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DISTORTION FUNCTIONS

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DISTORTION FUNCTIONS

There is a pressing need for a better description of the simple, ***non-resonant*** distortion of a beam in a proton storage ring, created by non-linear (multipole) elements. The ***distortion functions*** provide a new technique for predicting the major part of this distortion.

A very large superconducting storage ring cannot be designed solely by linear theory. The quality of its performance is largely determined by the non-linear properties of the magnets. The large size, very slow cycle, slow recovery from beam-induced quenches, need for sophisticated diagnosis of minor faults - all demand a ***rational*** beam behaviour; and that means a linear behaviour. On the other hand it is apparent that one can easily push too far on the reduction of magnet size (cost) creating a really ***irrational*** device.

Tracking, following a particle for many turns through computer simulated fields, is the usual approach to non-linear problems. Tracking is at its best when demonstrating that a design is very good and that nothing significant has been overlooked. For this purpose it is irreplacable. It is not so good when a design is marginal, particularly if the problem is multiple random arrays of magnet errors. There is difficulty with cause and effect and therefore little guidance as to cures. Tracking is not a design tool in the sense that linear theory guides the layout of a focussing lattice.

In linear theory tracking is not necessary, there is a much more direct way of finding the size and shape of beams than by following particles for many turns. This is done by describing the ***shape of the beam that exactly repeats after one turn.*** In analogy with "the closed orbit" we will call this "the closed beam shape". The beta functions - β,α - are simply the shape parameters for this particular beam. The beta functions vary around the ring but they also must "close".

Linear theory is much more than a computational device for avoiding tracking. We do not design by guessing at a complete lattice and then finding the beam size. We think in terms of beta and we understand the patterns of focussing which generate beta. This is what we mean by "a design tool".

We will propose a set of ***closed distorion functions*** which are generated by multipole distributions in much the same way that beta functions are generated by quadrupole distributions. These functions describe a portion of non-linear behaviour, ***the closed, amplitude-dependent beam shape***. The functions themselves do not depend on amplitude. There are not too many and they are easy to calculate by simple familiar algorithms, including their variation around the ring.

The relation between a distortion function and its multipole distribution is formally the same as the relation of orbit distortion to dipole error, or of dispersion to $\Theta(\Delta P/P)$. Also, more complex non-linear phenomena can usually be described as a further interaction of the simple closed beam ***distortion*** with the multipole arrays. This means that, after a little experience, we should have a new design tool to supplement tracking.

The reader may well complain that this is an old problem (it is), that it has been completely solved by ..... (it has, many names), and that everyone knows that beam distortion is caused by the influence of nearby resonances (not everyone). The solution starts with an evaluation of "driving terms" for all the simple resonances, involving the usual resonance phase combinations $(m\psi_x + n\psi_y)$. Resonances produce closed beam shapes which are combined by a double summation. If necessary, one proceeds to the next order of resonances which involve products of the multipole strengths. Numerical factors in the combination greatly enhance the effect of the nearest resonances.

Consider a numerical example: a ring of 200 simple FODO cells near 60^o (tunes ~33.4), with a string of 11 cells containing a specific pattern of 10 sextupoles. (This example will be explained later). Figure 1 is a plot, without multipoles, for a particle with equal amplitudes in x and y. Figure 2 is a plot at the center of the sextupole array. It shows a characteristic effect of sextupoles, a triangular x shape and a smearing of both x and y, clearly there is a large nearby thirds resonance. Figure 3 is a plot at a position outside the array of exactly the same particle, but it shows only a small higher-order distortion. The problem is that there cannot be a large ***resonance*** at one point in a ring which doesn't appear at most other points.

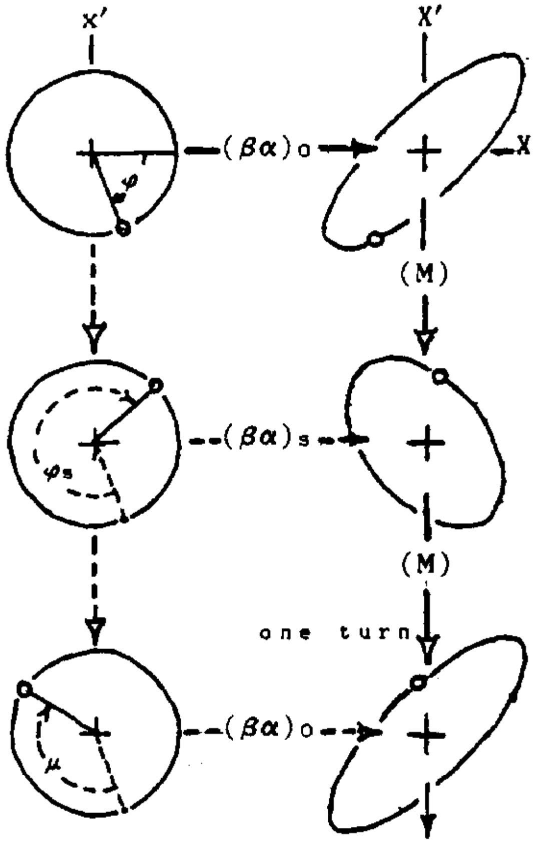
The word ***resonance*** in the above is a mistake; replace it by ***harmonic*** - same arithmetic but different connotation. The example is a very precise arrangement of sextupoles in 200 cells. It requires something like 200 harmonics to describe it, and to describe the effects. This ***distortion bump*** has been constructed as a dramatic example, but it is not unfair. Much more than a few nearby ***harmonics*** are required to give an adequate description of any particular beam distortion, which is why this theory has never become a useful design tool.

