepsilon)\infty$, all the terms in the double sum in (B.9) for which $E_i \neq E_0$ and $E_j \neq E_0$ will be exponentially small compared to the single term $i = j = 0$, and may be ignored:

$$\lim_{\varepsilon\to 0_+} Z[J] = \lim_{\varepsilon\to 0_+} \Big\{ \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} \int dQ'dQdq_2dq_1$$

$$<Q'|E_0>e^{-iE_0T'} <E_0|q_2,t_2><q_2,t_2|q_1,t_1>^J<q_1,t_1|E_0>e^{iE_0T}<E_0|Q>\Big\}$$

$$= \lim_{\varepsilon\to 0_+} \Big\{[\int dq_2dq_1<E_0|q_2,t_2><q_2,t_2|q_1,t_1>^J<q_1,t_1|E_0>]$$

$$\cdot \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} [\int dQ'dQ<Q'|E_0>e^{-iE_0T'}\ e^{iE_0T}<E_0|Q>]\Big\}$$

$$= <E_0|E_0>^J$$

$$\cdot \lim_{\varepsilon\to 0_+} \Big\{ \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} \int dQ'dQ<Q'|E_0>e^{-iE_0T'}\ e^{iE_0T}<E_0|Q>\Big\} \tag{B.10}$$

We can add back the $i\neq 0$, $j\neq 0$ terms to (B.10) for the same reason we could remove them in the first place:

$$\lim_{\varepsilon\to 0_+} Z[J] = <E_0|E_0>^J \lim_{\varepsilon\to 0_+} \Big\{ \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} \sum_{i,j} \int dQ'dQ<Q'|E_i>e^{-iE_iT'}\ e^{iE_jT}<E_j|Q>\Big\}$$

$$= <E_0|E_0>^J \lim_{\varepsilon\to 0_+} \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} \sum_i \int dQ'dQ<Q',T'|E_i><E_i|Q,T>$$

$$= <E_0|E_0>^J \lim_{\varepsilon\to 0_+} \lim_{\substack{T\to -(1-i\varepsilon)\infty \\ T'\to +(1-i\varepsilon)\infty}} \int dQ'dQ<Q',T'|Q,T> \quad . \tag{B.11}$$

Comparing (B.11) with the definition (B.5), we conclude that the groundstate-to-groundstate amplitude in the presence of an external source is given by

$$\langle E_0 | E_0 \rangle^J = \frac{\lim_{\varepsilon \to 0_+} Z[J]}{\lim_{\varepsilon \to 0_+} Z[0]} \quad . \tag{B.12}$$

In terms of path integrals, Z[J] can be written (provided $\hat{H}$ is quadratic in the canonical momentum) as

$$Z[J] = \lim_{\varepsilon \to 0} \int \mathcal{D}q \, \exp(iS[q(t)] + i \int_{-(1-i\varepsilon)\infty}^{+(1-i\varepsilon)\infty} dt \, J(t) q(t)) \quad , \tag{B.13}$$

where t is complex,

$$t = (1 - i\varepsilon)\tau \quad , \qquad \tau \text{ real} \quad , \tag{B.14}$$

and where the path integral is over <u>all</u> possible paths q(t), with no restrictions on the values of q(t) at $t \to \pm(1-i\varepsilon)\infty$. (S[q(t)] is the classical action, a functional of the path q(t)).

The foregoing material is, quite likely, familiar to the reader, and has been presented in essentially identical form in refs. [9,35]. Our purpose in repeating it here is to remind the reader that the rule

$$t \to (1 - i\varepsilon)t \tag{B.15}$$

applies whenever the quantum system under consideration has

1) a Hamiltonian not explicitly dependent on t and 2) a unique groundstate. The system we consider in this paper -- small gravitational fluctuations about $M^n \otimes S^N$ -- satisfies (1) by inspection, and (2) by ansatz. (See discussion in section IV.) In particular, the fact that the bare Lagrangian of our system has tachyonic eigenvalues

$$k_\mu k^\mu + \lambda_j \quad , \quad \lambda_j < 0 \quad , \tag{B.16}$$

and "wrong-sign" eigenvalues

$$- (k_\mu k^\mu + \lambda_j) \tag{B.17}$$

makes no difference.

We shall not repeat here the analysis [9,10] which leads from Z[J] to the effective action Γ, or the expression for Γ in the one-loop approximation, except to remind the reader that the condition (2.10) is independent of the loop expansion.

APPENDIX C

Values of Various Quantities

${}_J A_{\nu,m}$

$\nu = 1$:

$${}_0A_{1,0} = 1 ;$$
$${}_1A_{1,-1} = -2 , \quad {}_1A_{1,0} = 2 ;$$
$${}_2A_{1,-1} = -8 , \quad {}_2A_{1,0} = 2 .$$

$\nu = 2$:

$${}_0A_{2,0} = -\frac{1}{12} , \quad {}_0A_{2,1} = \frac{1}{12} ;$$
$${}_1A_{2,0} = -\frac{4}{3} , \quad {}_1A_{2,1} = \frac{1}{3} ;$$
$${}_2A_{2,0} = -\frac{27}{4} , \quad {}_2A_{2,1} = \frac{3}{4} .$$

$\nu = 3$:

$${}_0A_{3,0} = \frac{1}{90} , \quad {}_0A_{3,1} = -\frac{1}{72} , \quad {}_0A_{3,2} = \frac{1}{360} ;$$
$${}_1A_{3,0} = \frac{3}{20} , \quad {}_1A_{3,1} = -\frac{1}{6} , \quad {}_1A_{3,2} = \frac{1}{60} ;$$
$${}_2A_{3,0} = \frac{16}{18} , \quad {}_2A_{3,1} = -\frac{17}{18} , \quad {}_2A_{3,2} = \frac{1}{18} .$$

$\nu = 4$:

$${}_0A_{4,0} = -\frac{9}{7!} , \quad {}_0A_{4,1} = \frac{49}{4\times 7!} , \quad {}_0A_{4,2} = -\frac{7}{2\times 7!} , \quad {}_0A_{4,3} = \frac{1}{4\times 7!} ;$$
$${}_1A_{4,0} = -\frac{128}{7!} , \quad {}_1A_{4,1} = \frac{168}{7!} , \quad {}_1A_{4,2} = -\frac{42}{7!} , \quad {}_1A_{4,3} = \frac{2}{7!} ;$$
$${}_2A_{4,0} = -\frac{100}{576} , \quad {}_2A_{4,2} = \frac{129}{576} , \quad {}_2A_{4,2} = -\frac{30}{576} , \quad {}_2A_{4,3} = \frac{1}{576} .$$

$\nu = 5$:

$$(\alpha_{50} \equiv \frac{2}{10!} \ ; \quad \alpha_{51} \equiv \frac{1}{9!} \ ; \quad \alpha_{52} \equiv \frac{108}{10!}) .$$

$$ {}_0A_{5,0} = \alpha_{50} \times 576 \ , \quad {}_0A_{5,1} = -\alpha_{50} \times 820 \ , \quad {}_0A_{5,2} = \alpha_{50} \times 273 \ , $$

$$ {}_0A_{5,3} = -\alpha_{50} \times 30 \ , \quad {}_0A_{5,4} = \alpha_{50} \ ; $$

$$ {}_1A_{5,0} = \alpha_{51} \times 1{,}800 \ , \quad {}_1A_{5,1} = -\alpha_{51} \times 2{,}522 \ , \quad {}_1A_{5,2} = \alpha_{51} \times 798 \ , $$

$$ {}_1A_{5,3} = -\alpha_{51} \times 78 \ , \quad {}_1A_{5,4} = \alpha_{51} \ ; $$

$$ {}_2A_{5,0} = \alpha_{52} \times 1{,}296 \ , \quad {}_2A_{5,1} = -\alpha_{52} \times 1{,}800 \ , \quad {}_2A_{5,2} = \alpha_{52} \times 553 \ , $$

$$ {}_2A_{5,3} = -\alpha_{52} \times 50 \ , \quad {}_2A_{5,4} = \alpha_{52} \ . $$

$\nu = 6$:

$$(\alpha_{60} \equiv \frac{1}{12!} \ ; \quad \alpha_{61} \equiv \frac{2}{11!} \ ; \quad \alpha_{62} \equiv \frac{154}{12!}) .$$

$$ {}_0A_{6,0} = -\alpha_{60} \times 28{,}800 \ , \quad {}_0A_{6,1} = \alpha_{60} \times 42{,}152 \ , \quad {}_0A_{6,2} = -\alpha_{60} \times 15{,}290 \ , $$

$$ {}_0A_{6,3} = \alpha_{60} \times 2{,}046 \ , \quad {}_0A_{6,4} = -\alpha_{60} \times 110 \ , \quad {}_0A_{6,5} = 2 \times \alpha_{60} \ ; $$

$$ {}_1A_{6,0} = -\alpha_{61} \times 20{,}736 \ , \quad {}_1A_{6,1} = \alpha_{61} \times 30{,}096 \ , \quad {}_1A_{6,2} = -\alpha_{61} \times 10{,}648 \ , $$

$$ {}_1A_{6,3} = \alpha_{61} \times 1{,}353 \ , \quad {}_1A_{6,4} = -\alpha_{61} \times 66 \ , \quad {}_1A_{6,5} = \alpha_{61} \ ; $$

$$ {}_2A_{6,0} = -\alpha_{62} \times 28{,}224 \ , \quad {}_2A_{6,1} = \alpha_{62} \times 40{,}756 \ , \quad {}_2A_{6,2} = -\alpha_{62} \times 14{,}192 \ , $$

$$ {}_2A_{6,3} = \alpha_{62} \times 1{,}743 \ , \quad {}_2A_{6,4} = -\alpha_{62} \times 79 \ , \quad {}_2A_{6,5} = \alpha_{62} \ . $$

$d_i(n)$:

$$d_1(n) = d_2(n) = d_3(n) = d_7(n) = d_8(n) = 1 \quad ;$$

$$d_4(n) = d_5(n) = d_9(n) = n \quad ;$$

$$d_6(n) = \frac{n(n+1)}{2} - 1 \quad ;$$

$$d_{10\pm}(n) = \pm 1 \quad .$$

J_i :

$$J_1 = 2 \quad ;$$

$$J_2 = J_4 = J_7 = 1 \quad ;$$

$$J_3 = J_5 = J_6 = J_8 = J_9 = J_{10\pm} = 0 \quad .$$

γ_i^2 :

$(\nu \equiv \frac{N-1}{2})$ For $N \geq 3$,

$$\gamma_1^2 = -3\nu^2 + 2\nu + \lambda \quad , \qquad \gamma_2^2 = -3\nu^2 + 4\nu + 1 + \lambda$$

$$\gamma_3^2 = -3\nu^2 + 6\nu + \lambda \quad , \qquad \gamma_4^2 = -3\nu^2 + 1 + \lambda$$

$$\gamma_5^2 = -3\nu^2 + 2\nu + \lambda \quad , \qquad \gamma_6^2 = -3\nu^2 - 2\nu + \lambda$$

$$\gamma_7^2 = (\nu+1)^2 \quad , \qquad \gamma_8^2 = \nu^2 + 4\nu$$

$$\gamma_9^2 = \nu^2 \quad , \qquad \gamma_{10_\pm}^2 = -3\nu^2 + 2\nu + \lambda \pm 4\nu\left[1 - \frac{n(2\nu+1)}{2(n+2\nu-1)}\right]^{\frac{1}{2}}$$

L_i :

$$L_1 = L_2 = L_3 = L_7 = \nu + 2 \quad ;$$

$$L_4 = L_5 = L_8 = L_9 = \nu + 1 \quad ;$$

$$L_6 = L_{10_\pm} = \nu \quad .$$

Q_i and q_i $\quad (\|\gamma_i^2\| = \{(\gamma_i^2)^* \gamma_i^2\}^{\frac{1}{2}}$, $[x]$ = integer part of x)

$i = 1,\ldots,9$:

If $\|\gamma_i^2\| < L_i^2$: $\quad Q_i = L_i, \quad q_i = L_i - 1$

If $\|\gamma_i^2\| = L_i^2$ and $\gamma_i^2 > 0$: $\quad Q_i = L_i + 1, \quad q_i = L_i - 1$

If $\|\gamma_i^2\| = L_i^2$ and $\gamma_i^2 < 0$: $\quad Q_i = L_i + 1, \quad q_i = L_i - 1$

If $\|\gamma_i^2\| > L_i$ and $(\|\gamma_i^2\|)^{\frac{1}{2}} \neq [(\|\gamma_i^2\|)^{\frac{1}{2}}]$ and $\gamma_i^2 > 0$:

$$Q_i = [(\|\gamma_i^2\|)^{\frac{1}{2}}] + 1, \quad q_i = [(\|\gamma_i^2\|)^{\frac{1}{2}}]$$

If $\|\gamma_i^2\| > L_i^2$ and $(\|\gamma_i^2\|)^{\frac{1}{2}} = [(\|\gamma_i^2\|)^{\frac{1}{2}}]$ and $\gamma_i^2 > 0$:

$$Q_i = [(\|\gamma_i^2\|)^{\frac{1}{2}}] + 1, \quad q_i = [(\|\gamma_i^2\|)^{\frac{1}{2}}] - 1$$

If $\|\gamma_i^2\| > L_i^2$ and $\gamma_i^2 < 0$: $\quad Q_i = [\|\gamma_i^2\|^{\frac{1}{2}}] + 1, \quad q_i = [(\|\gamma_i^2\|)^{\frac{1}{2}}]$

$i = 10_\pm$:

If $\|\gamma_{10_\pm}^2\| < \nu^2$: $\quad Q_{10_\pm} = \nu, \quad q_{10_\pm} = \nu - 1$

If $\|\gamma_{10_\pm}^2\| \geq \nu^2$: $\quad Q_{10_\pm} = [(\|\gamma_{10_\pm}^2\|)^{\frac{1}{2}}] + 1, \quad q_{10_\pm} = [(\|\gamma_{10_\pm}^2\|)^{\frac{1}{2}}]$

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TABLE 1

Features of "Unstable Solutions"

N	$\bar{\Lambda}/\bar{L}_p^{-2}$	$r/\bar{L}_p$	$Re\tilde{V}_Q/\bar{L}_p^{-4}$	$Im\tilde{V}/\bar{L}_p^{-4}$	r/L_p *	$\alpha \equiv e^2/4\pi$ *	$\rho/\bar{L}_p^{-4}$ *
3	1.9800×10^1	7.8000×10^{-1}	-1.8585×10^0	-5.2909×10^1	2.3873×10^0	1.3989×10^0	1.0582×10^2
3	3.0077×10^1	1.2043×10^0	-1.7793×10^1	-2.0653×10^3	7.0706×10^0	1.5999×10^{-1}	4.1296×10^4
5	5.9447×10^1	1.1720×10^0	-6.1223×10^1	-4.7358×10^4	9.7045×10^0	1.2743×10^{-1}	9.4716×10^4
5	5.9447×10^1	1.2920×10^0	-1.0548×10^2	-8.8956×10^4	1.3650×10^1	6.43603×10^{-2}	1.7791×10^5
5	5.0750×10^1	8.7175×10^{-1}	-4.1840×10^{-3}	-1.8944×10^3	3.4443×10^0	1.0594×10^3	3.7888×10^3
7	5.2930×10^1	1.1100×10^0	-2.5281×10^1	-3.0643×10^3	9.1137×10^0	1.9255×10^{-1}	6.1286×10^3
9	6.6210×10^1	1.5100×10^0	-7.1709×10^2	-2.9343×10^5	4.8714×10^1	8.4278×10^{-3}	5.8686×10^5
11	7.8928×10^1	1.2352×10^0	-2.2239×10^1	-8.3937×10^2	1.5800×10^1	9.61233×10^{-2}	1.6787×10^3
13	9.2320×10^1	1.3440×10^0	-4.6420×10^1	-1.1914×10^3	2.6599×10^1	3.95768×10^{-2}	2.3828×10^3

*These are bare quantities; see section IV. Here $e^2 \equiv [\frac{N+1}{2}][\frac{N(N-1) - \lambda}{f(\lambda)}]$; see ref.[4].

TABLE 2*

One graviton is worth a thousand scalars

N	$f(\lambda)/[\frac{(N+4)(N+1)}{2}]$	$h(\lambda)/[\frac{(N+4)(N+1)}{2}]$	$f(0)/[\frac{(N+4)(N+1)}{2}]$	$h(0)/[\frac{(N+4)(N+1)}{2}]$	$C_N^{(0)}$	$C_N^{(\frac{1}{2})}/2^{\frac{N+3}{2}}$
3	-4.9138×10^{-2}	-1.3989×10^{0}	-8.0029×10^{-2}	-3.6110×10^{-4}	7.5687×10^{-5}	2.4313×10^{-5}
	-2.67291×10^{0}	-3.1031×10^{2}				
5	-4.2782×10^{0}	-3.3093×10^{3}	1.9295×10^{0}	-1.5916×10^{-2}	4.2830×10^{-4}	-7.1275×10^{-6}
	-1.0885×10^{1}	-9.1804×10^{3}				
	-8.9496×10^{-5}	-4.0520×10^{1}				
7	-8.7222×10^{-1}	-1.0572×10^{2}	-2.4807×10^{1}	2.8279×10^{-1}	8.1588×10^{-4}	1.8621×10^{-6}
9	-5.7355×10^{1}	-2.3470×10^{4}	2.5169×10^{2}	-2.7426×10^{0}	1.1339×10^{-3}	-4.6746×10^{-7}
11	-5.7519×10^{-1}	-2.1710×10^{1}	-2.2568×10^{3}	2.1623×10^{1}	1.3293×10^{-3}	1.1545×10^{-7}
13	-1.2728×10^{0}	-3.2667×10^{1}	2.0297×10^{4}	-1.5460×10^{2}	1.3740×10^{-3}	-2.8292×10^{-8}

*Contributions per degree of freedom to the quantum part of the effective potential, $\tilde{V}_Q$, on $M \otimes S^N$. For gravitons, $\tilde{V}_Q = f(\lambda)/r^4 + ih(\lambda)/r^4$, $\lambda = \bar{\Lambda} r^2$. The values in the first and second columns are for the "unstable solutions" (see table 1); those in the third and fourth columns are for $\lambda = 0$. For scalars, $\tilde{V}_Q = C_N^{(0)}/r^4$, and for Dirac spinors, $\tilde{V}_Q = C_N^{(\frac{1}{2})}/r^4$. $C_N^{(0)}$ and $C_N^{(\frac{1}{2})}$ are independent of r and $\bar{\Lambda}$. The values in columns five and six are from ref.[4].

A Guide to the Graviton Graphs

Figs. 1 - 8: Real part of the effective potential density $\tilde{V}$ for each of the "unstable solutions" in table 1.

Figs. 9 - 16: Same as 1 - 8, showing the behavior of $\mathrm{Re}\tilde{V}$ for large and small r.

Figs. 17-22: $h(\lambda) = r^4 \times \mathrm{Im}\tilde{V}_Q$, N = 3,5,...,13 (with some close-ups).

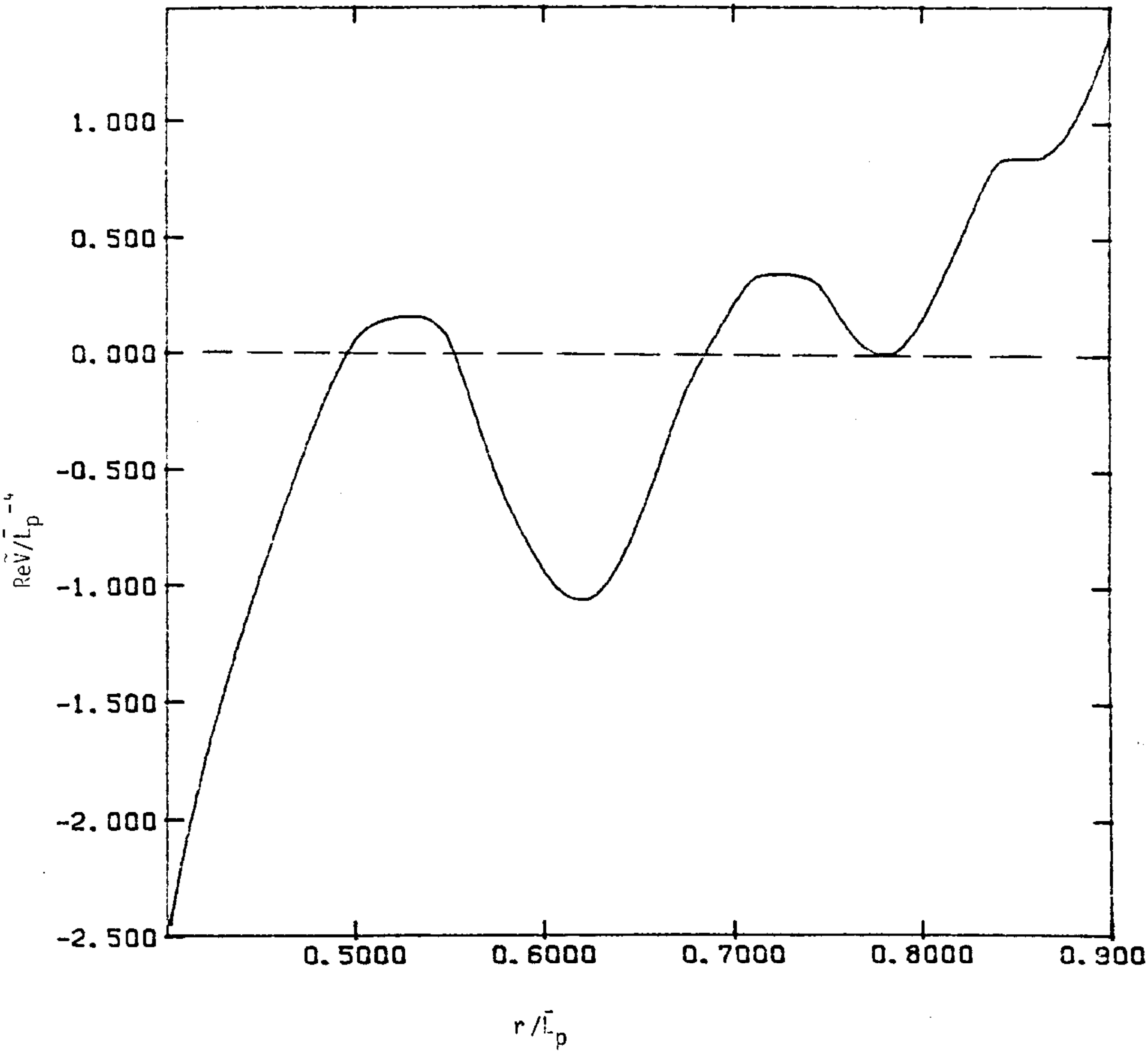
Figs. 23-28: Same as 17 - 22, showing the behavior of $h(\lambda)$ for $\lambda \gg 0$ and $\lambda \ll 0$.

Figs. 29-34: $f(\lambda) = r^4 \times \mathrm{Re}\tilde{V}_Q$, N = 3,5,...,13.

Figs. 35-40: Same as 29 - 34, showing the behavior of $f(\lambda)$ for $\lambda \gg 0$ and $\lambda \ll 0$.