There is an analogy to orbit distortion. A few harmonics tell us nothing about the actual displacement at a particular point. Actually, we can calculate the displacement directly with no more effort than would be needed for one harmonic, and this also applies to closed beam distortion!

The existence of another perturbation solution for a solved problem is not suprising. The claim is that the new solution provides a long-sought design tool. The proof will come only when others, after some instruction, actually use that tool. This paper then is a how-to-use-it exposition.

LINEAR RINGS

The betatron theory for linear lattices is well-known. Although not strictly non-linear, ***skew quadrupoles*** are ***not*** allowed. The theory starts with the observation that a plot for a single particle on successive turns will trace an ellipse in X,X′ space.



One can proceed immediately to the concept of starting a beam of particles which lie on this particular ellipse. After one turn the beam shape is exactly the same, but individual particles have a new position. We describe this ***closed beam shape*** by parameters β,α in the equation

$$X^2+(\beta X'+\alpha X)^2=\beta\epsilon, \quad \pi\epsilon = \text{emittance.}$$

There is a standard transformation from real space to a slightly abstract ***circular space***:

$$x = X\,(\beta_0/\beta_x)^{1/2}$$
$$x'-\alpha x = X'\,(\beta_0\beta_x)^{1/2}$$

$$x^2+x'^2=\beta_0\,\epsilon$$

(β_0 is simply a scaling value required because we have changed the independent variable to φ from s.) In this space we can see exactly what we mean by beam amplitude and tune ν_x ($\mu=2\pi\nu$) and initial phase φ and lattice phase φ_s.

Once we have established the ***closed*** values for β,α at one point in the ring (from the one turn matrix), we can easily propagate the ellipse to other points and develop a table of functions for the ring. This table will also include the lattice phase φ_s. One never needs to "track" a particle in a linear ring. One simply calculates directly from the circles (dotted line) using the established beta functions.

All of the above is repeated for the other plane. The motion is described by two independent circles, which are the projection of a particular surface in four-dimensional phase space - a hyper-sphere.

MULTIPOLE DISTORTION

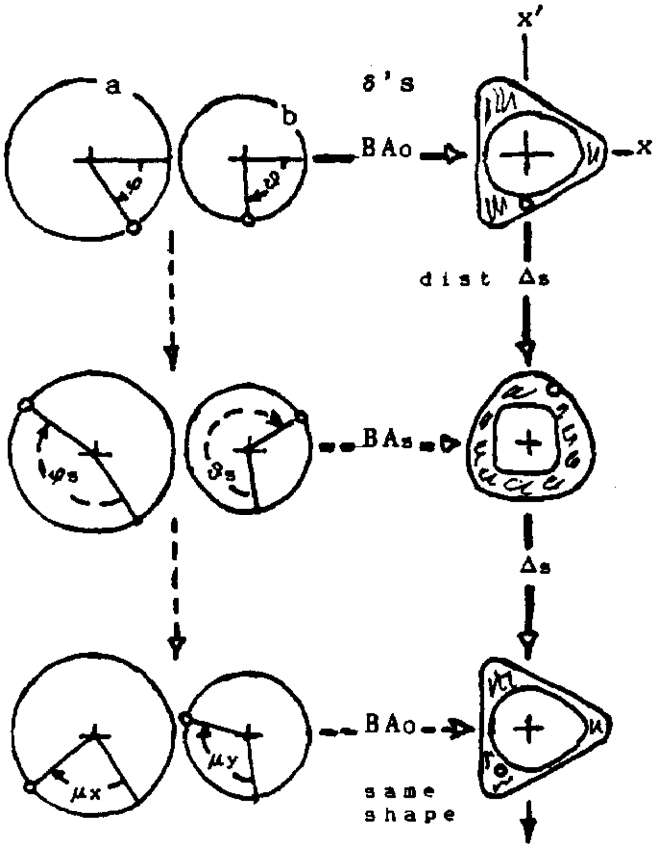
In a ring with non-linear (or skew quadrupole) elements, multiple turns of a single particle do not trace a clean ellipse, nor clean circles when we make the standard transformation. If we ***avoid resonances*** there will still be a ***closed beam shape***, but the particles move on a distorted surface in four dimensional phase space (call it the hyper-egg). Our two dimensional plots are simply projections of this surface and show bands rather than lines.