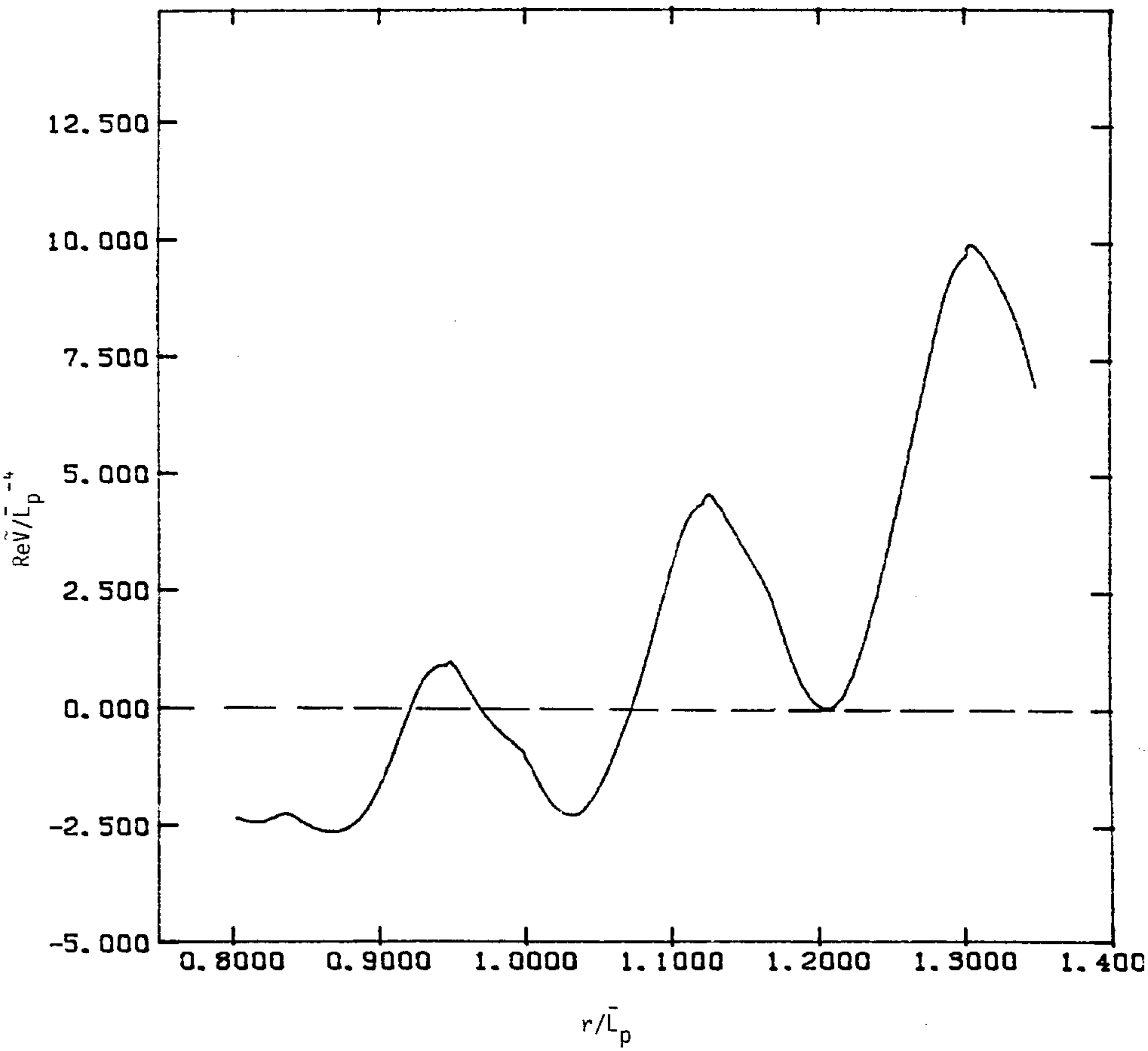
Fig. 41: Real part of the "Euclideanized" effective potential density, N = 13.

Fig. 1



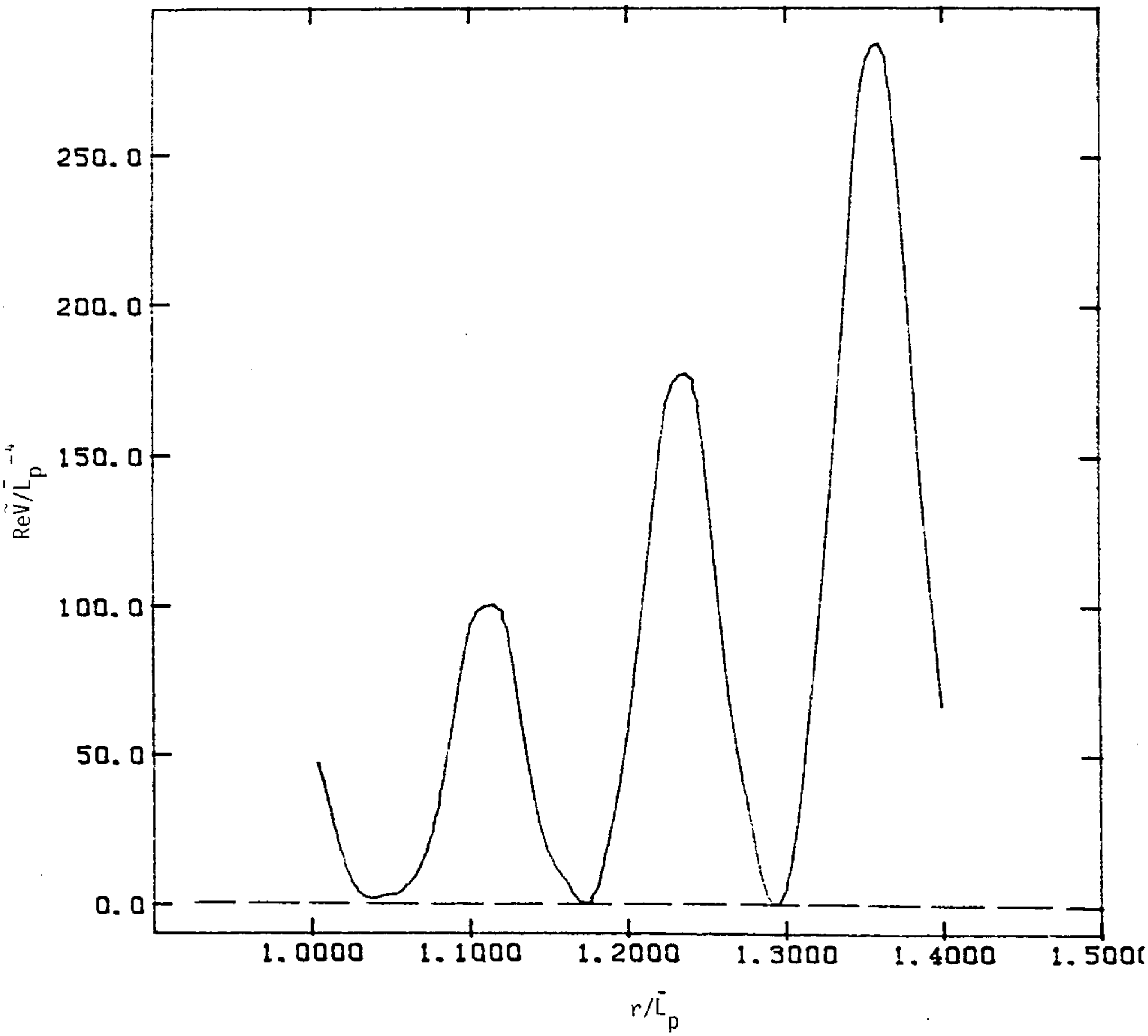
$Re\tilde{V}$: $N = 3$, $\bar{\Lambda} = 19.800$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 0.78000$

Fig. 2



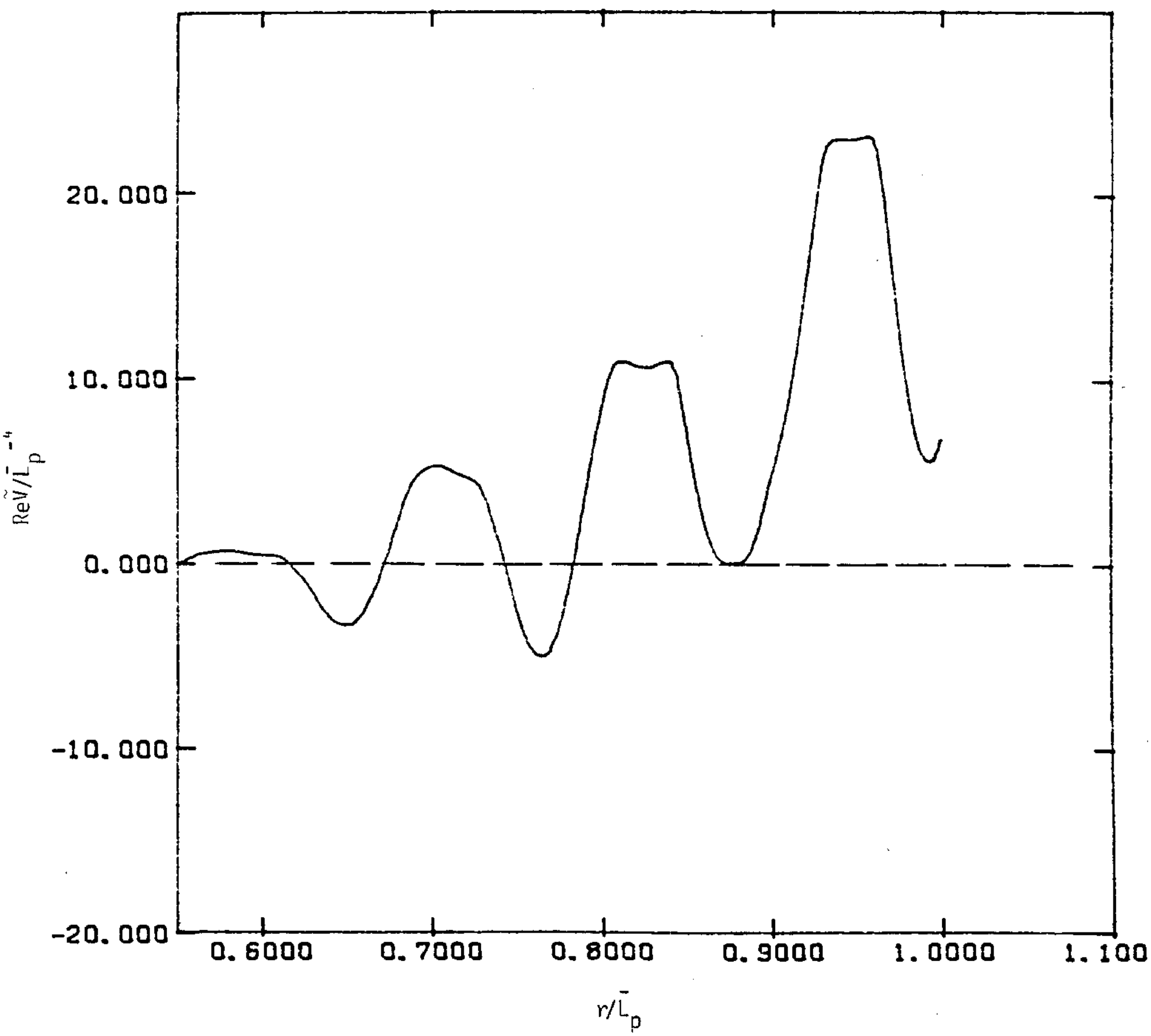
$\mathrm{Re}\tilde{V}$: $N = 3$, $\bar{\Lambda} = 30.077$, $\mathrm{Re}\tilde{V} = \frac{\partial}{\partial r}\mathrm{Re}\tilde{V} = 0$ at $r = 1.2043$

Fig. 3



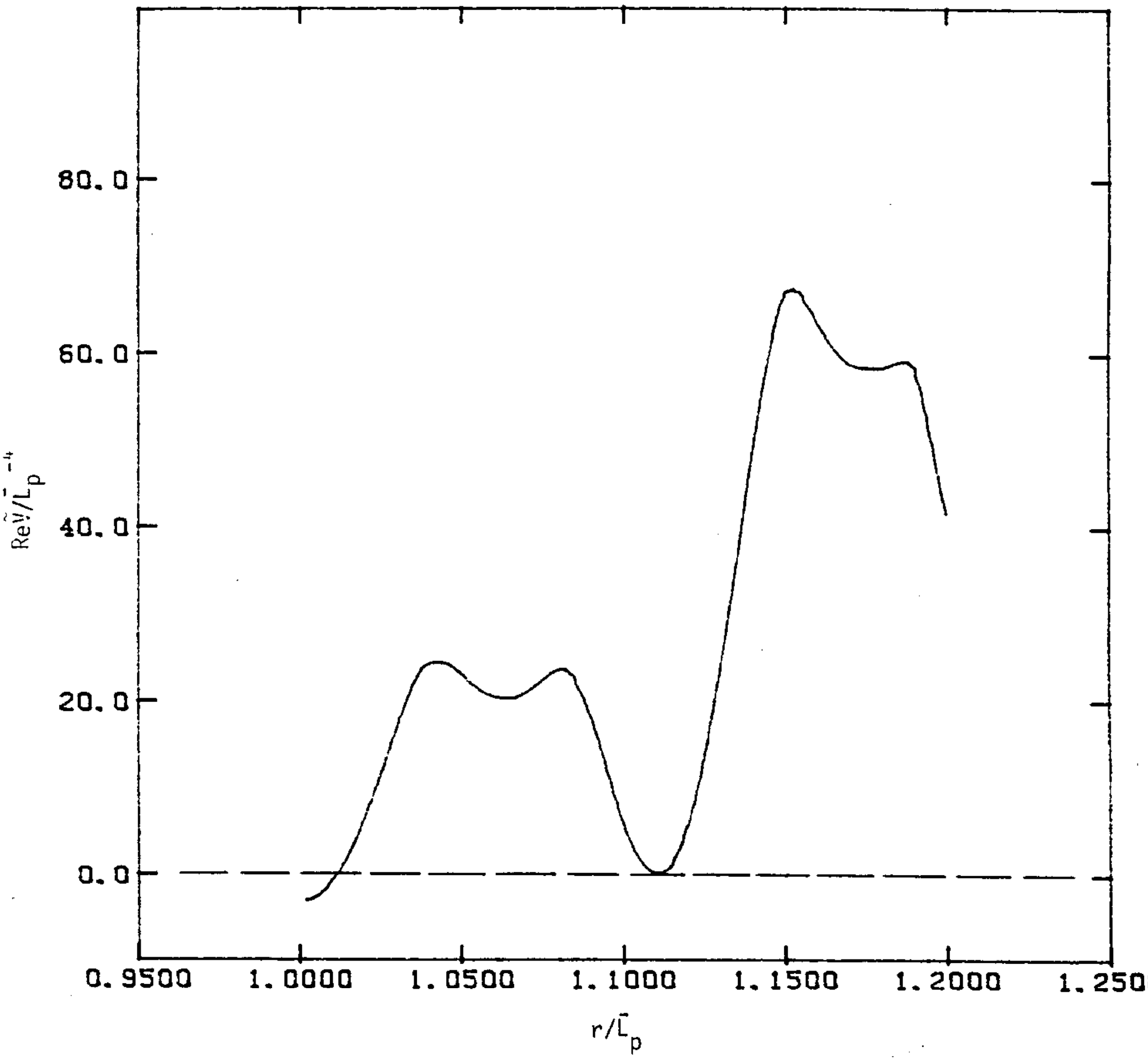
$\text{Re}\tilde{V}$: $N = 5$, $\bar{\Lambda} = 59.447$, $\text{Re}\tilde{V} = \frac{\partial}{\partial r}\text{Re}\tilde{V} = 0$ at $r = 1.1720$ and 1.2920

Fig. 4



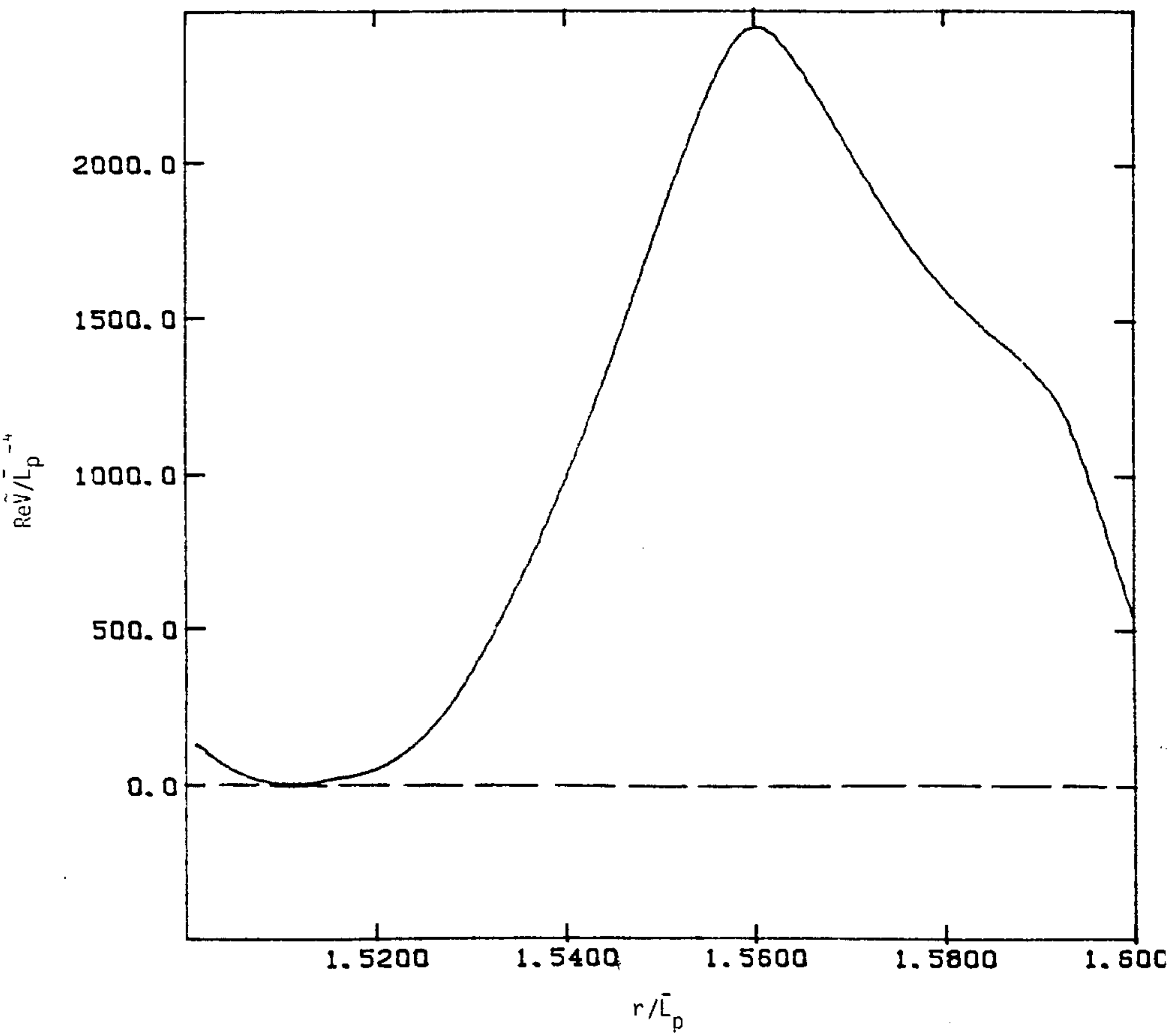
$\mathrm{Re}\tilde{V}$: $N = 5$, $\bar{\Lambda} = 50.750$, $\mathrm{Re}\tilde{V} = \frac{\partial}{\partial r}\mathrm{Re}\tilde{V} = 0$ at $r = 0.87175$

Fig. 5



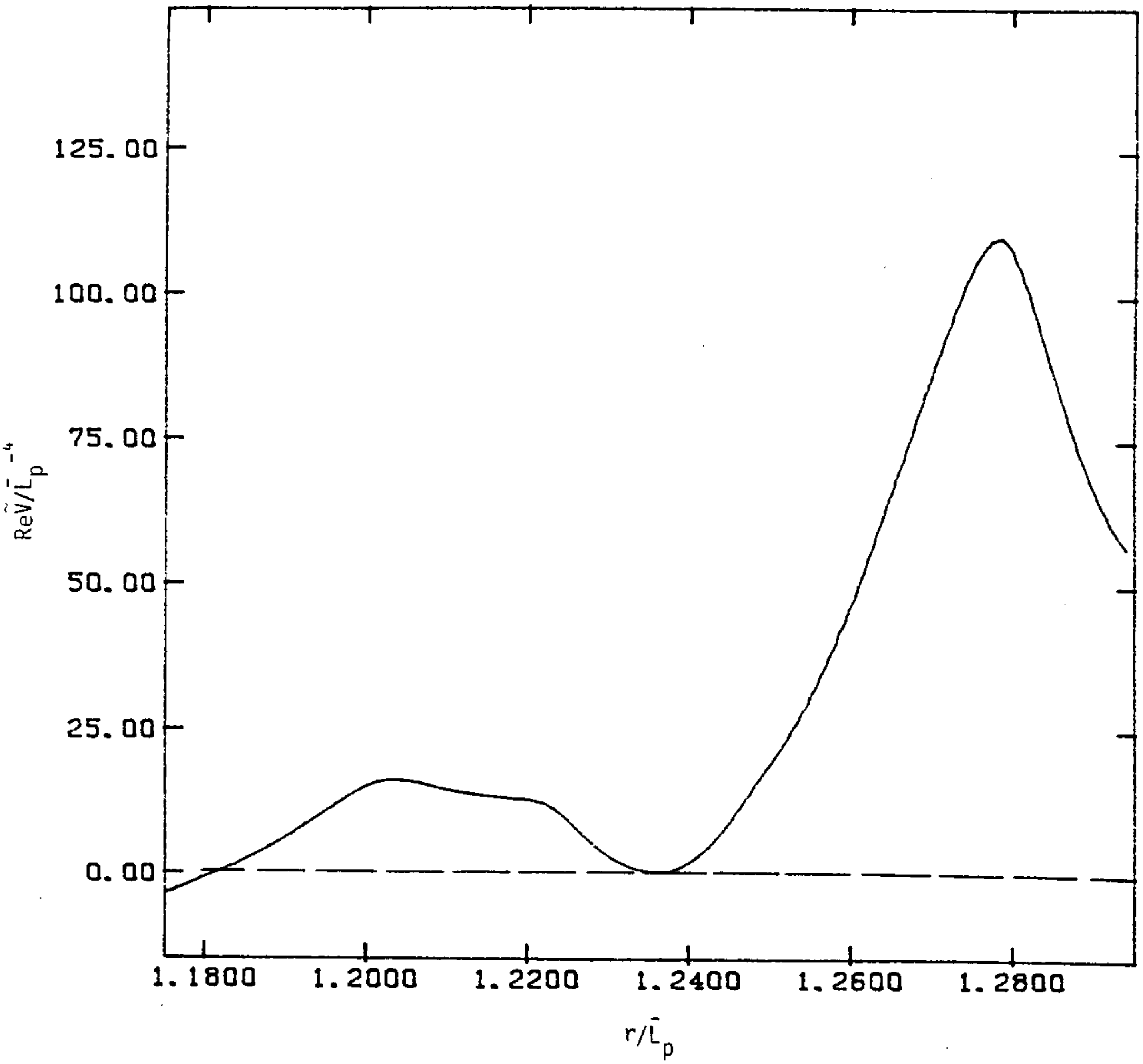
$Re\tilde{V}$: $N = 7$, $\bar{\Lambda} = 52.930$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 1.1100$

Fig. 6



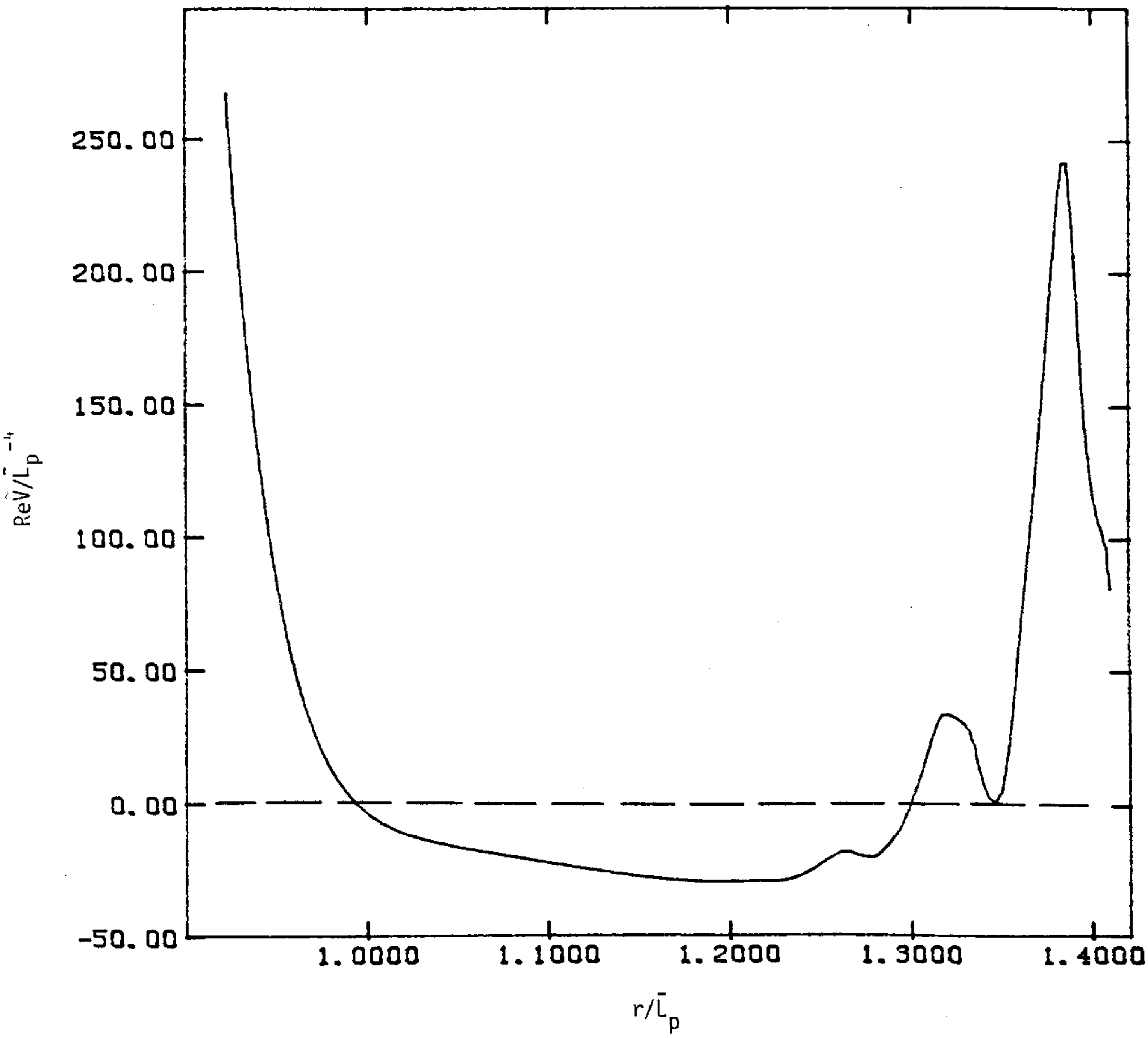
$\mathrm{Re}\tilde{V}$: $N = 9$, $\bar{\Lambda} = 66.210$, $\mathrm{Re}\tilde{V} = \frac{\partial}{\partial r}\mathrm{Re}\tilde{V} = 0$ at $r = 1.5100$

Fig. 7



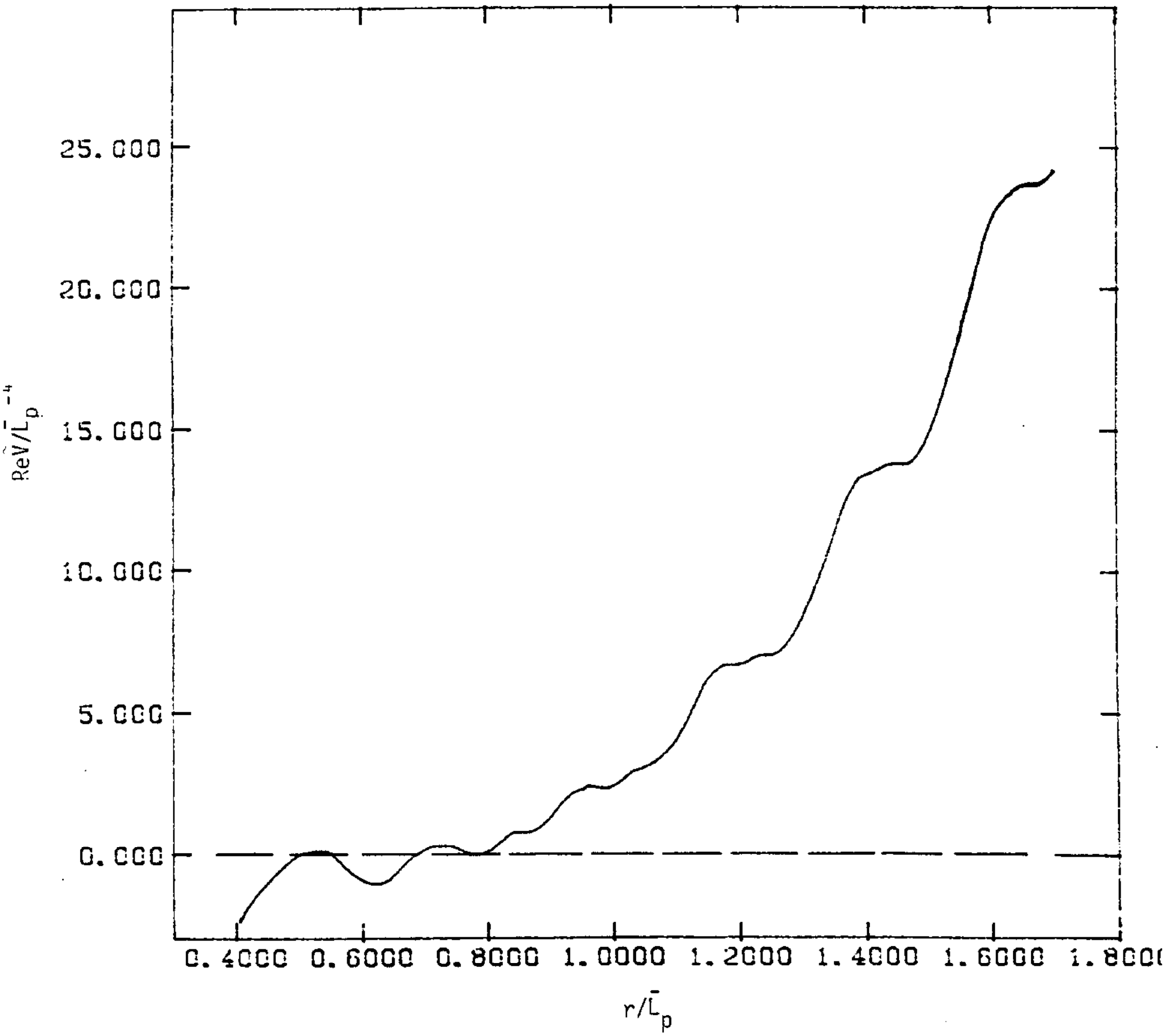
$Re\tilde{V}$: $N = 11$, $\bar{\Lambda} = 78.828$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 1.2352$

Fig. 8



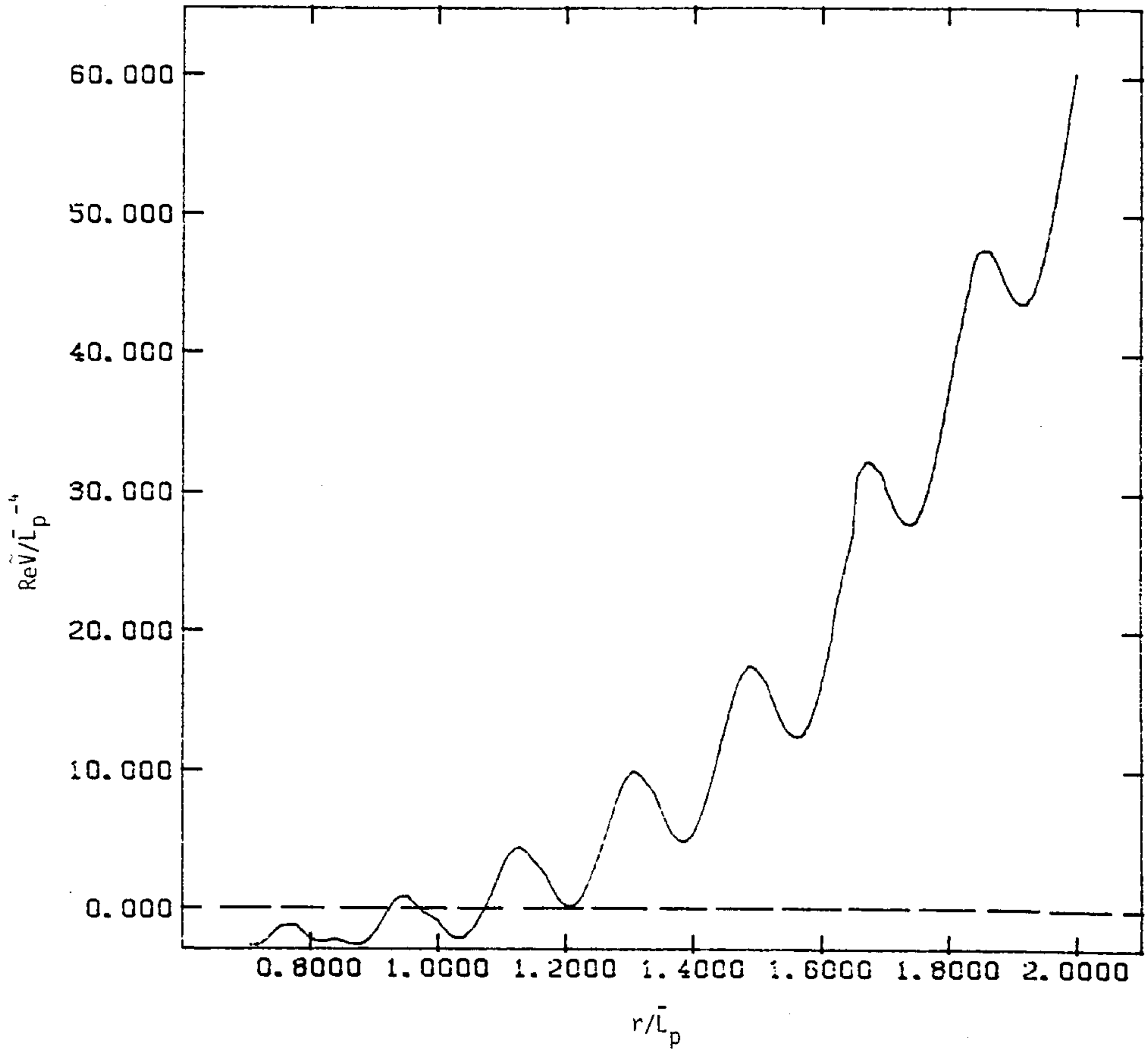
$Re\tilde{V}$: $N = 13$, $\bar{\Lambda} = 92.320$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 1.3440$

Fig. 9



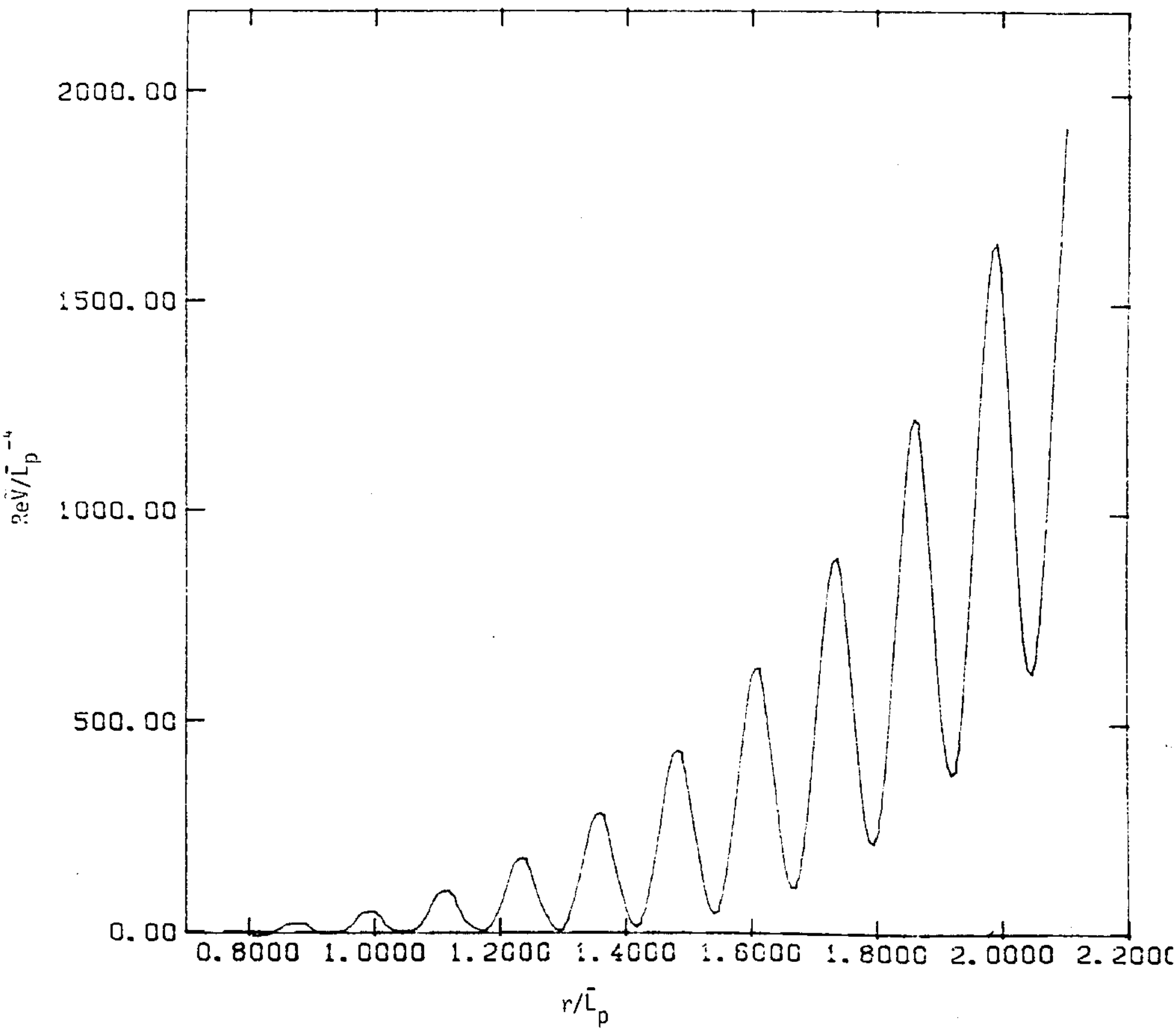
$Re\tilde{V}$: $N = 3$, $\bar{\Lambda} = 19.800$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 0.70000$

Fig. 10



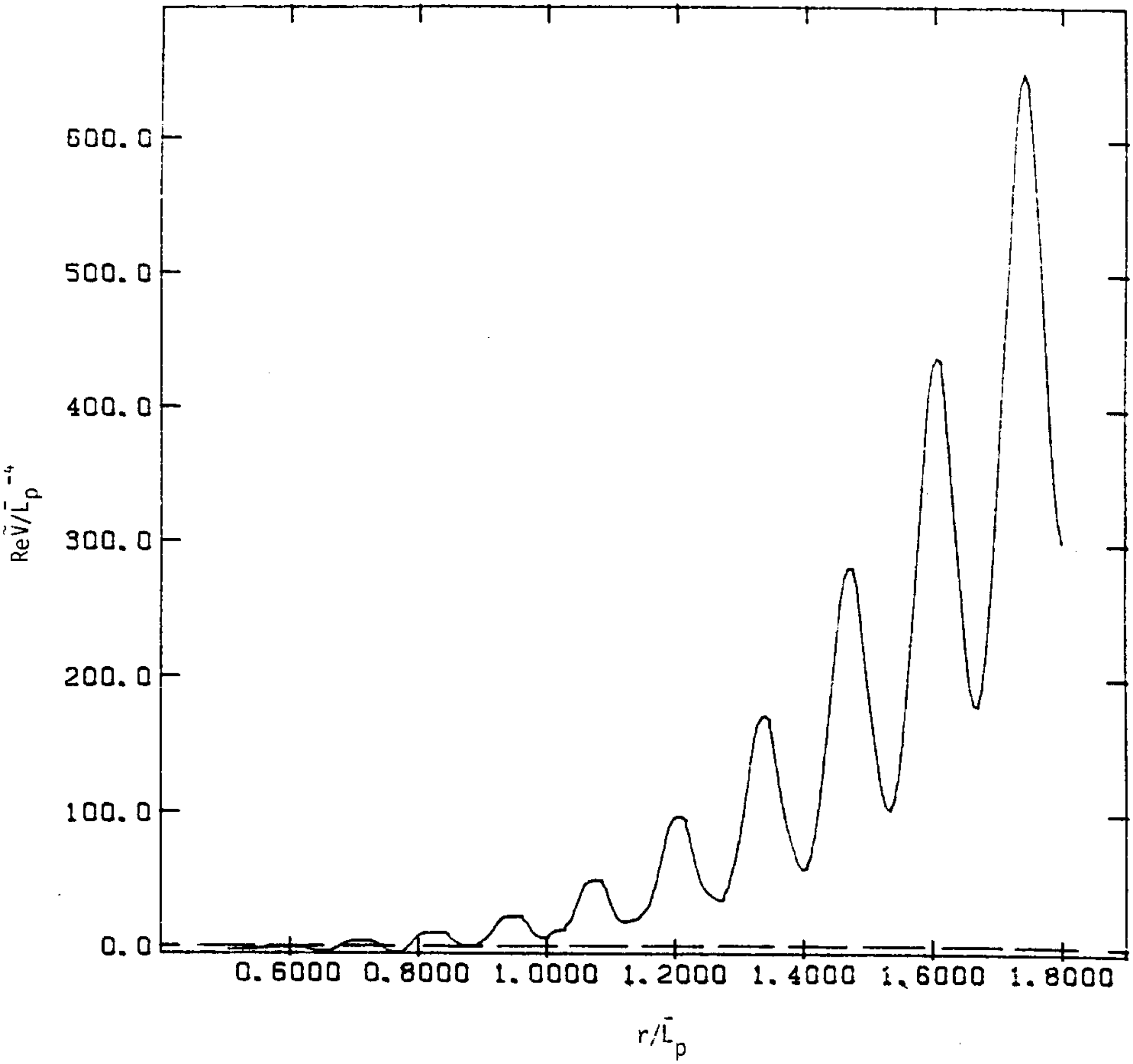
$Re\tilde{V}$: $N = 3$, $\bar{\Lambda} = 30.077$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$, at $r = 1.2043$

Fig. 11



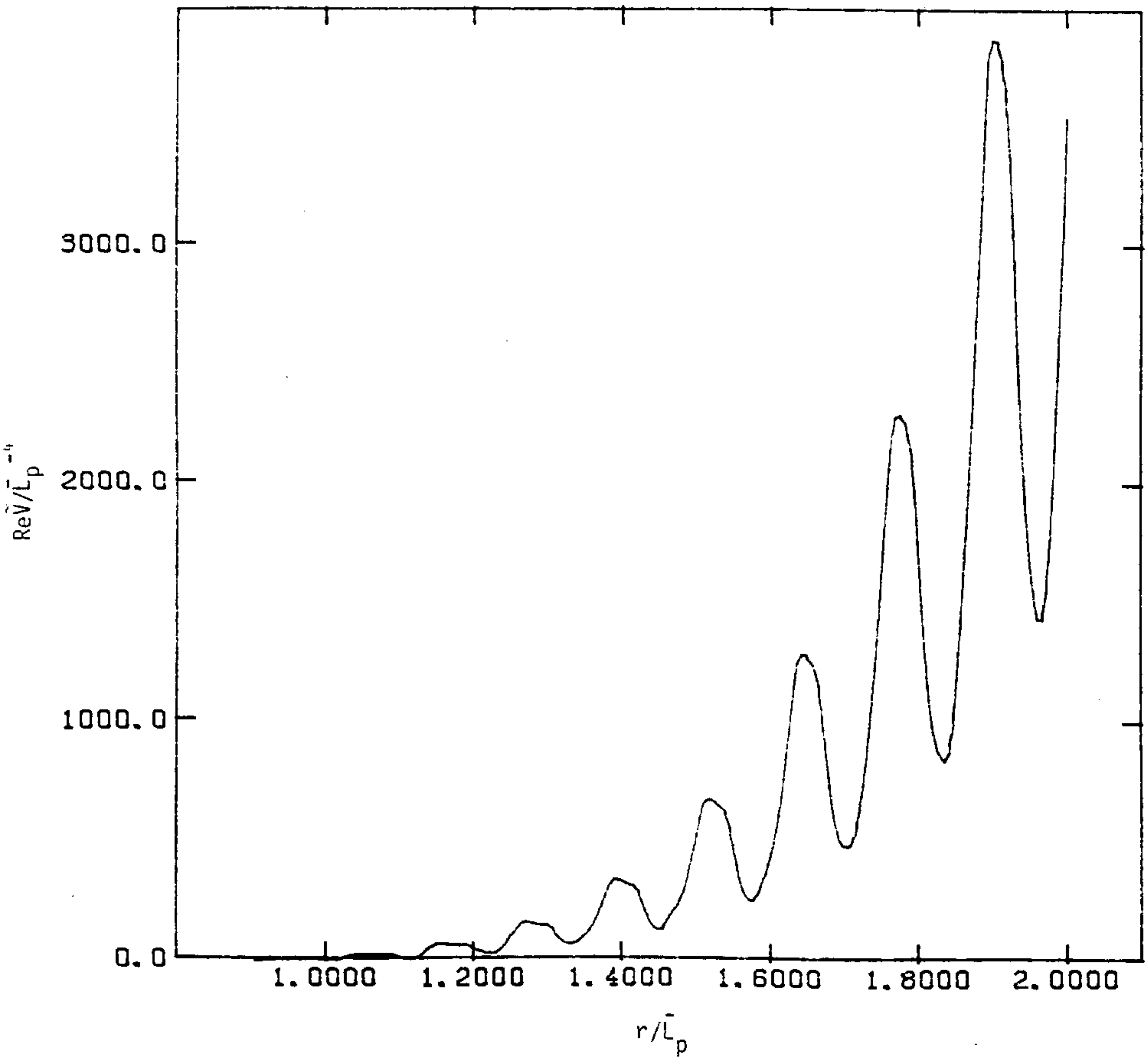
$\mathrm{Re}\tilde{V}$: $N = 5$, $\bar{\Lambda} = 59.447$, $\mathrm{Re}\tilde{V} = \frac{\partial}{\partial r}\mathrm{Re}\tilde{V} = 0$ at $r = 1.1720$ and 1.2920

Fig. 12



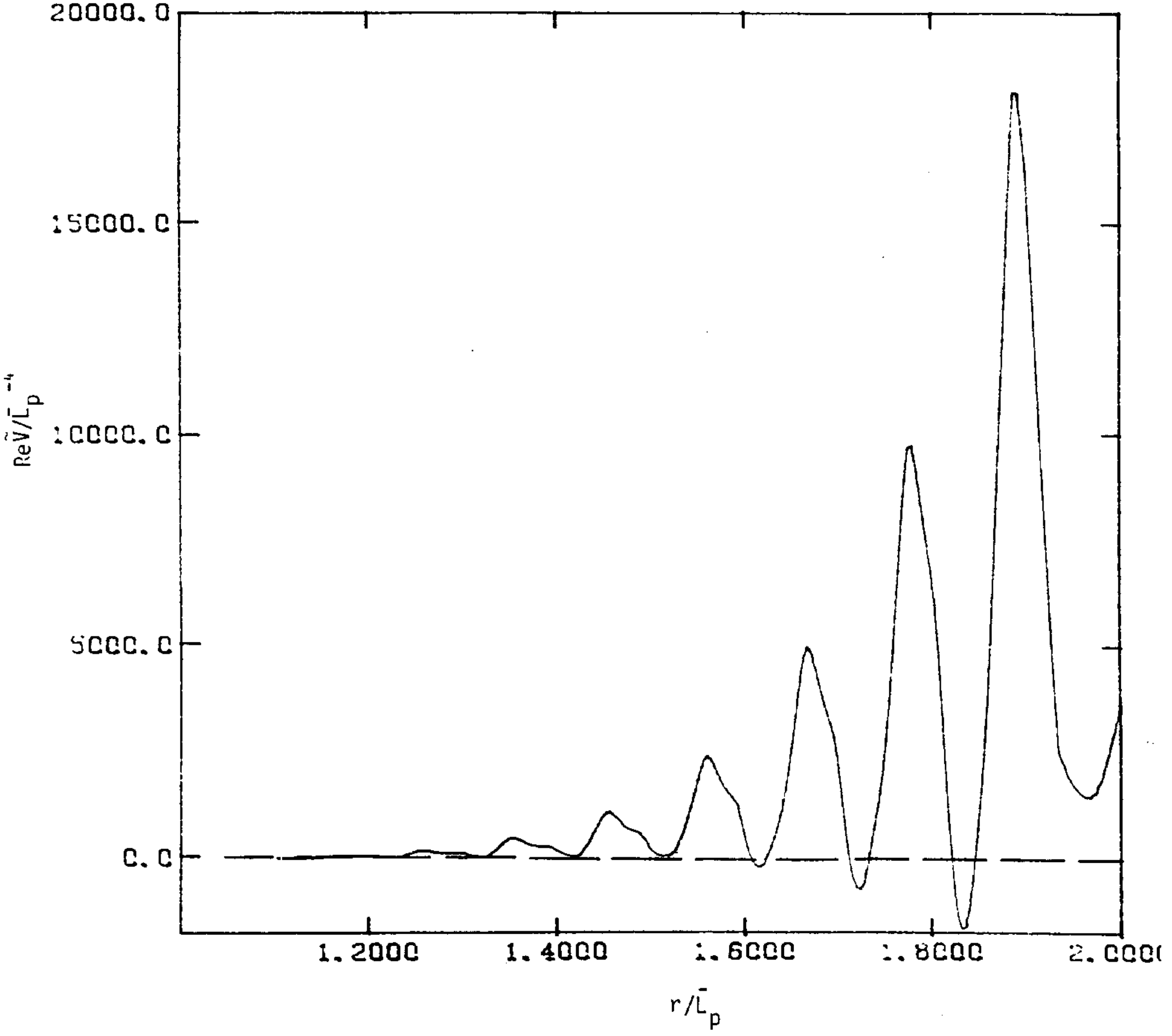
$\mathrm{Re}\tilde{V}$: $N = 5$, $\bar{\Lambda} = 50.750$, $\mathrm{Re}\tilde{V} = \frac{\partial}{\partial r}\mathrm{Re}\tilde{V} = 0$ at $r = 0.87175$