In order to restore precision to the concepts of amplitude and phase, it ***is*** necessary to start with two simple circles (a hypersphere). Without distortion we would write

$$x = a\cos\varphi \qquad\qquad y = b\cos\vartheta$$
$$x' = -a\sin\varphi \qquad\qquad y' = -b\sin\vartheta,$$

however with distortion we are first going to squeeze our hyper-sphere into the hyper-egg by making local changes in a, b, φ, ϑ, so

$$x = x_0 + (a+\delta a)\cos(\varphi+\delta\varphi) \qquad y = y_0 + (b+\delta b)\cos(\vartheta+\delta\vartheta)$$
$$x' = x'_0 - (a+\delta a)\sin(\varphi+\delta\varphi) \qquad y' = y'_0 - (b+\delta b)\sin(\vartheta+\delta\vartheta).$$



(Some multipoles produce a shift of the beam center, x_0 etc., by a field average over the amplitude.)

The δ's are expressions which contain several pairs of coefficients (B,A). Because the distortion varies around the ring, these ***distortion functions***, the B,A's, also vary but they must close. (To be useful, the distortion functions must not depend on amplitude, only on the multipole distribution.)

Consider $\delta a(a,b,\varphi,\vartheta)$ and follow the solid line. To this initial distortion each multipole adds its own Δa, for a one-turn total of $\sum(\Delta a)_s$. On the other hand, following the dotted line we have $\delta a(a,b,\varphi+\mu_x,\vartheta+\mu_y)$, (the μ's may be shifted from linear values.) Thus, symbolically, the closure condition for each δ expression is

$$\delta(\mu) - \delta(0) = \sum\Delta_s \ .$$

The closure condition on the δ expressions cannot be fulfilled in a closed form. We use the usual expansion in a power series of multipoles. The ***first-order distortion*** will use only the first power of the multipole strengths. This means that we will use only $x = a\cos(\varphi+\varphi_s)$ and $y = b\cos(\vartheta+\vartheta_s)$ when we evaluate the terms in $\sum\Delta$. We do not include the initial distortion δ's because they already contain a first power of multipoles.

Once we have obtained the proper ***closed*** first-order distortion equations then we can directly (dotted line) express the additional, closed, first-order displacement at each multipole and accumulate second-order Δ's, which in turn give closed second-order distortions; and so on.

The first-order distortion expressions contain familiar combinations of φ and ϑ. The second order contains all sums and differences of the first-order angles, a substantial increase in the number of terms. The second-order distortion functions involve local products of the first-order functions and the multipole strength.

The word "order" is often used, but with quite different meanings. We use it in its strongest sense as the mathematical power of multipole strengths. In transport theory it is used as the power of the displacement variables; thus sextupoles produce second-order terms, octupoles produce third order (x^3) etc. This would be confusing in our context. There is also a rather sloppy, weak usage where simple multiples of the tune are referred to as orders; thus a simple sextupole resonance at a fractional tune of 1/3 is called third order. In our terms this is a first-order sextupole resonance at 1/3 . At the same tune one can have a second-order sextupole resonance (2/6 with an amp^2 dependence), and a first-order decapole resonance (1/3 with amp^4, from expanding cos^5). To make our usage more specific "order" is followed by "distortion" or "resonance", or else these words are clearly implied.

It is clear from our expansion in orders that the key to avoiding complex difficulties is to control the closed, first-order distortion. If it is small then the higher order distortion is negligible. Because we can easily calculate the first-order distortion functions, using algorithms which follow and without "tracking", they become the design tool for studying systematic multipole arrays and for specifying permissable random errors.

There is one second-order term of great significance (for some multipoles), amplitude-dependent tune shift. It arises when $\Delta\varphi$ or $\Delta\vartheta$ does not depend on φ or ϑ, usually from sin^2 or cos^2 terms. Tune-shift of course is not a distributed function and it can be calculated as a simple extension of the first-order distortion algorithms.