Fig. 13



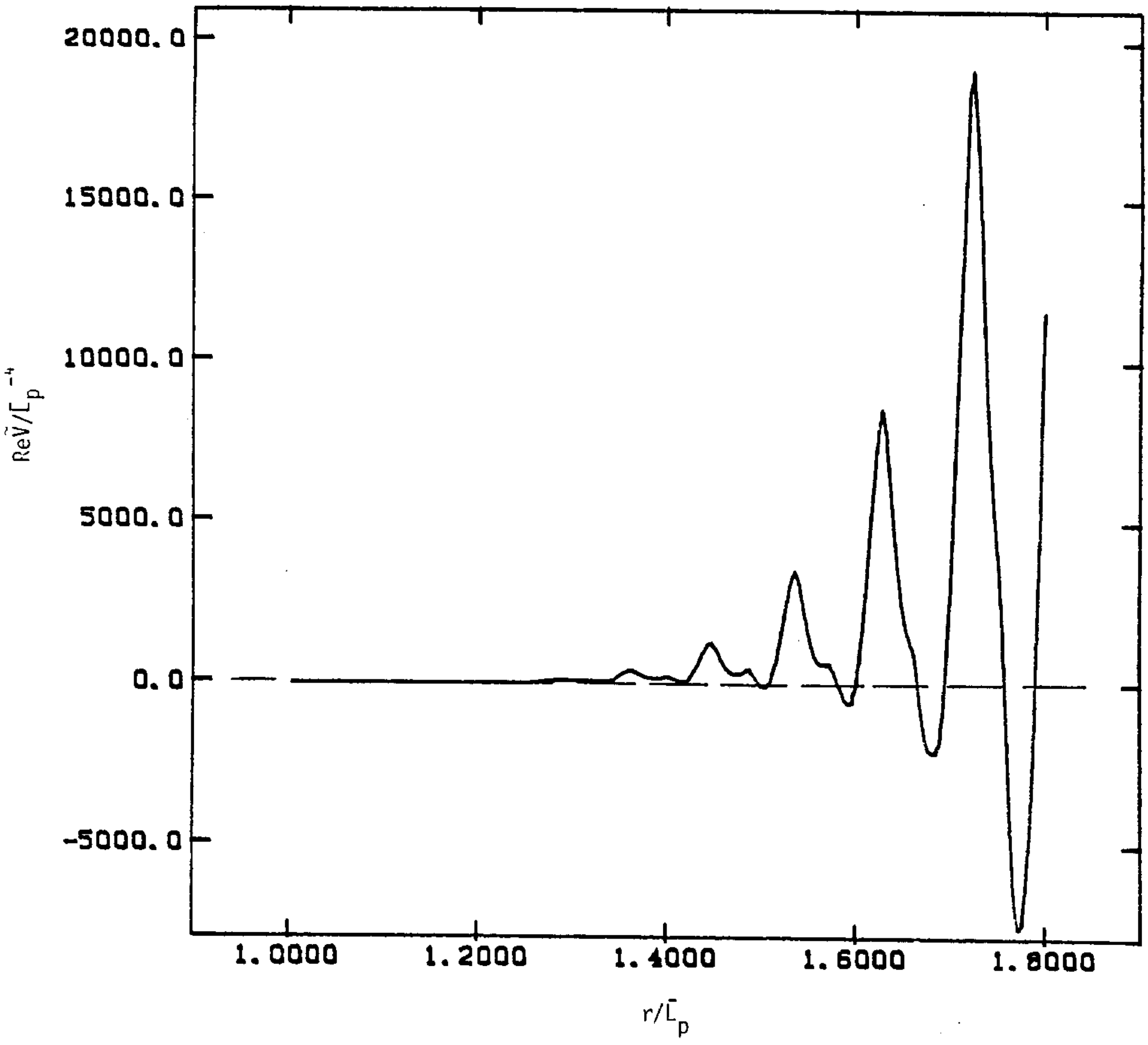
$Re\tilde{V}$: $N = 7$, $\bar{\Lambda} = 52.930$. $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at 1.1100

Fig. 14



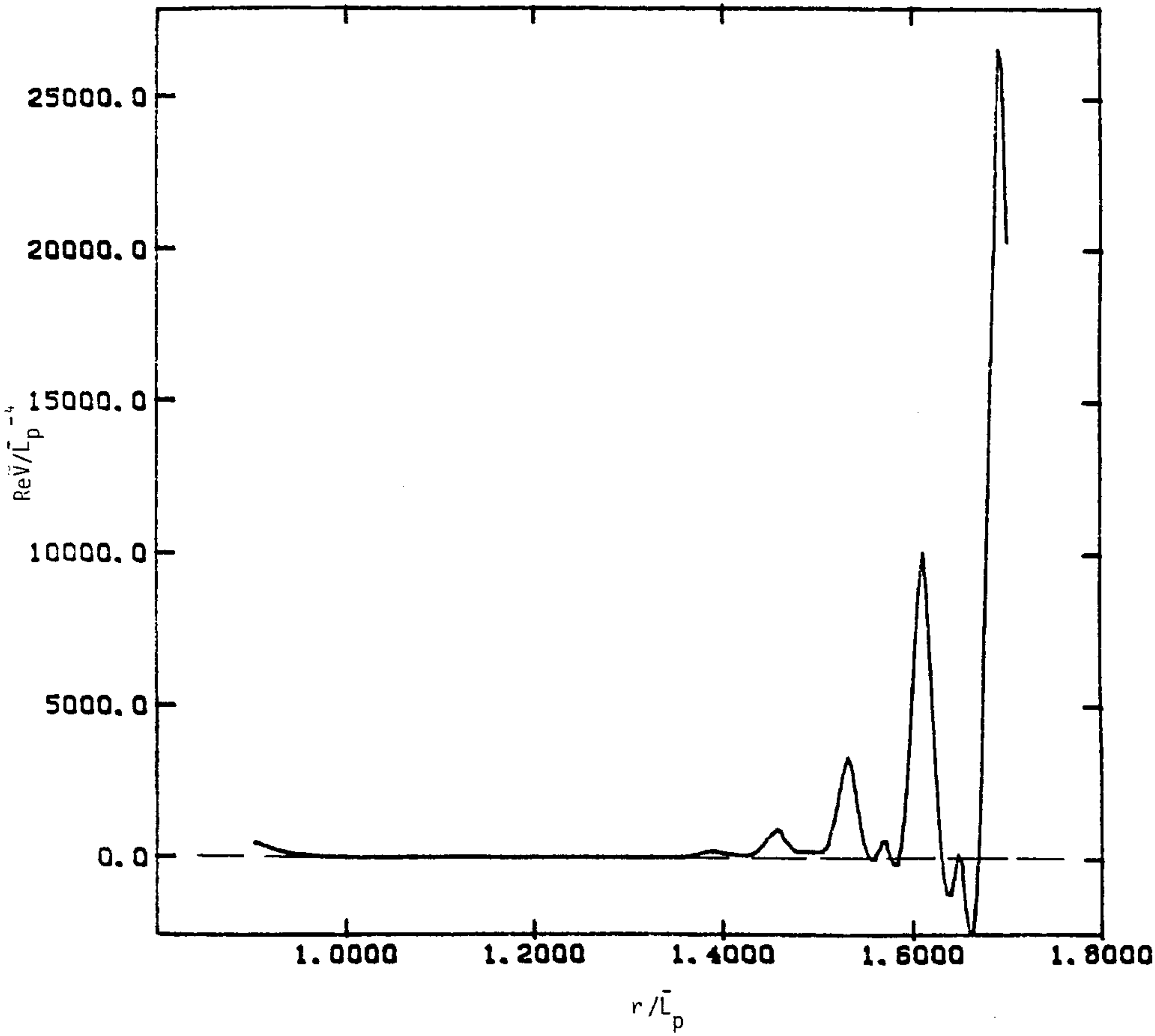
$Re\tilde{V}$: $N = 9$, $\bar{\Lambda} = 66.210$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 1.5100$

Fig. 15



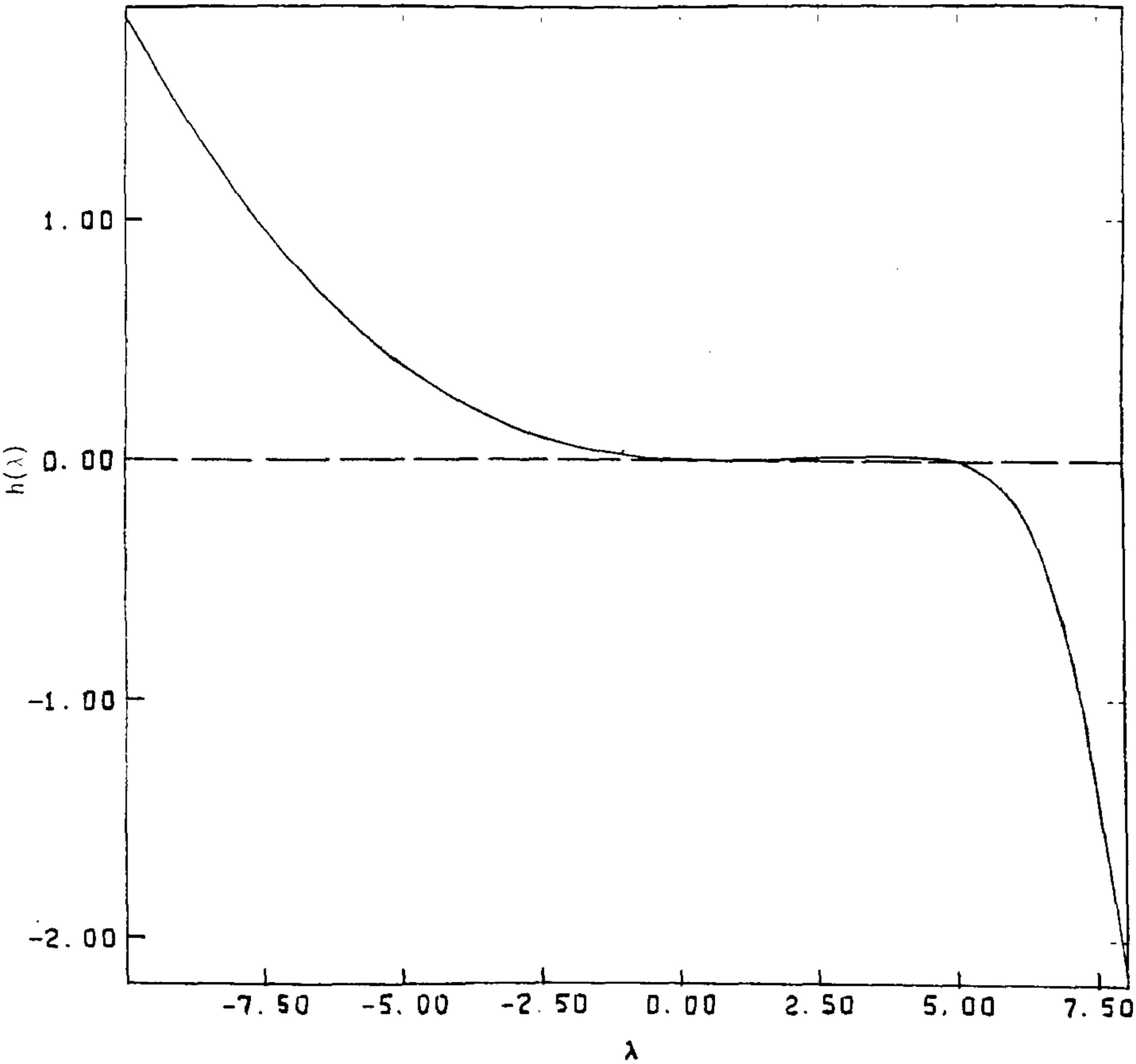
$Re\tilde{V}$: $N = 11$, $\bar{\Lambda} = 78.828$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$ at $r = 1.2352$

Fig. 16



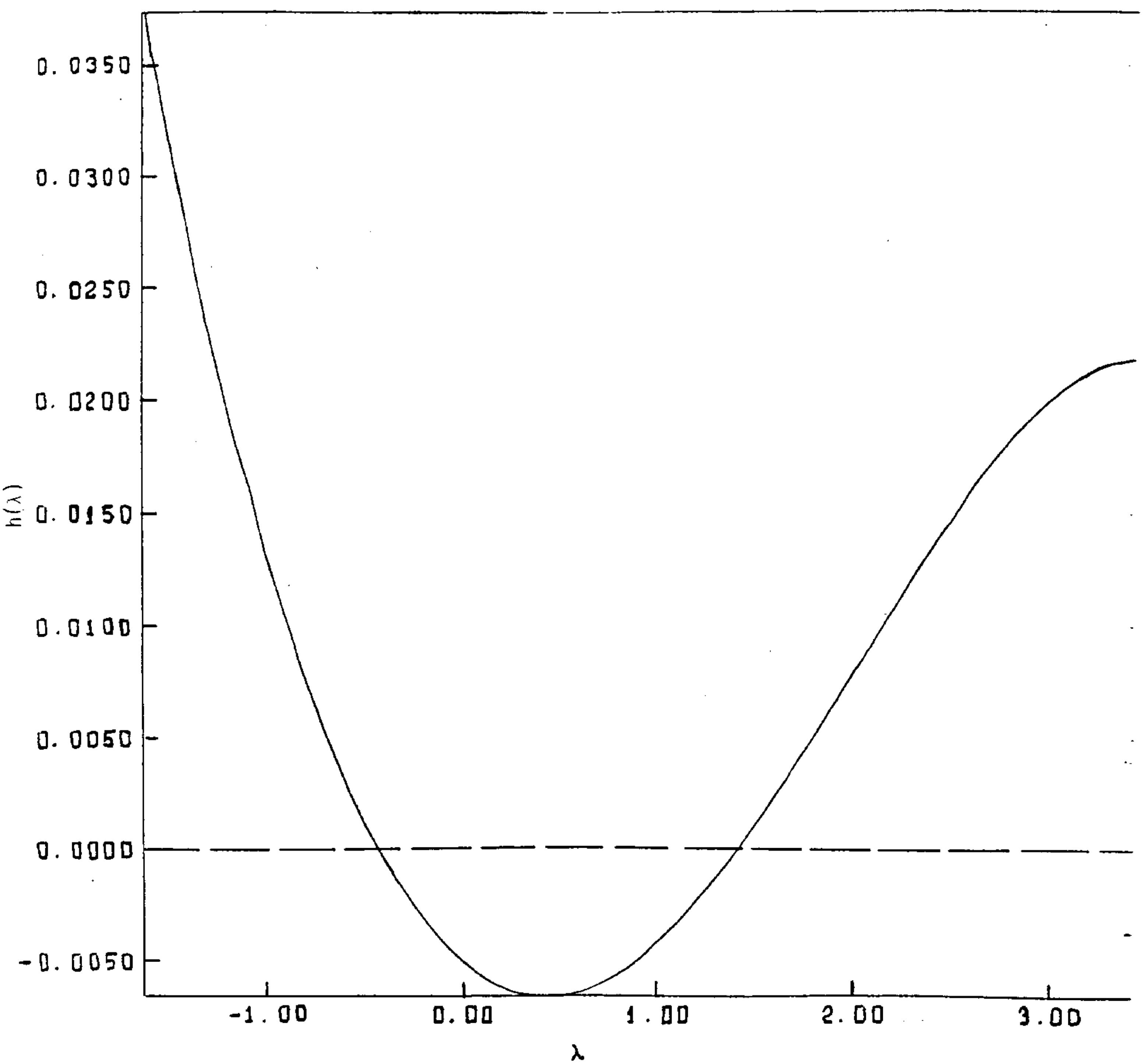
ReṼ: N = 1 , $\bar{\Lambda} = 92.320$, $Re\tilde{V} = \frac{\partial}{\partial r} Re\tilde{V} = 0$, at $r = 1.3440$

Fig. 17



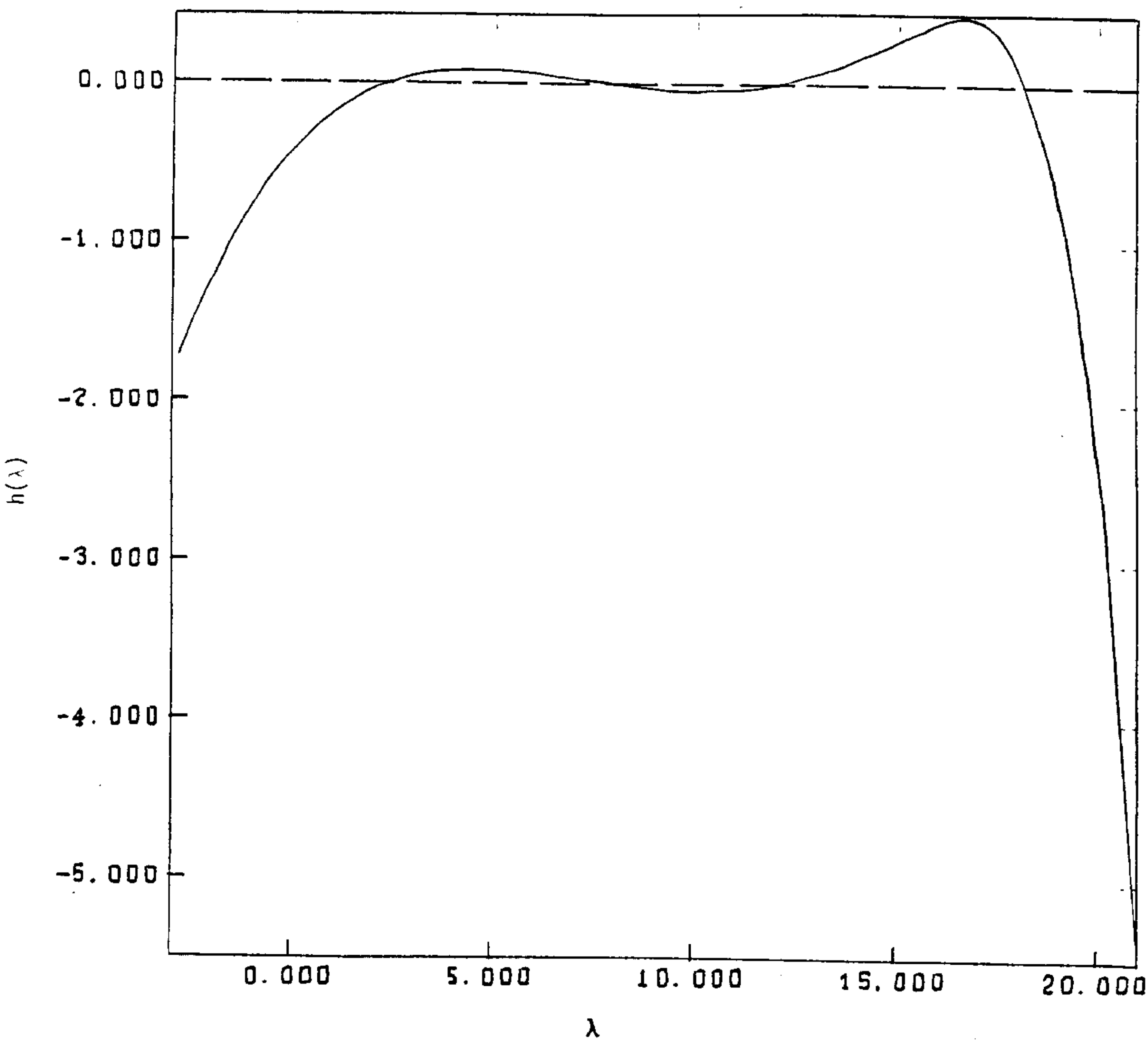
$h(\lambda) = r^4 \times \mathrm{Im}\ \tilde{V}_0$, N = 3.

Fig. 17a



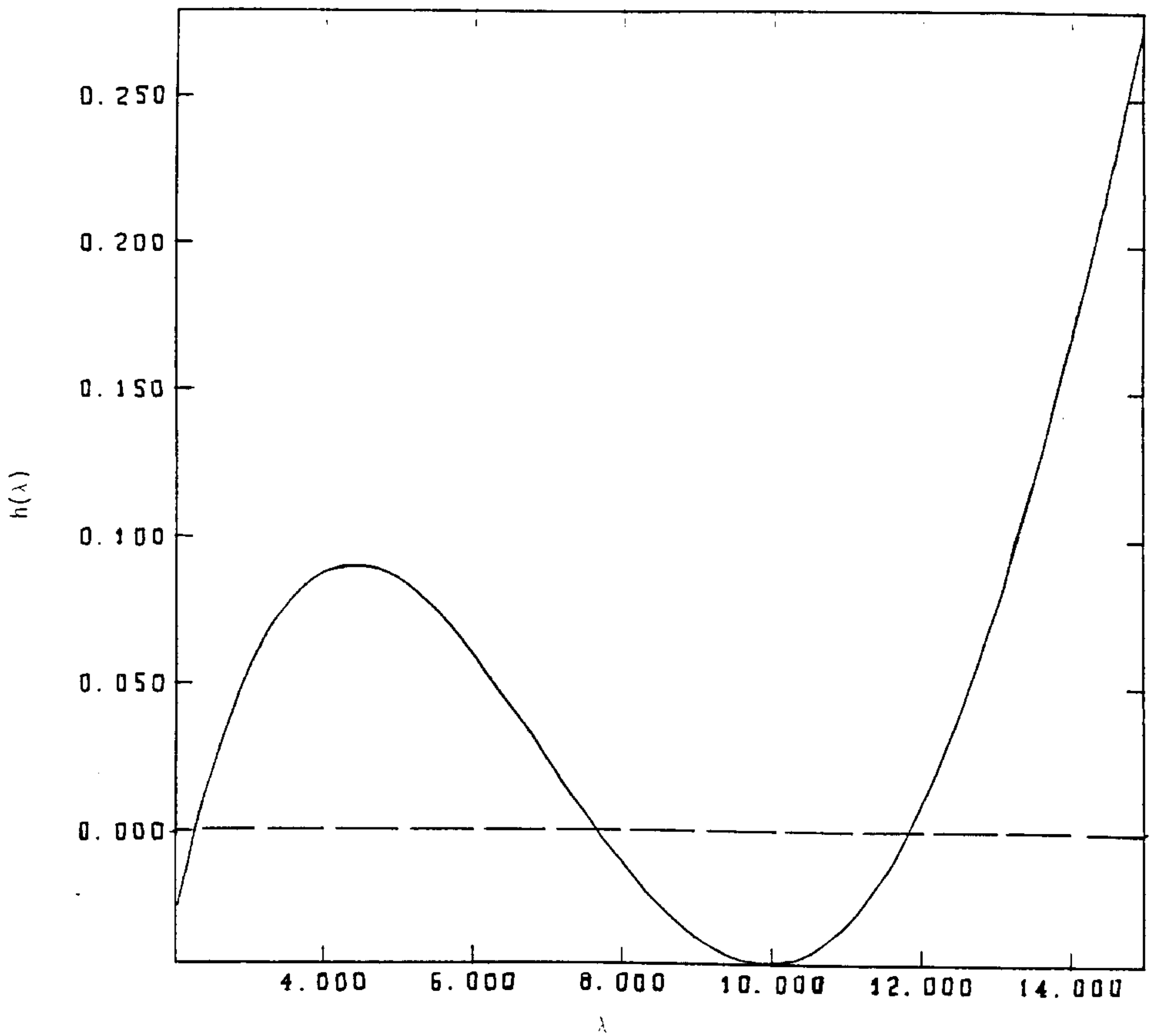
$h(\lambda) = r^4 \times \mathrm{Im}\ \check{V}_{\Omega}$, N = 3

Fig. 19



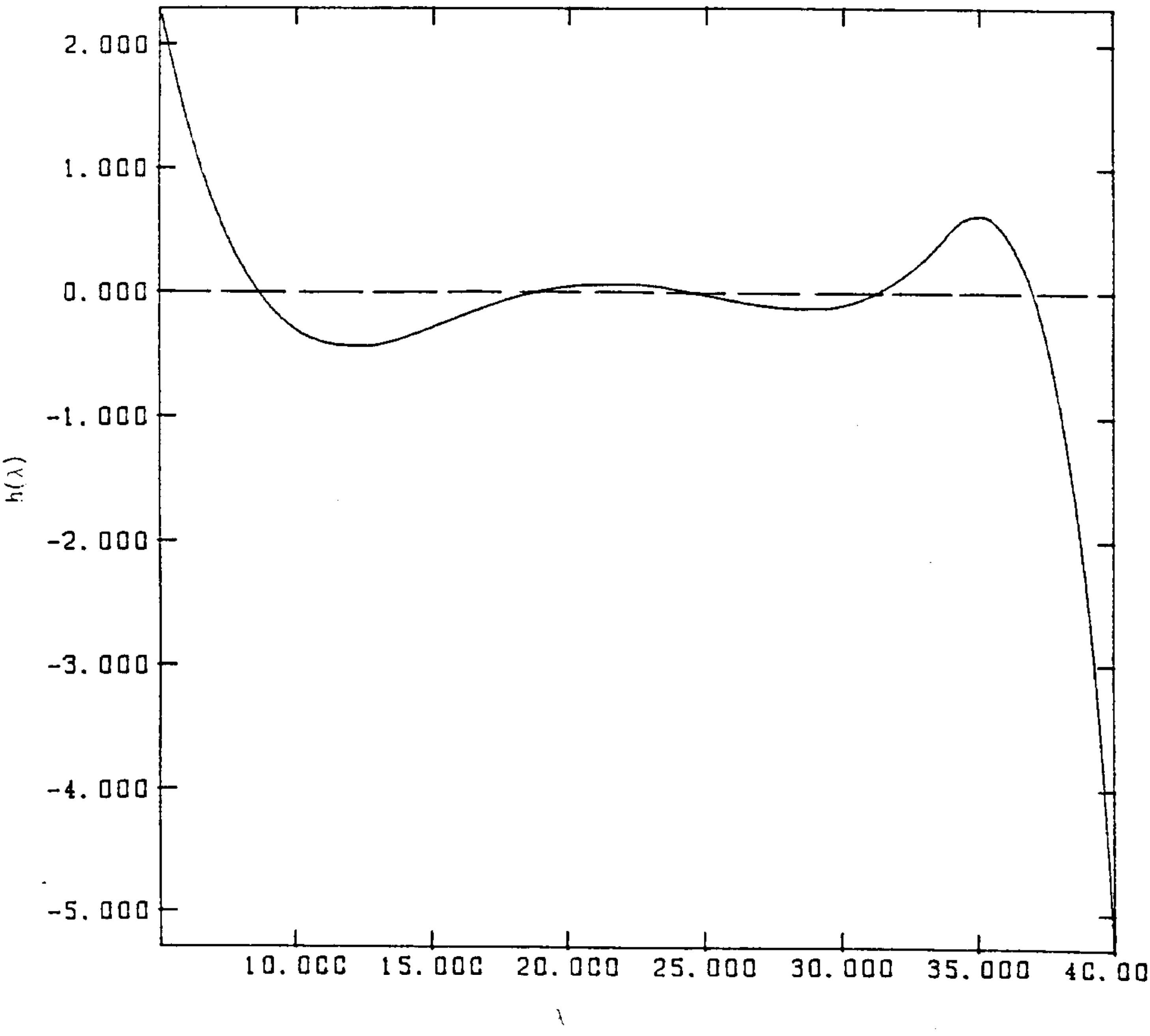
$h(\lambda) = r^4 \times \text{Im}\, \tilde{V}_0$, $N = 5$

Fig. 18a



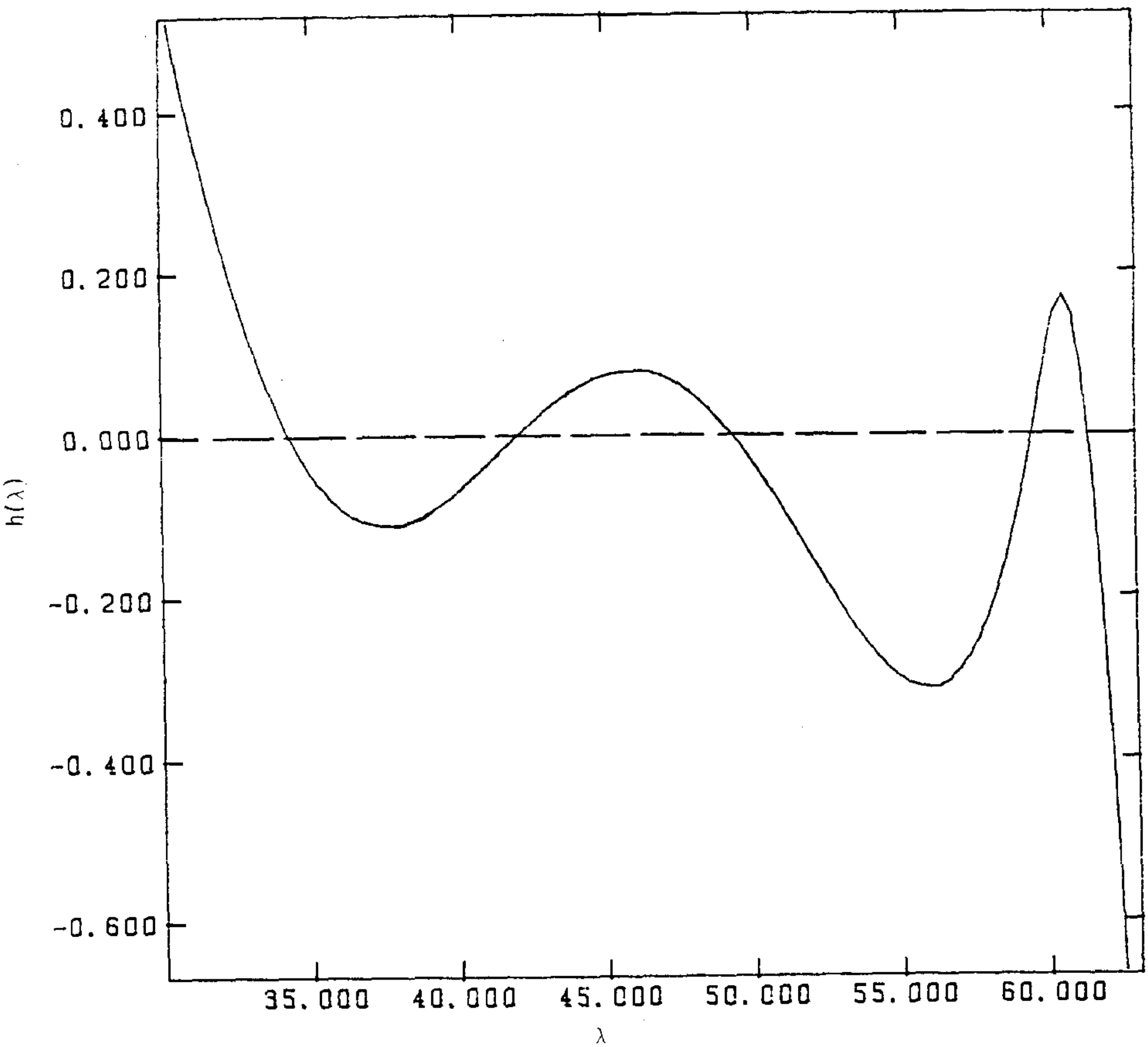
$h(\lambda) = r^4 \times \mathrm{Im}\, \tilde{V}_0$, $N = 5$

Fig. 19



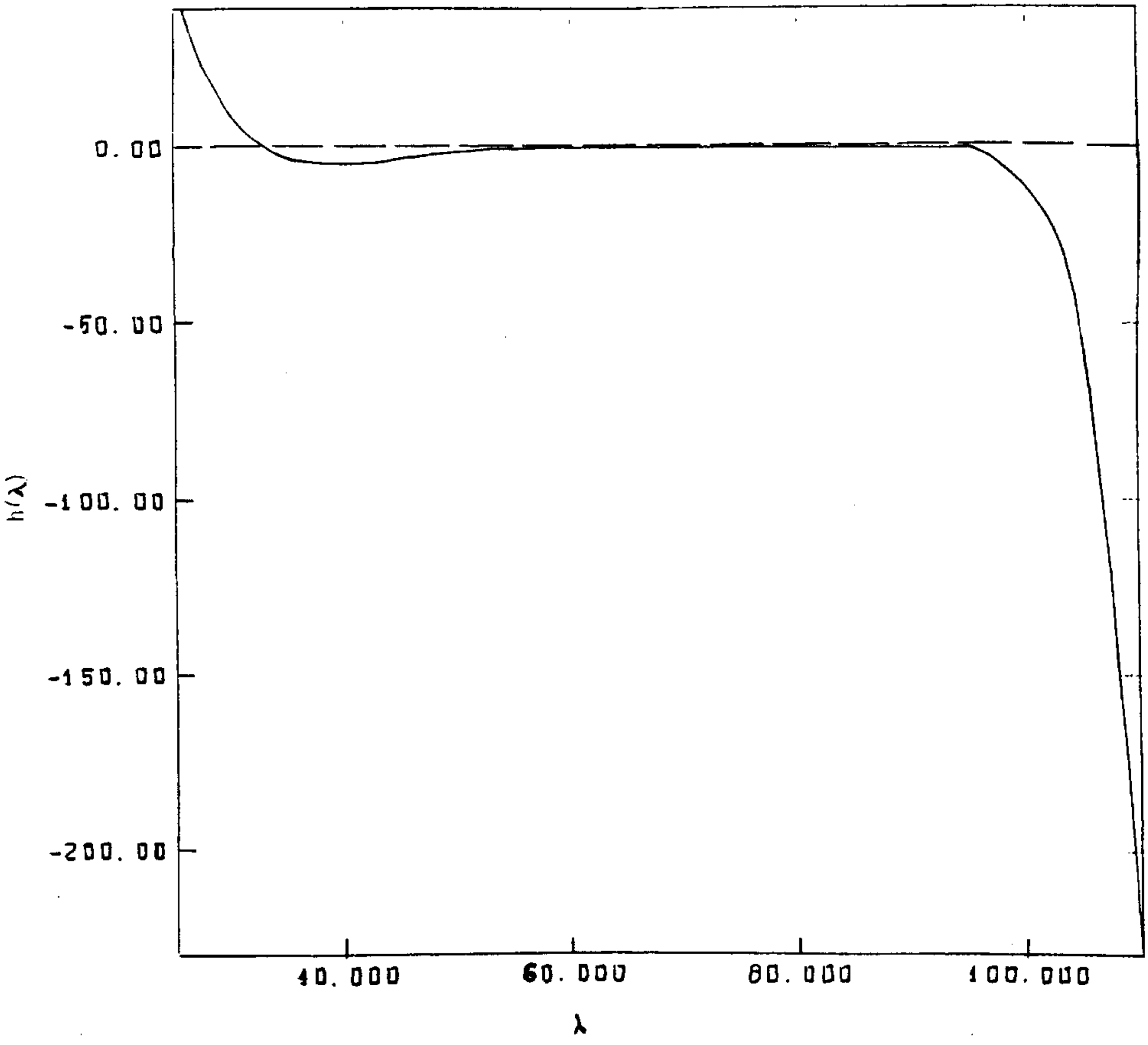
$h(\lambda) = r^4 \times \mathrm{Im}\ \tilde{V}_Q$, $N = 7$

Fig. 20



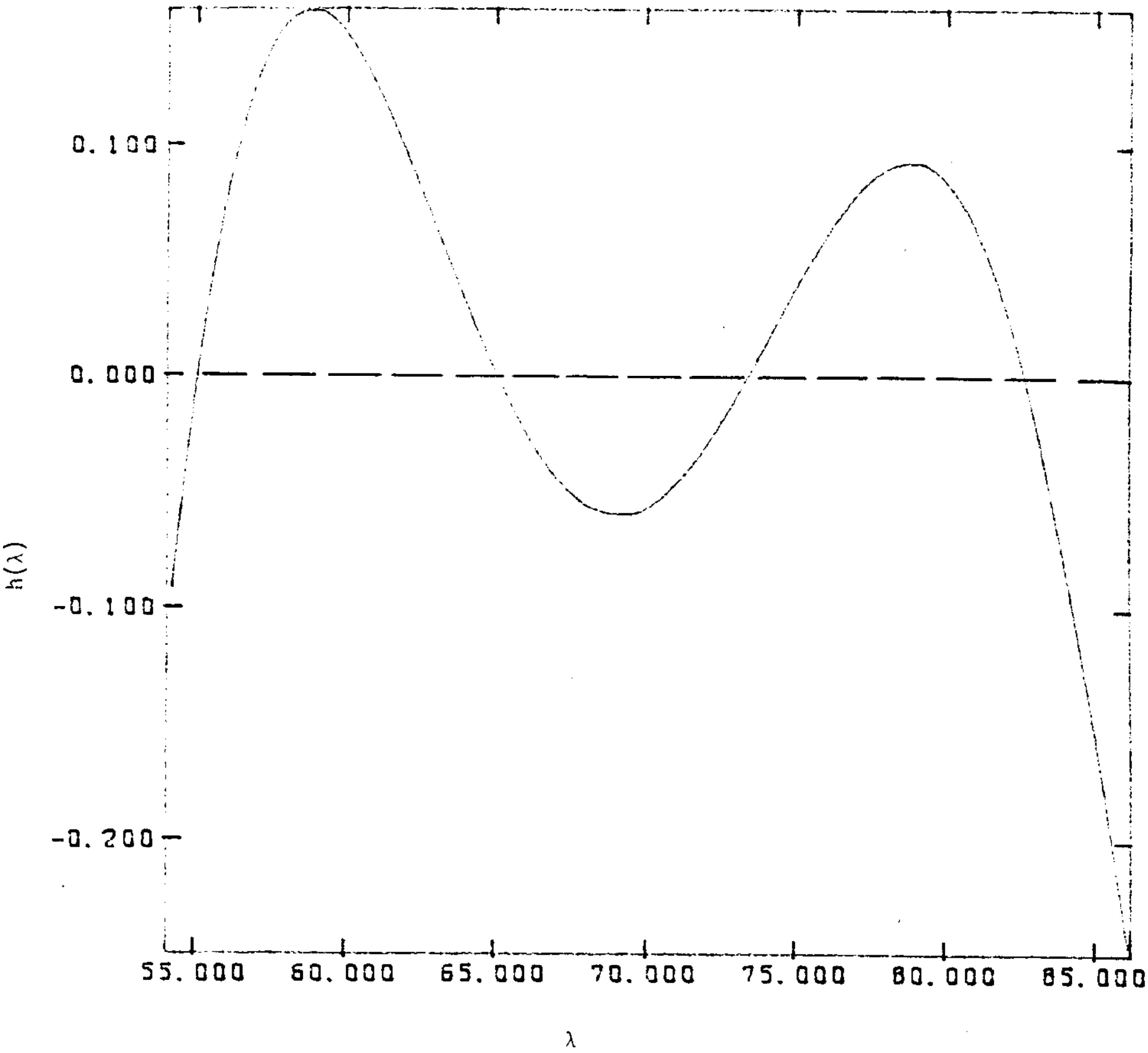
$h(\lambda) = r^4 \times \mathrm{Im}\ \check{V}_0$, $N = 9$

Fig. 21



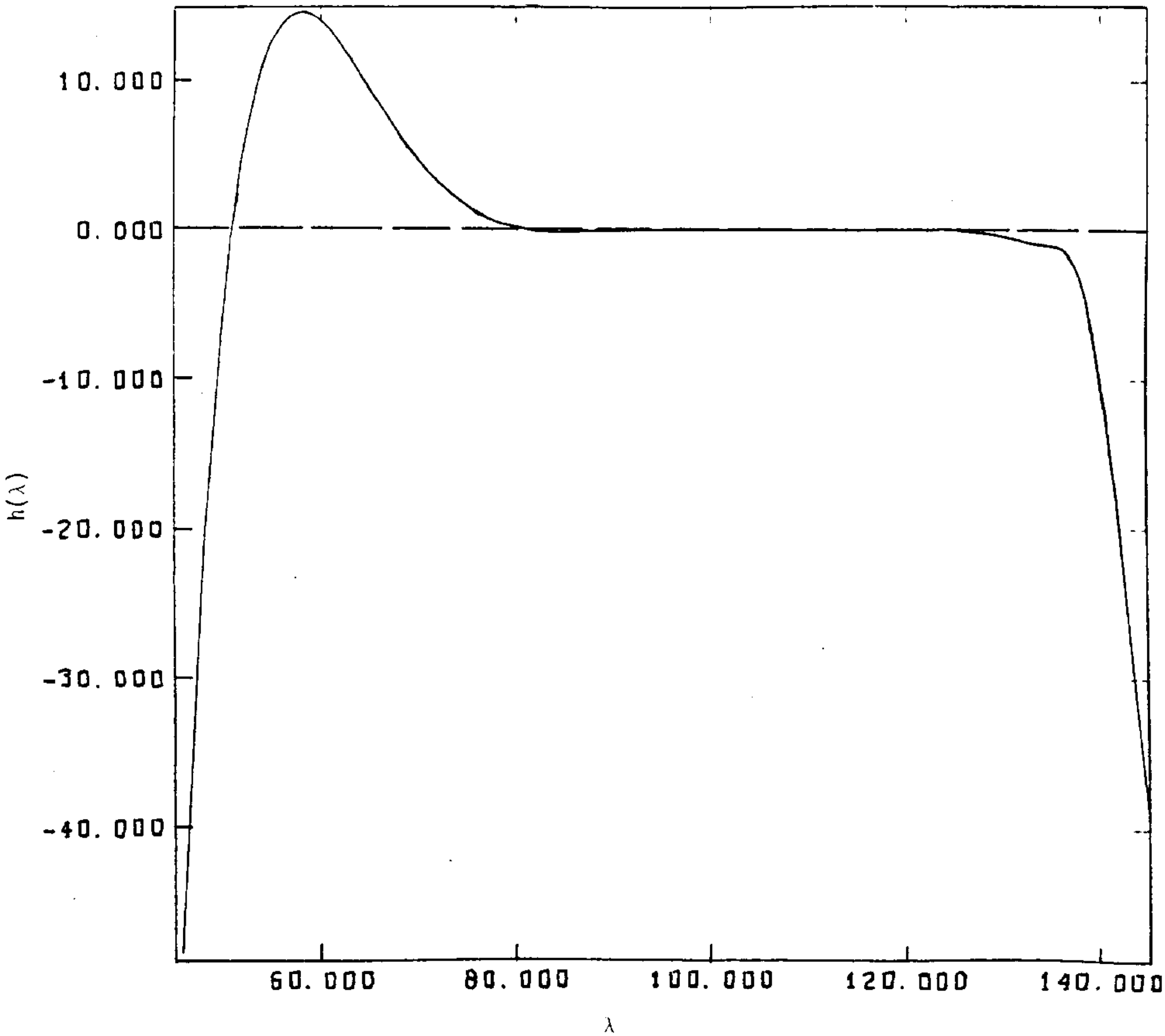
$h(\lambda) = r^4 \times \text{Im}\ \tilde{V}_0$, N = 11

Fig. 21a



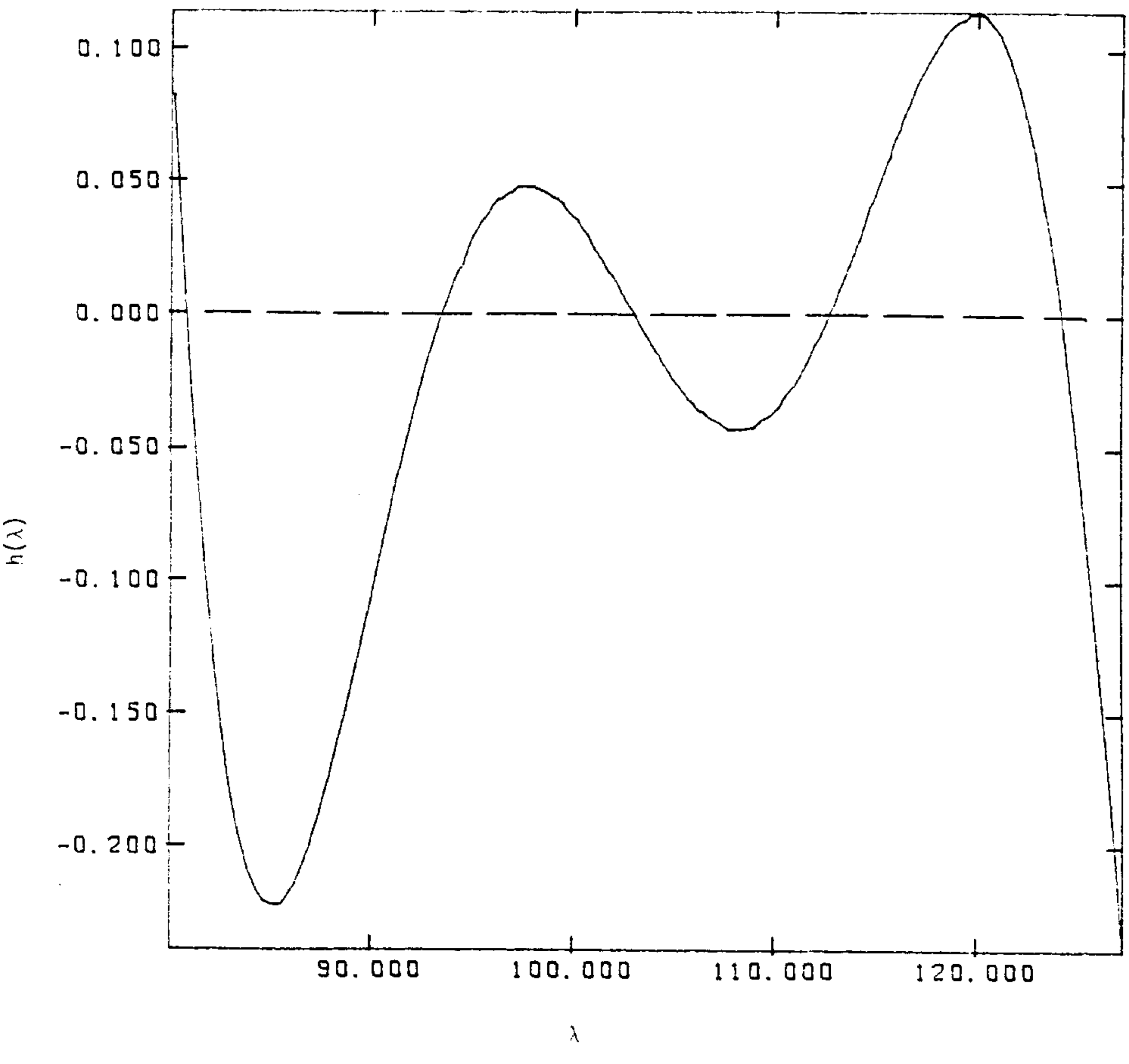
$h(\lambda) = r^4 \times \mathrm{Im}\tilde{V}_0$, $N = 11$

Fig. 22



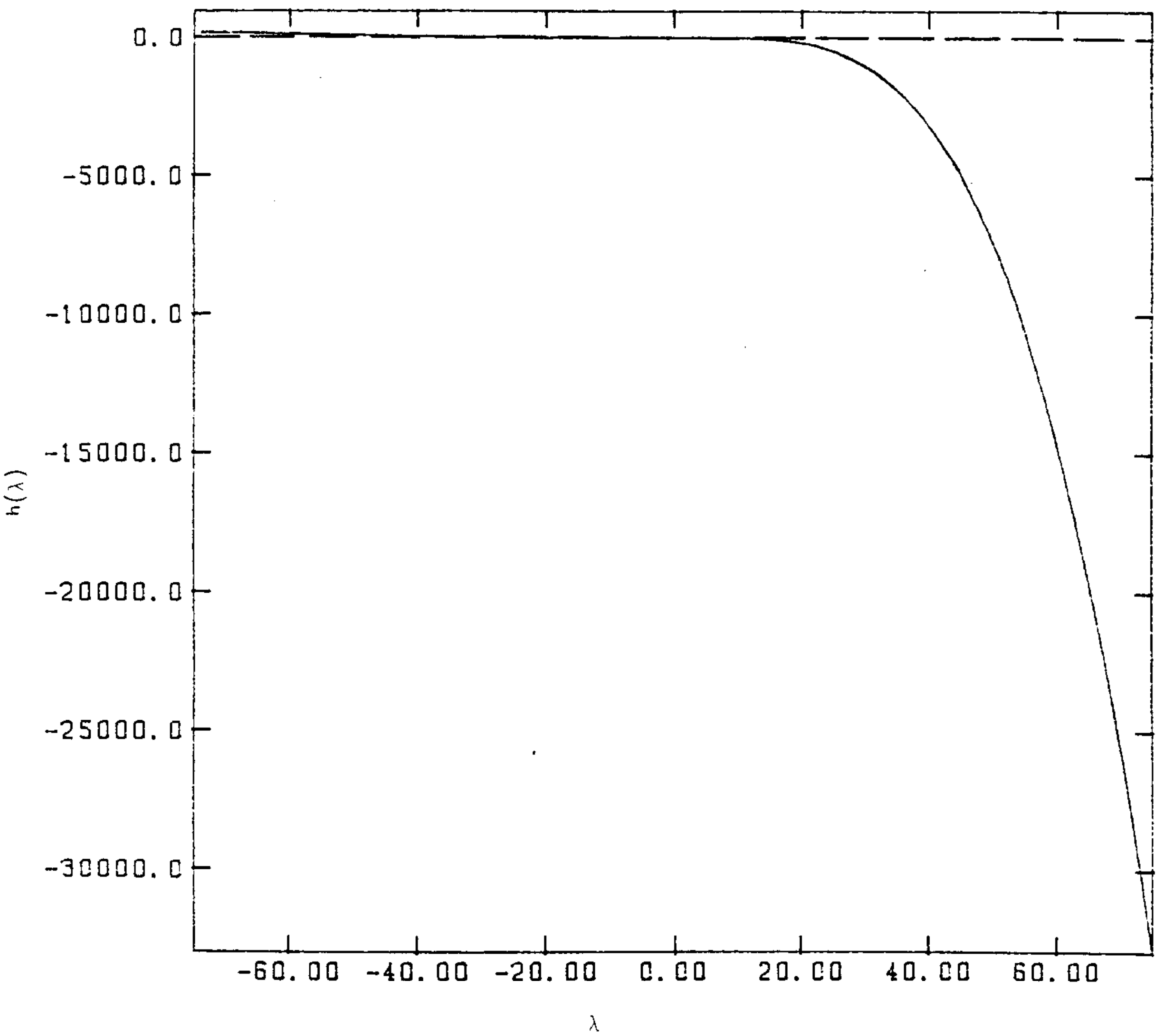
$h(\lambda) = r^4 \times \mathrm{Im}\, \tilde{V}_0$, N = 13

Fig. 22a



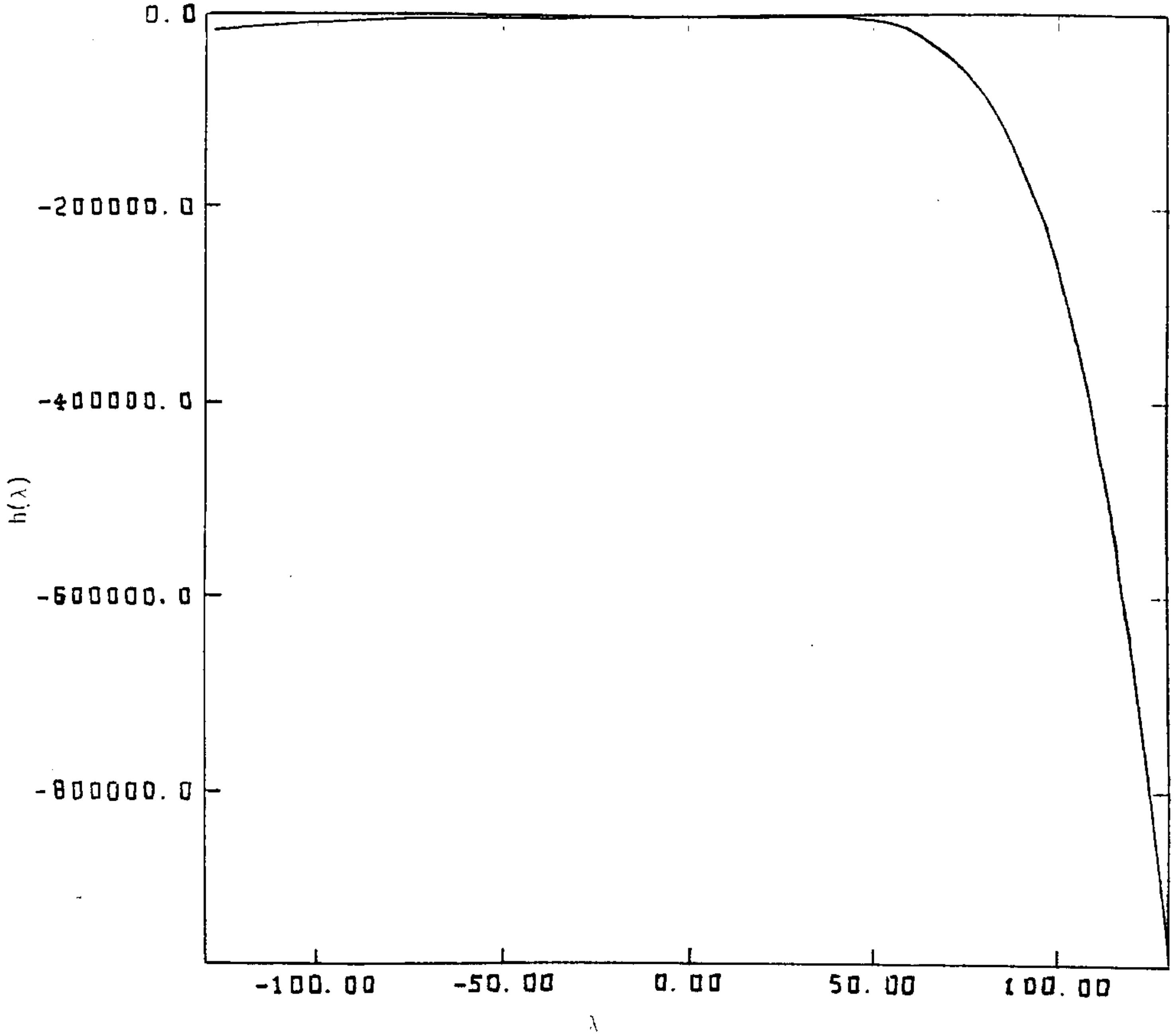
$h(\lambda) = r^4 \times \mathrm{Im}\tilde{V}_0$, N = 13

Fig. 23



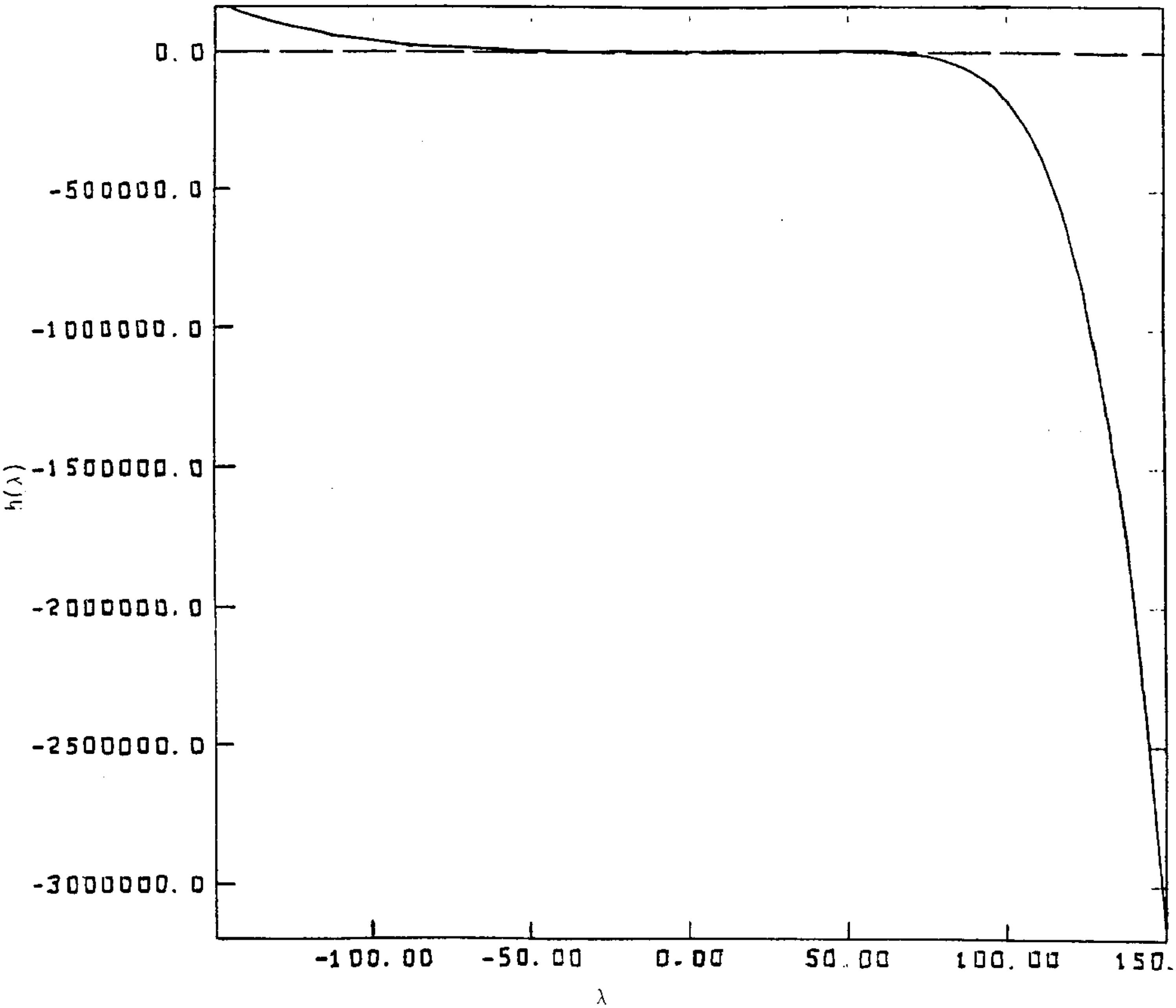
$h(\lambda) = r^4 \times \text{Im}\, \tilde{V}_0$, $N = 3$

Fig. 24



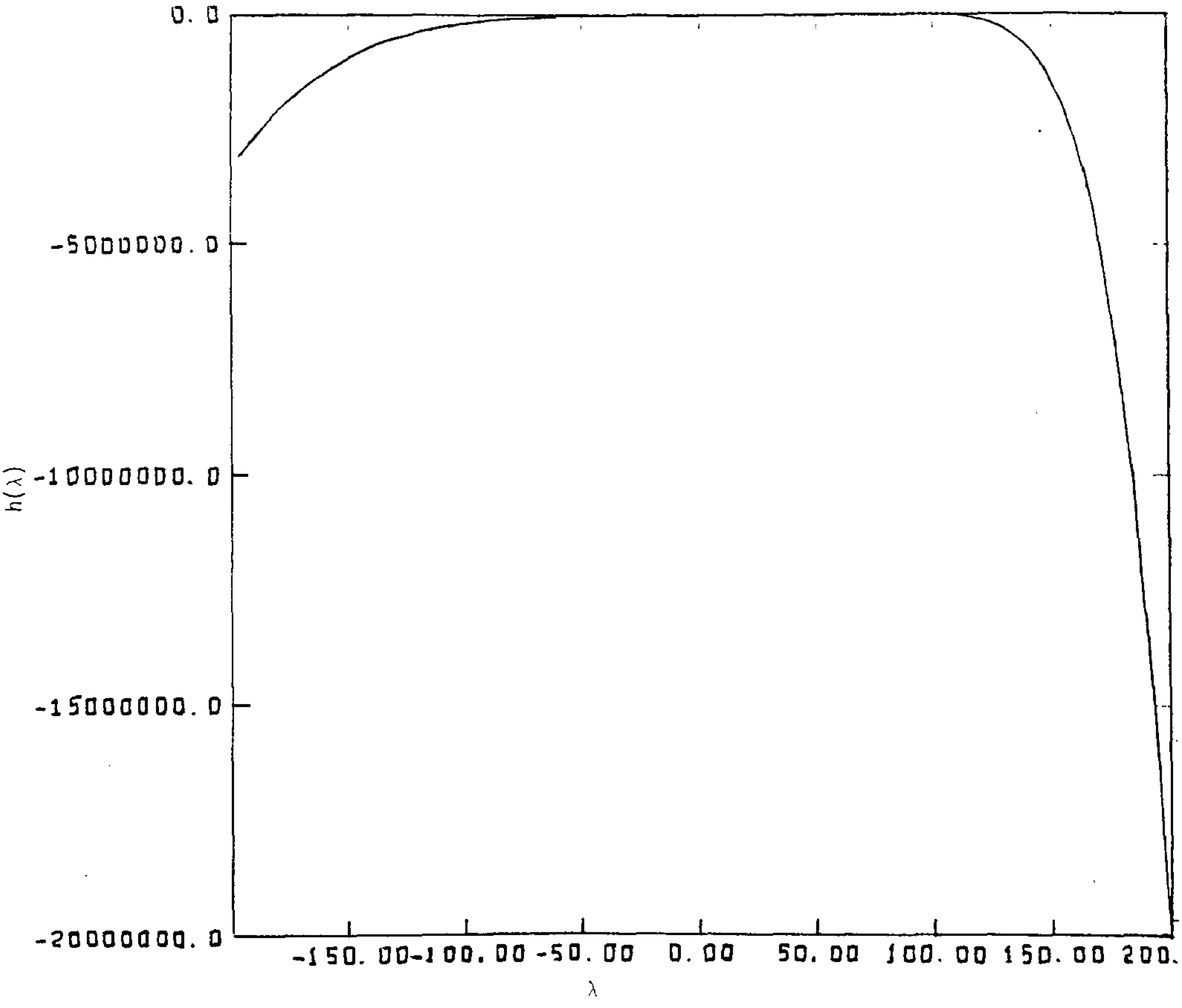
$h(\lambda) = r^4 \times \mathrm{Im}\, \tilde{V}_0$, N = 5

Fig. 25



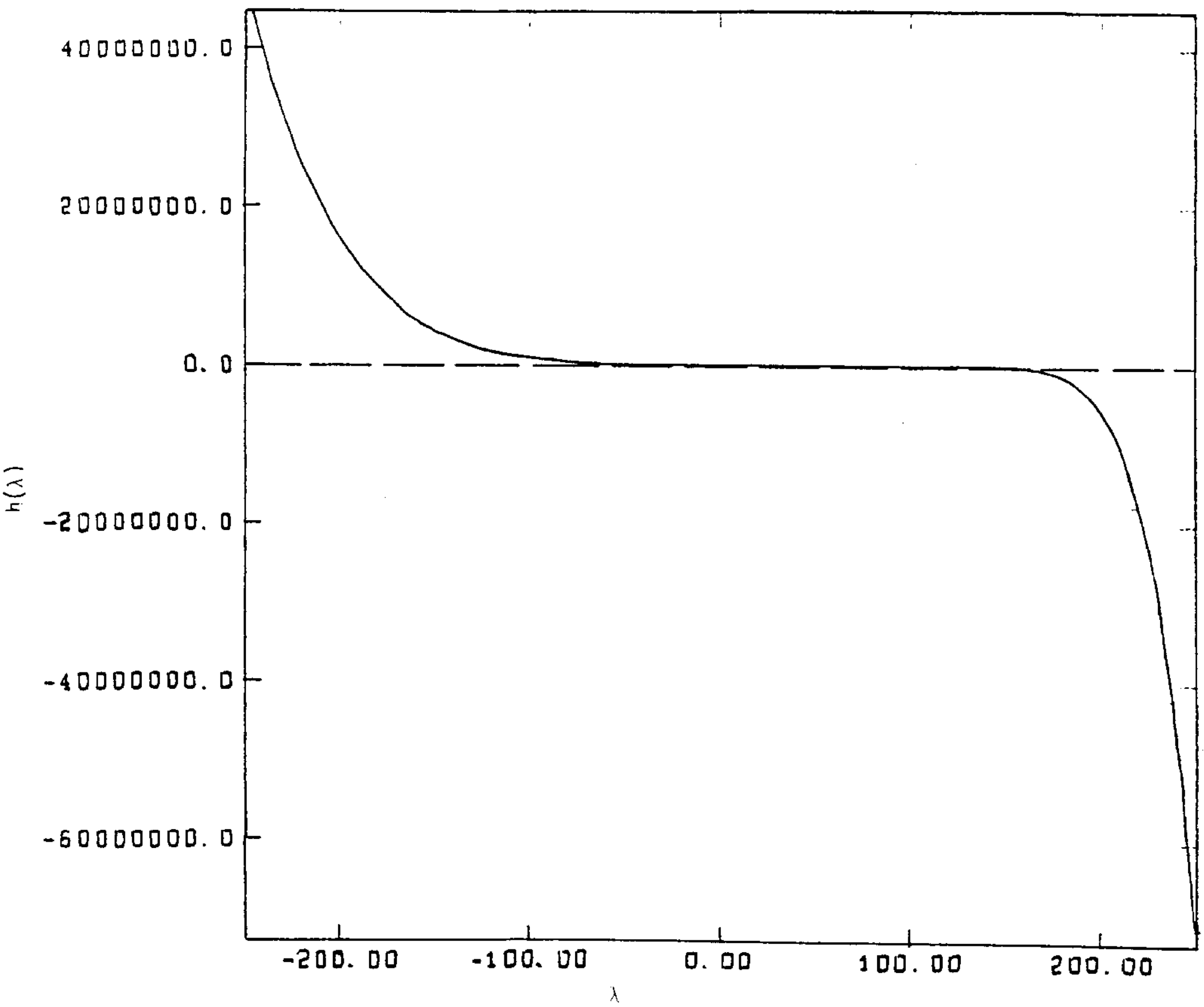
$h(\lambda) = r^4 \times \mathrm{Im}\ \tilde{V}_0$, N = 7

Fig. 26



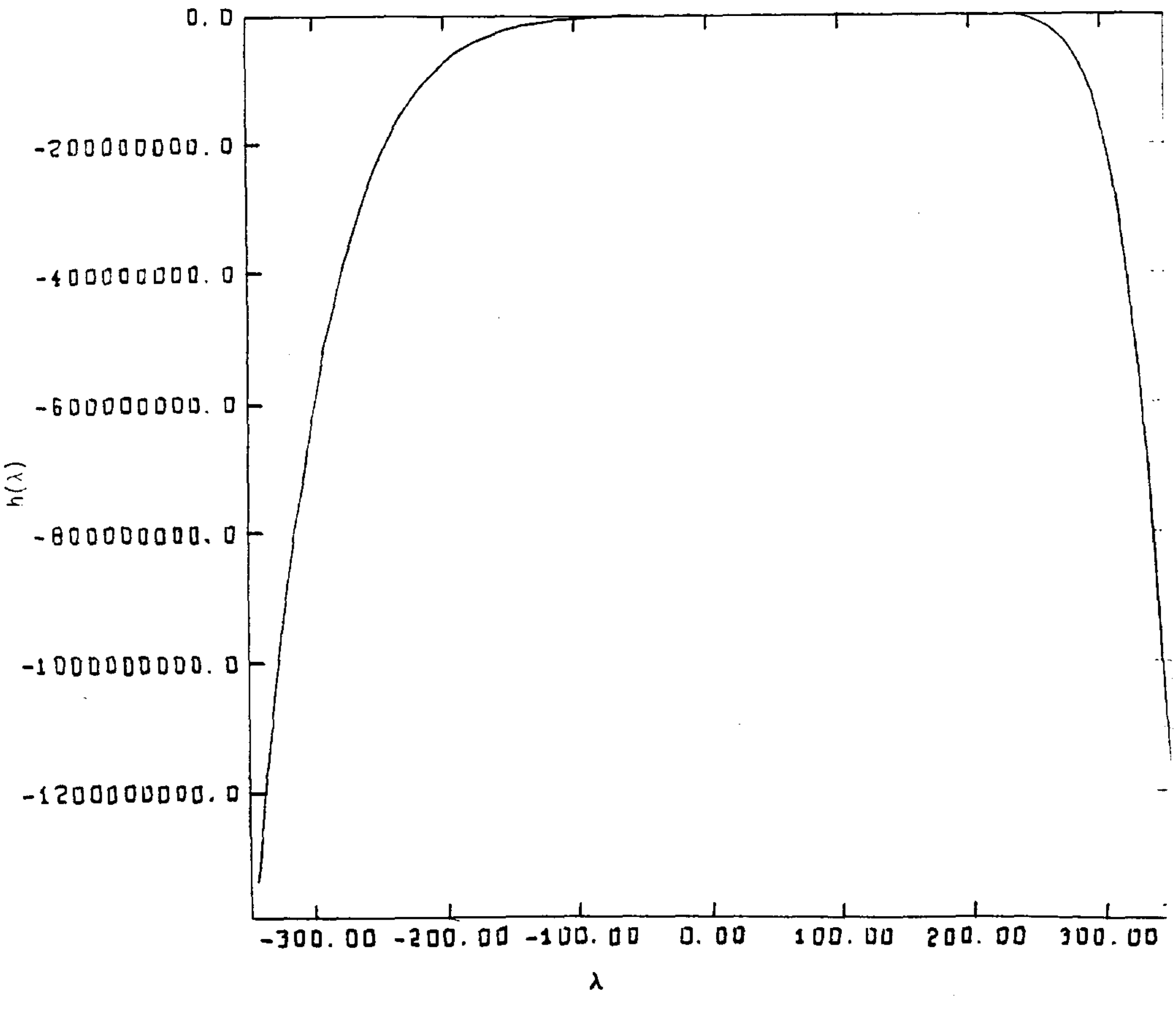
$h(\lambda) = r^4 \times \mathrm{Im}\ \tilde{V}_0$, N = 9

Fig. 27



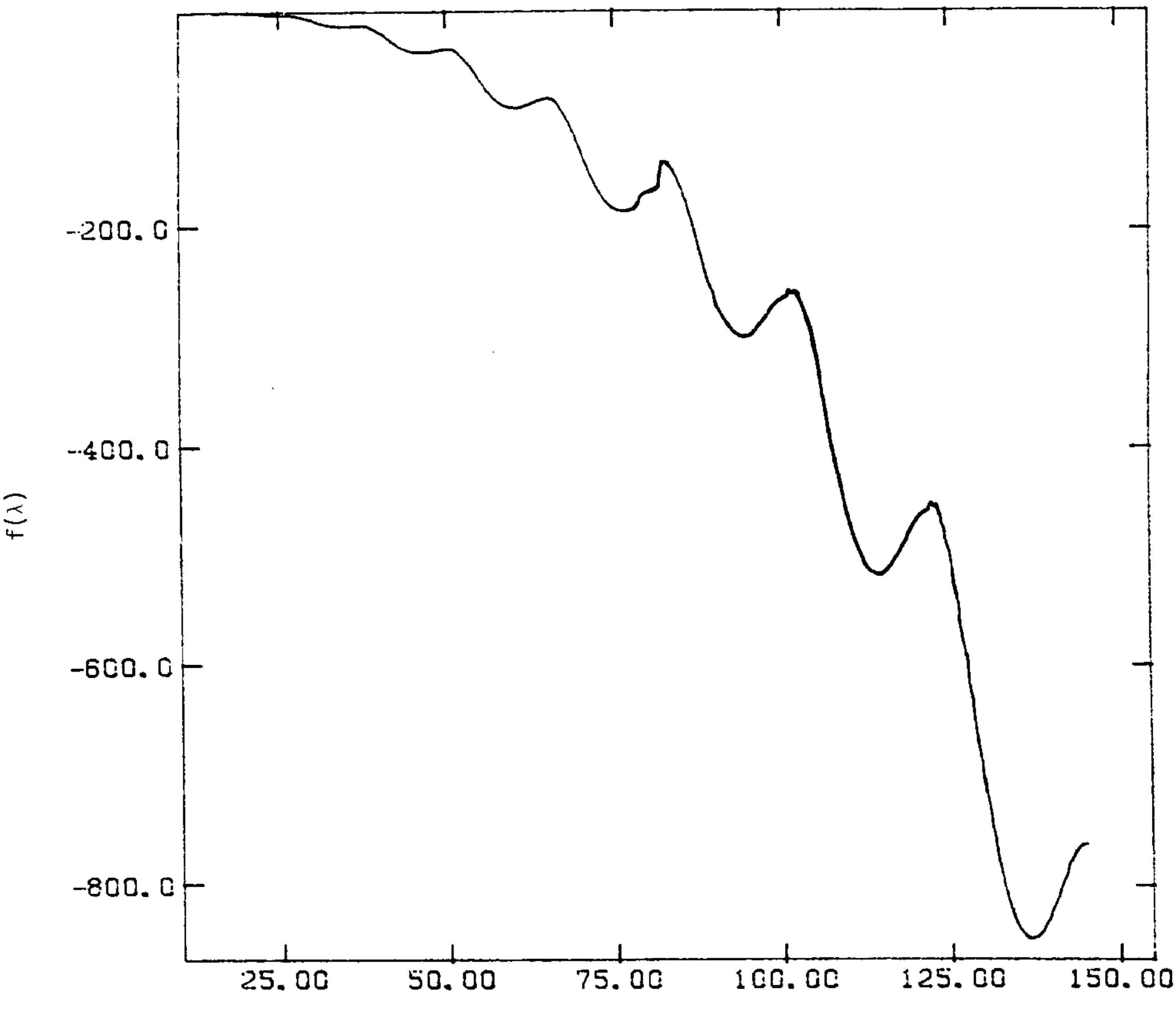
$h(\lambda) = r^4 \times \mathrm{Im}\,\tilde{V}_0$, $N = 11$

Fig. 28



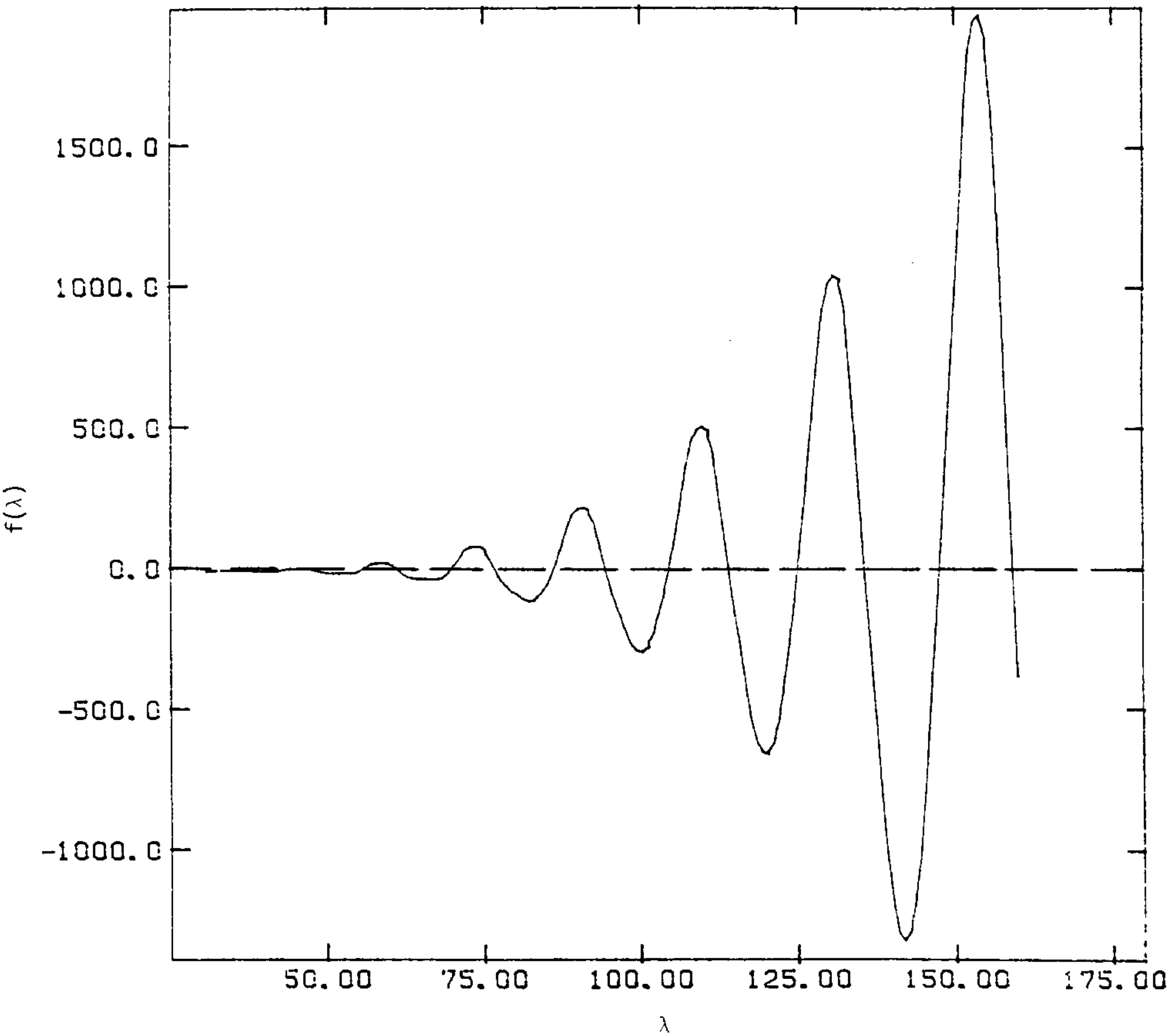
$h(\lambda) = r^4 \times \mathrm{Im}\ \tilde{V}_Q$, N = 13

Fig. 29



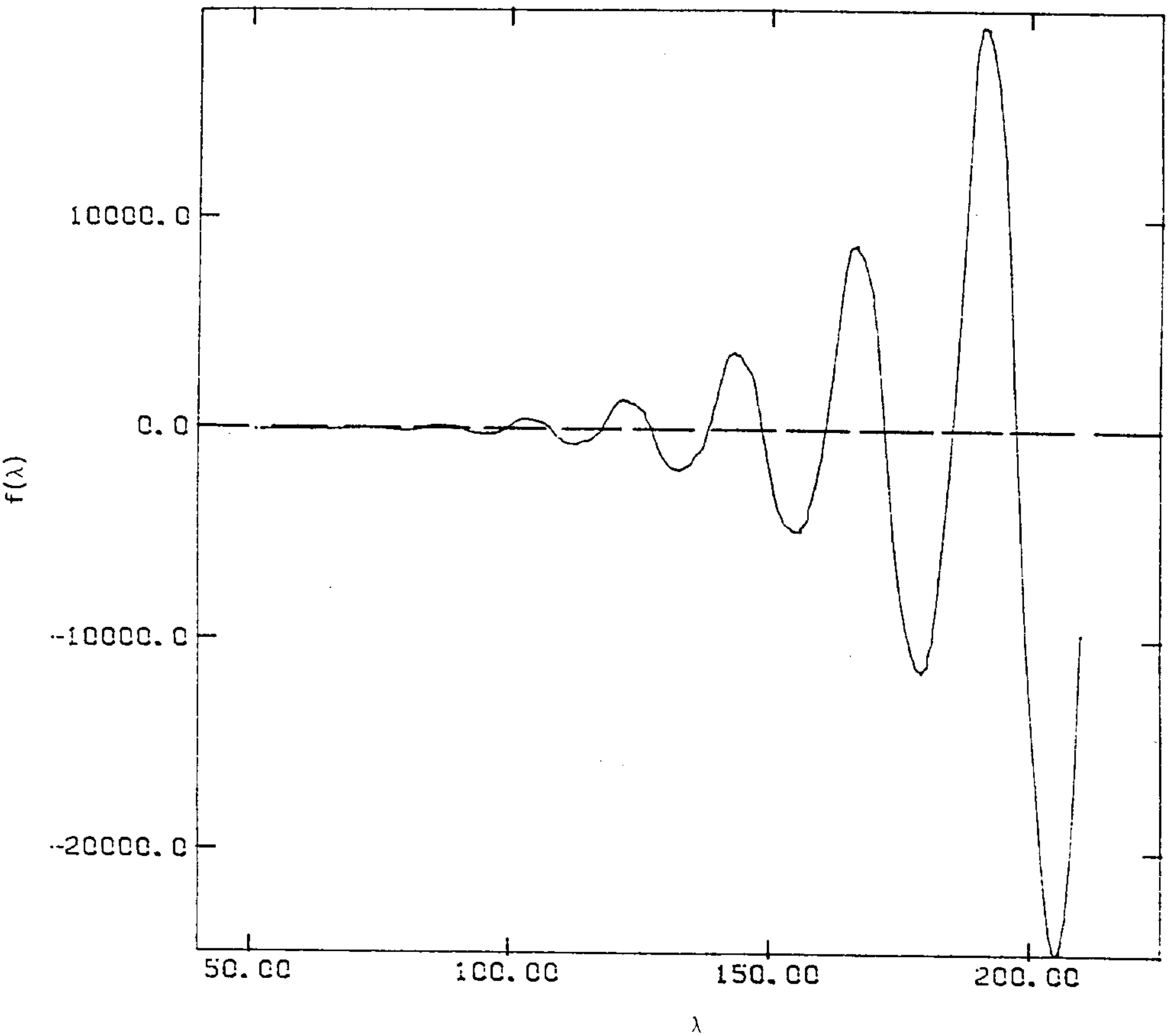
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_0,\ N = 3$

Fig. 30



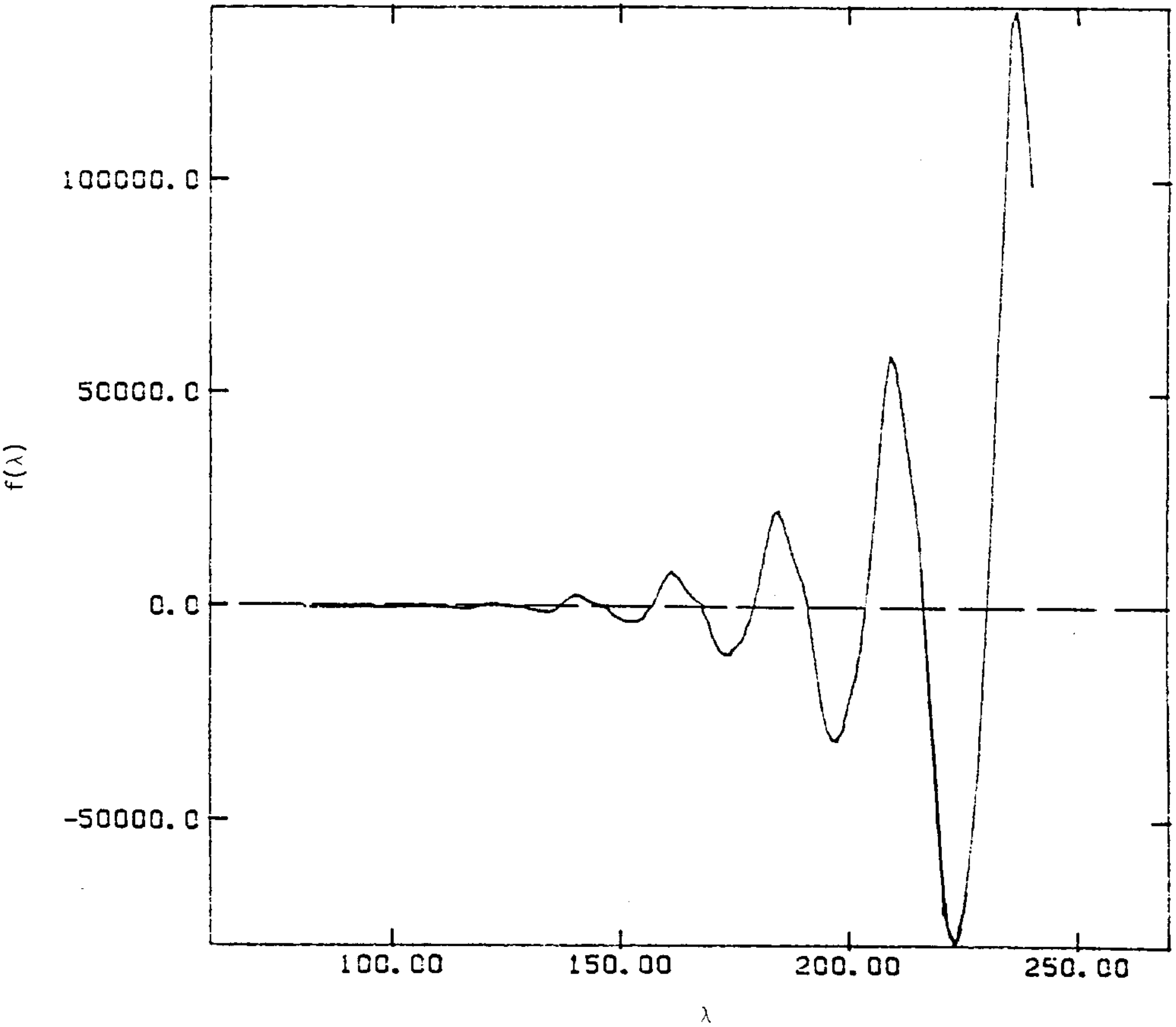
$f(\lambda) = r^4 \times \text{Re}\ \tilde{V}_0$, N = 5

Fig. 31



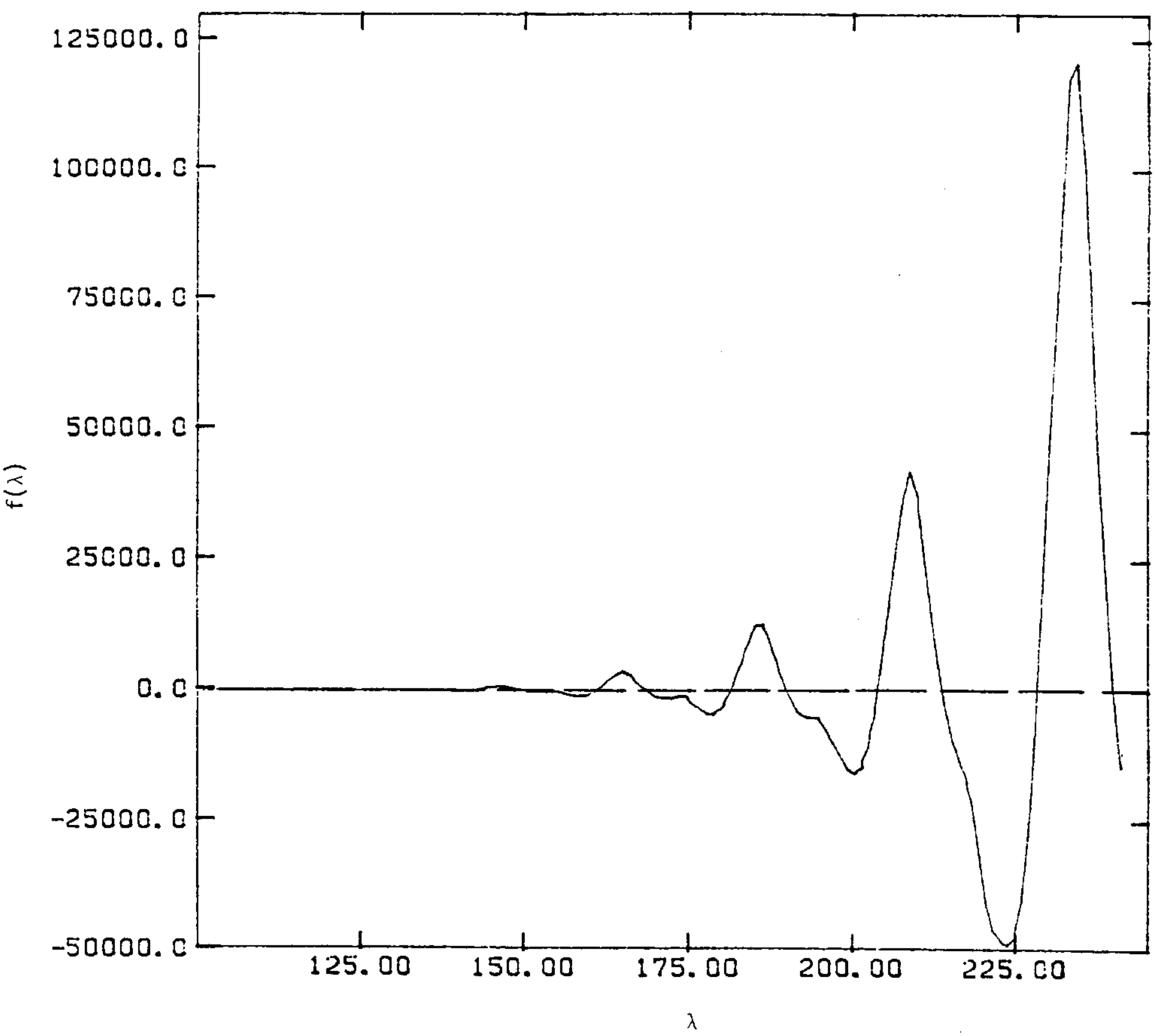
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_0$, N = 7

Fig. 32



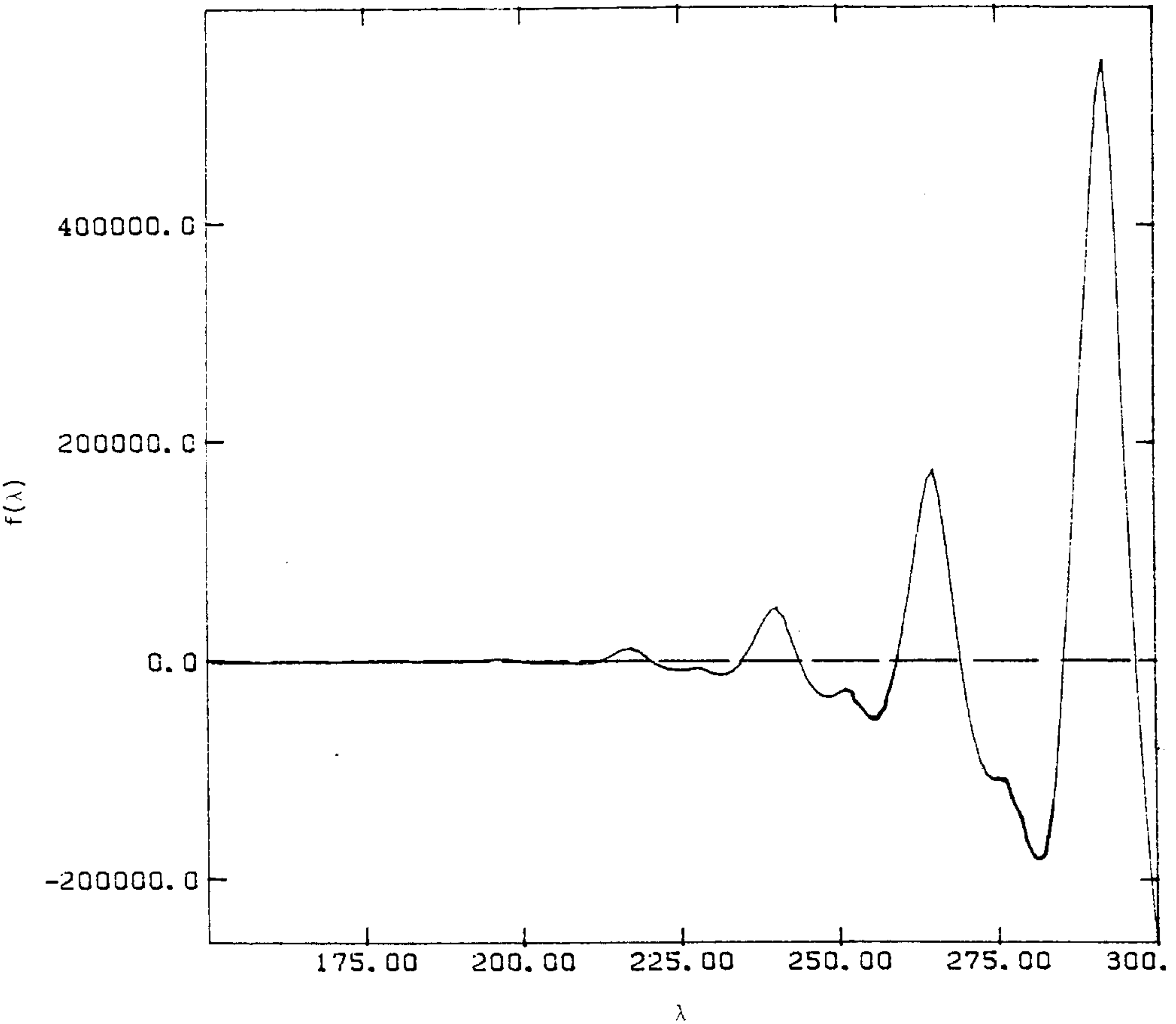
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_0$, $N = 9$

Fig. 33



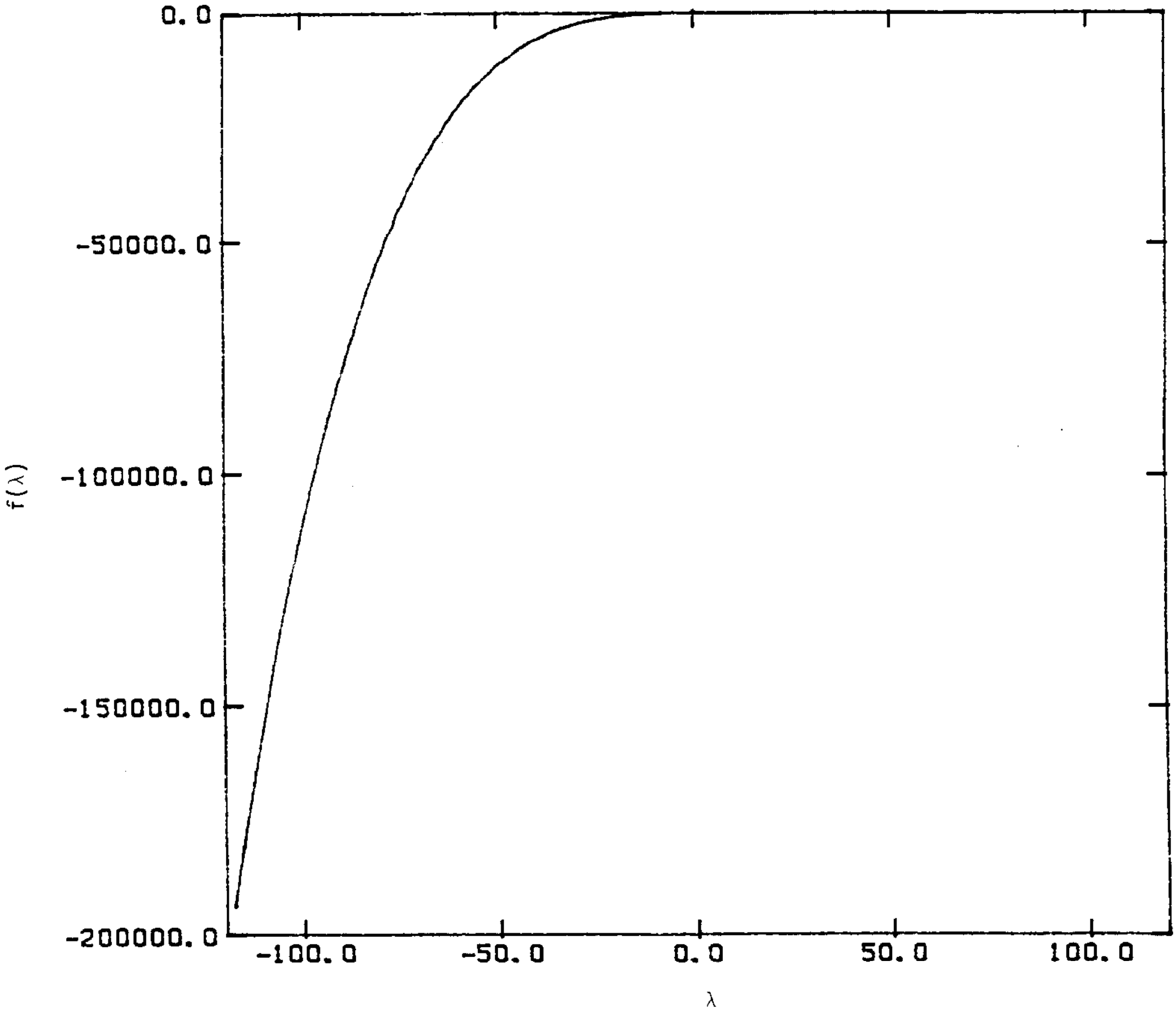
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_0$, N = 11

Fig. 34



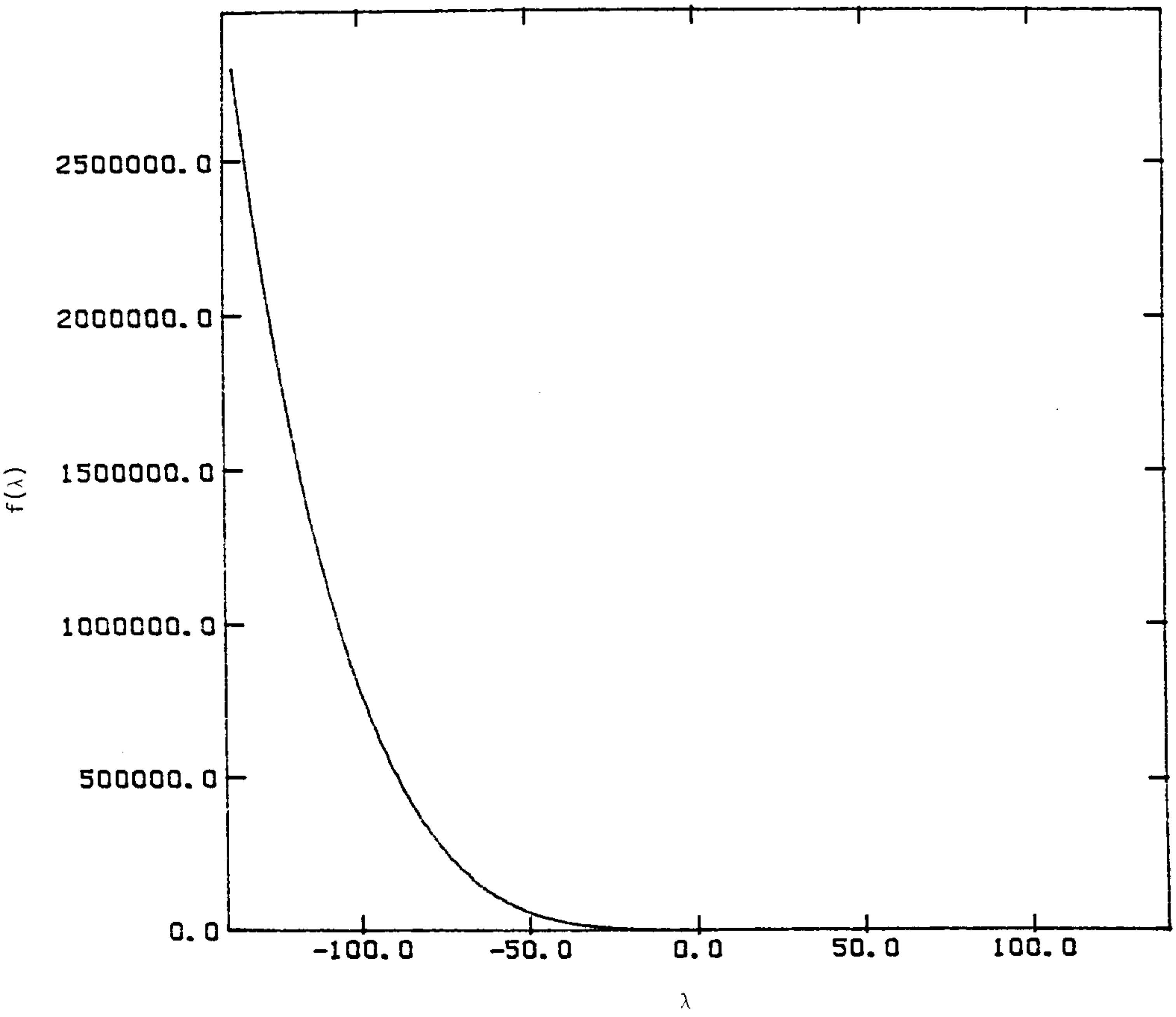
$f(\lambda) = r^4 \times \text{Re}\ \tilde{V}_0$, N = 13

Fig. 35



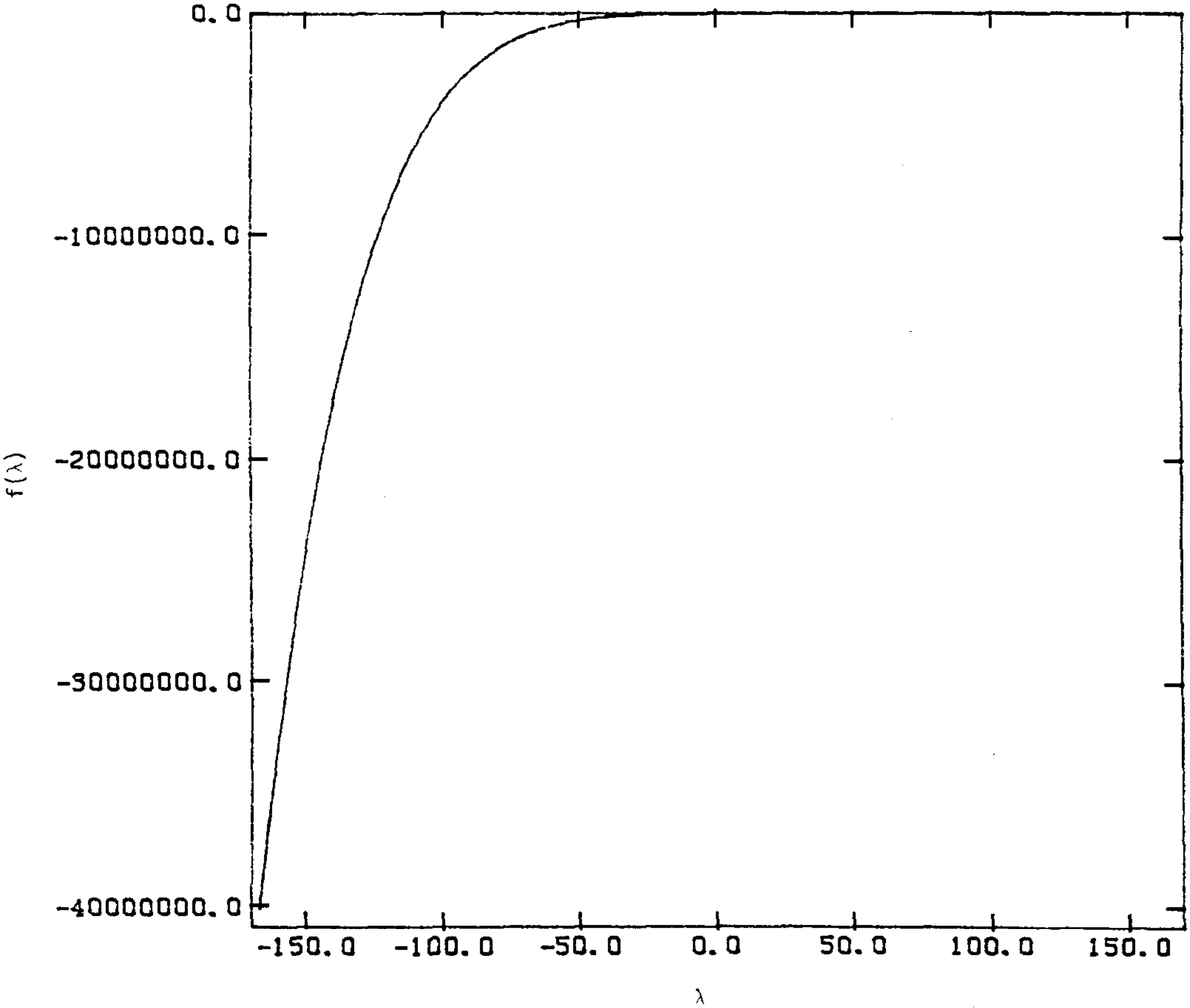
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_0$, N = 3

Fig. 36



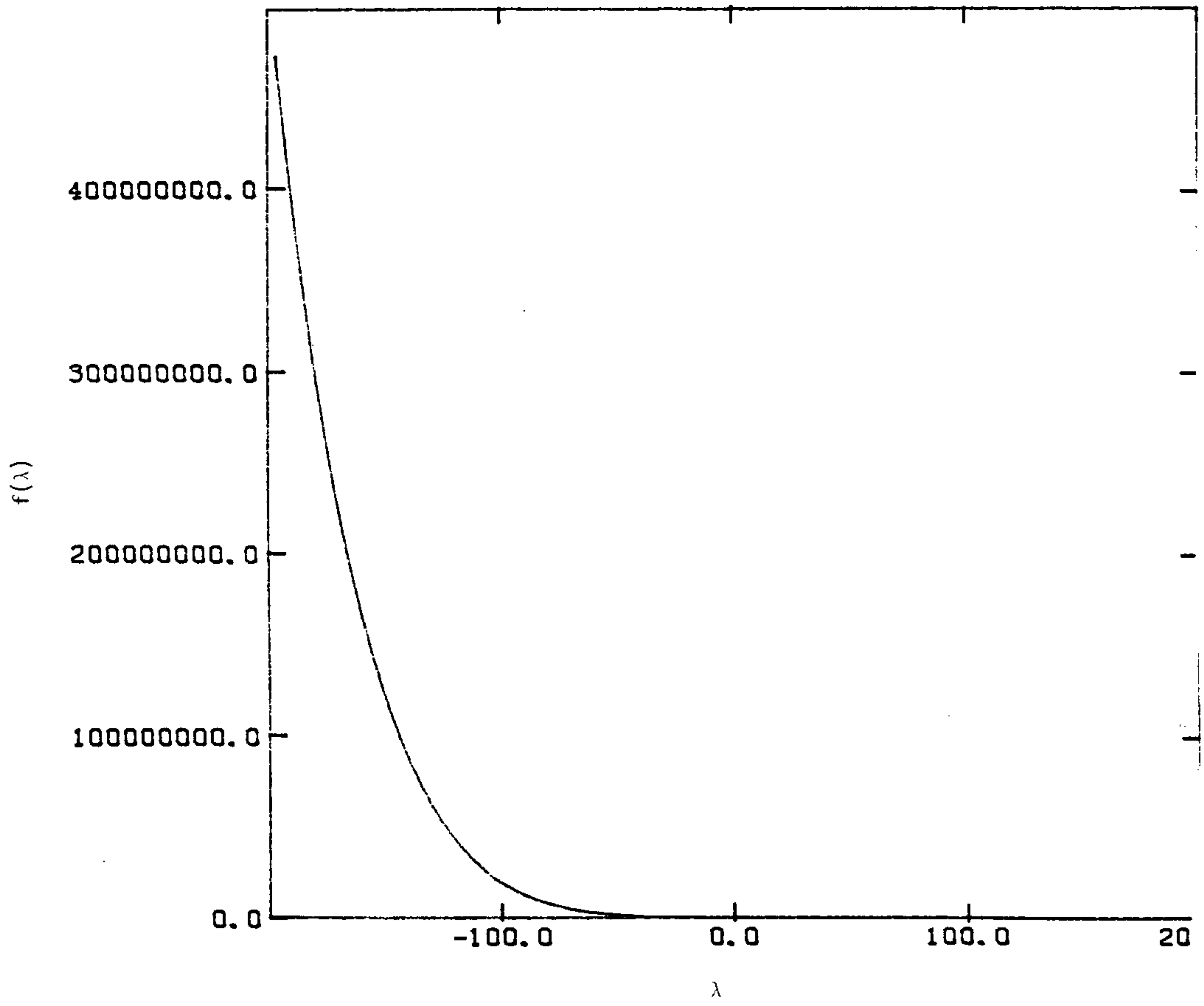
$f(\lambda) = r^4 \times \text{Re}\ \tilde{V}_Q$, N = 5

Fig. 37



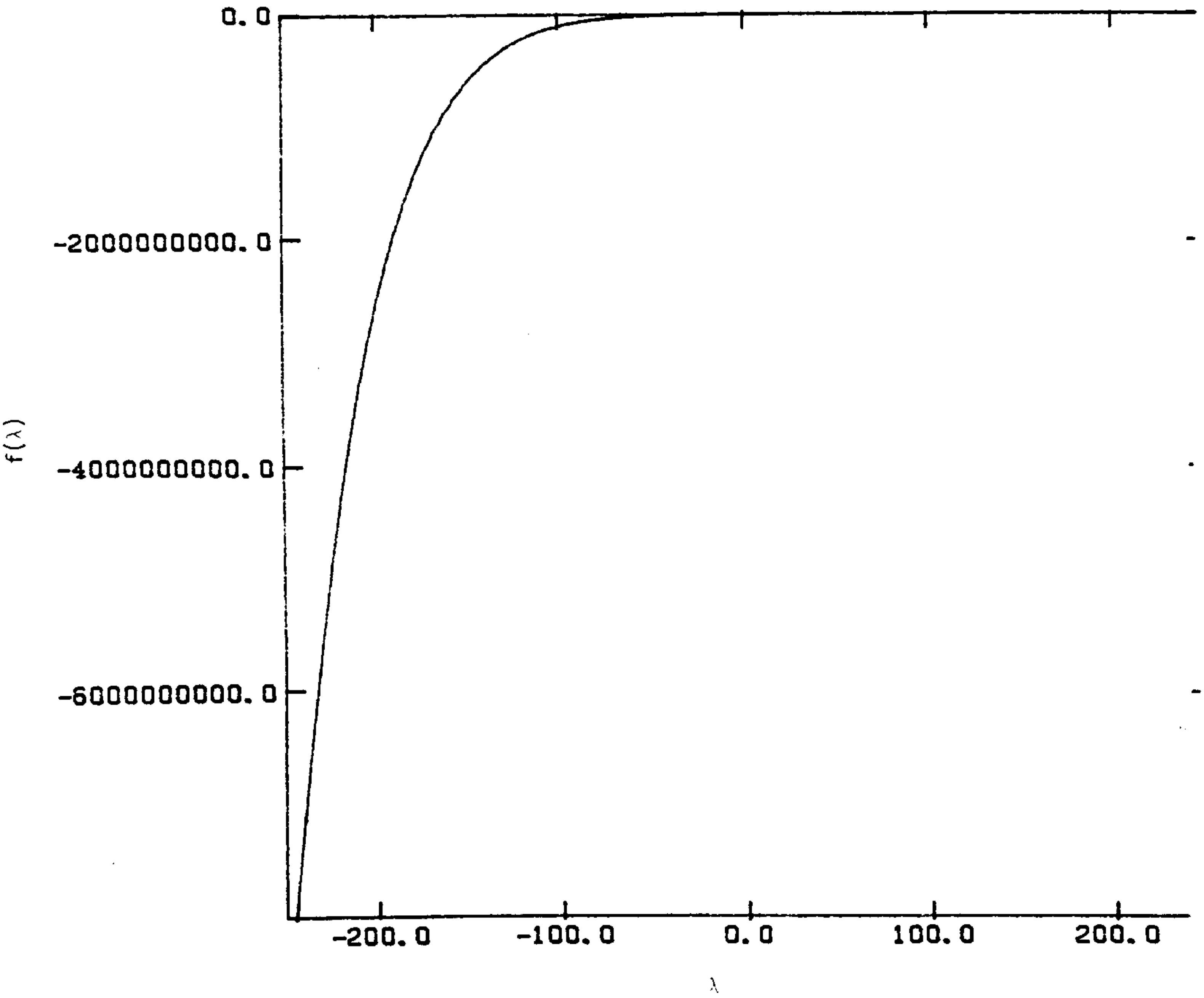
$f(\lambda) = r^4 \times \text{Re}\ \tilde{V}_0$, $N = 7$

Fig. 38



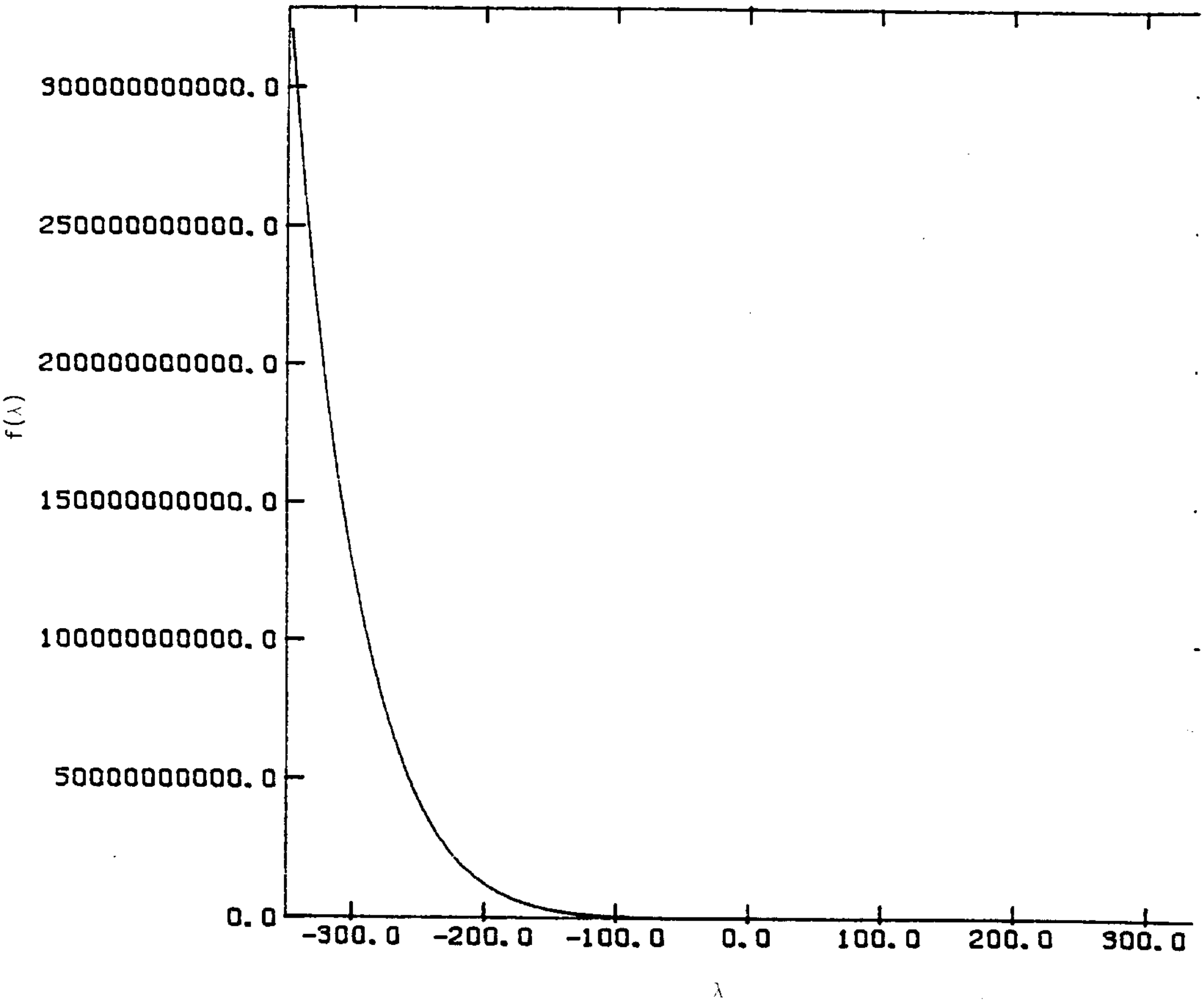
$f(\lambda) = r^4 \times \text{Re}\ \tilde{V}_0$, N = 9

Fig. 39



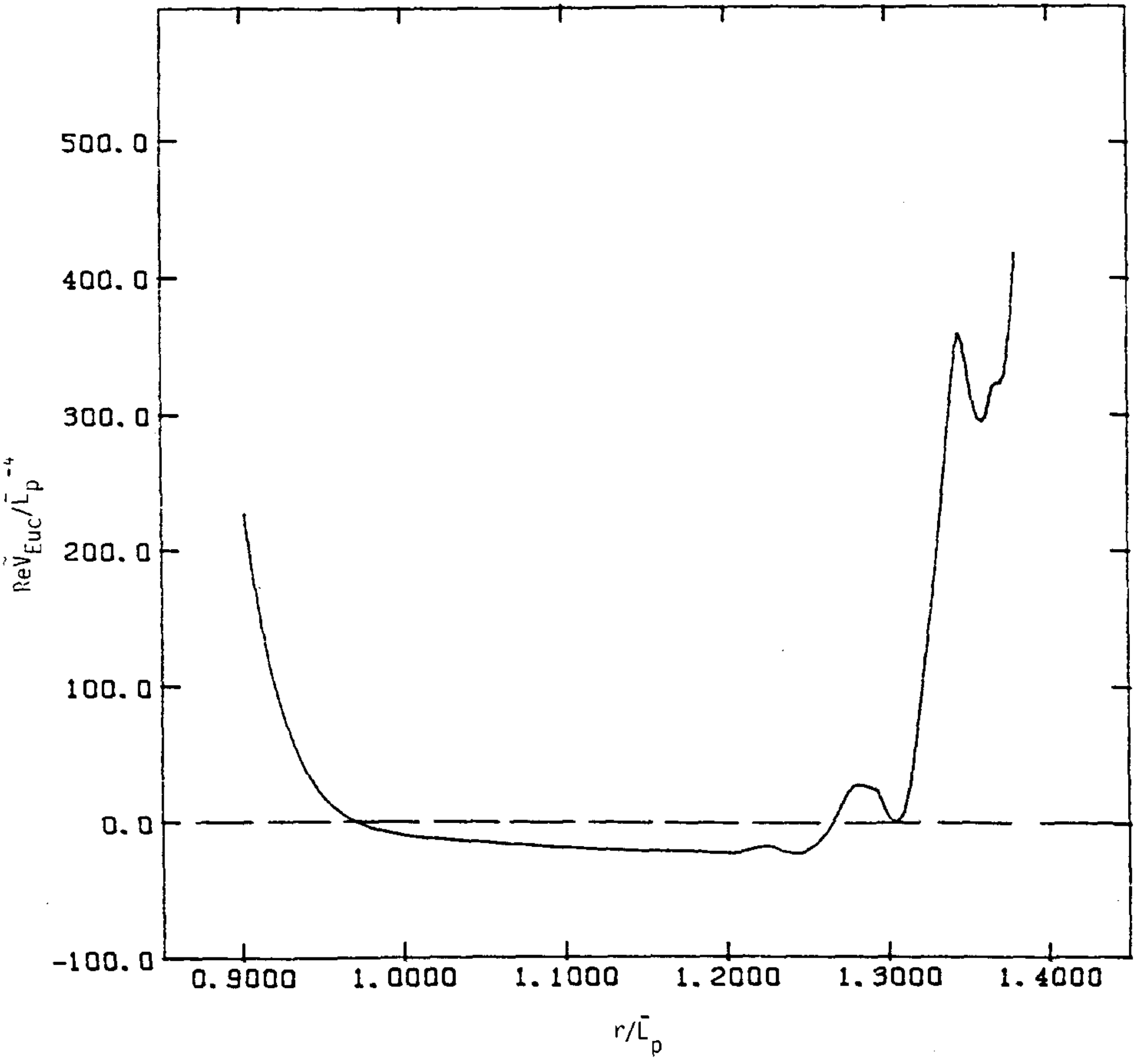
$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_Q$, N = 11

Fig. 40



$f(\lambda) = r^4 \times \mathrm{Re}\ \tilde{V}_Q$, N = 13

Fig. 41



Real part of "Euclideanized" effective potential density

$$\tilde{V}_{Euc} = \frac{\pi^{\frac{N-1}{2}}}{8\Gamma(\frac{N+1}{2})\bar{G}} [-N(N-1)r^{N-2} + \bar{\Lambda}r^{N}] + \tilde{V}_{C.M.} \text{ (see appendix A).}$$

$N = 13$, $\bar{\Lambda} = 98.000$, $\mathrm{Re}\,\tilde{V}_{Euc} = \partial/\partial r\, \tilde{V}_{Euc} = 0$ at $r = 1.3040\ \bar{L}_p$.