The GENERAL ALGORITHMS for DISTORTION FUNCTIONS

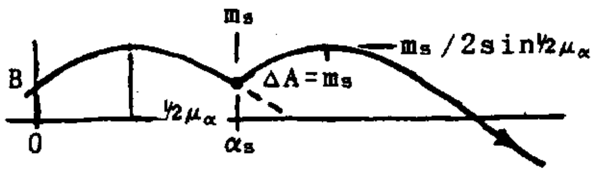
It is easy to express the effect of a ***single*** multipole in terms of Δa, Δb, $\Delta\varphi$, $\Delta\vartheta$. (If this is not familiar, see sextupole distortion). In "generic" terms

$\Delta_s = \ldots f(a,b)\ m_s\ \cos(\alpha+\alpha_s) \ldots$ where α is a combination
or $\quad \sin(\alpha+\alpha_s) \ldots$ of φ and ϑ

α is the phase at s=0, α_s is the lattice phase from 0 to s. m_s is the strength of the multipole at s, and contains a function of β_x, β_y. In general there is more than one m_s for each multipole. We must generate expressions $\delta a, \delta b, \delta\varphi, \delta\vartheta$ which describe the distortion of the hyper-egg (with respect to the hyper-sphere). In generic terms we had

$$\delta(\mu) - \delta(0) = \sum \Delta_s .$$

Define a vector B_α, A_α (where $A = dB/d\alpha$) to have the formal properties of a ***closed orbit***, with kicks $\Delta A = m_s$ and propagation with angle α, as shown for a single multipole. The combined value at $\alpha_s = 0$ will be



$$B_\alpha(0) = \sum m_s \cos(\alpha_s - ½\mu_\alpha) / 2\sin ½\mu_\alpha$$
$$A_\alpha(0) = \sum m_s \sin(\alpha_s - ½\mu_\alpha) / 2\sin ½\mu_\alpha$$

Expand the angles:

$B_\alpha(0) = ½S + ½C/\tan ½\mu_\alpha$ where $C = \sum m_s \cos \alpha_s$
$A_\alpha(0) = -½C + ½S/\tan ½\mu_\alpha$ $\quad S = \sum m_s \sin \alpha_s$

Now define two functions of the ***initial*** phase α:

$F(\alpha) = A_\alpha \cos\alpha + B_\alpha \sin\alpha$ and find $F(\alpha+\mu_\alpha) - F(\alpha)$, (also G)
$G(\alpha) = A_\alpha \sin\alpha - B_\alpha \cos\alpha$ using the values for B and A above.

$F(\alpha+\mu_\alpha) - F(\alpha) = C\cos\alpha - S\sin\alpha = \sum m_s \cos(\alpha+\alpha_s)$
$G(\alpha+\mu_\alpha) - G(\alpha) = S\cos\alpha + C\sin\alpha = \sum m_s \sin(\alpha+\alpha_s)$, which is what we want.

ALGORITHM: If the single multipole expression

Δ contains	then distortion δ contains
$f(a,b)\ m_s\ \cos(\alpha+\alpha_s)$	$f(a,b)\ F(\alpha, B_\alpha, A_\alpha)$
or $f(a,b)\ m_s\ \sin(\alpha+\alpha_s)$	$f(a,b)\ G(\alpha, B_\alpha, A_\alpha)$

where B_α, A_α is found by a closed orbit algorithm using m_s and α_s
We need one distortion vector for each combination of m_s and α_s
It is convenient to include common numerical factors in m_s.

One should note that the distortion functions do not depend on amplitude, only on the distribution of the multipoles, so one calculation can apply to many cases. The distortion itself of course depends on amplitude, often very strongly.

This theory has been clearly labelled as a ***non-resonance*** theory. We can now be more specific about its limitations. The resonances to be avoided are the primary resonances of the multipole under consideration. In these cases $\sin \frac{1}{2}\mu_\alpha$ in the denominator of our "closed orbit" expression approaches zero and the distortion functions become very large, just like ordinary orbit distortion near an integer. There is a difference, however, because orbit distortion near an integer is large but not inaccurate, whereas the distortion functions become large but not large enough.

When a tune differs from resonance by a very small amount we find that the phase shift generated by the multipoles is as big as the tune difference. This is the essential part of the "locking" phenomenon which creates fixed points, blow-up, islands and all the usual things that we associate with resonance - much more than simple division by a small number. None of this is in our expressions for the distortion functions.

The reason is simple. In order to solve the expressions, and in particular to make them amplitude independent, one persisted in using the unmodified tune. Actually the theory works well up to about one-half of the fixed-point amplitude for a particular tune, which means over all the tune space that we should be using, unless we are extracting the beam.

Resonance theory has much the same problem. It is solved by crossing out all terms except the resonant one, and assuming the tune difference is very small. It does not give correct distortion beyond about twice the "locking" tune difference. The two theories are complementary.

SEXTUPOLE DISTORTION

We define sextupole strength as $S = b_2\Theta$ where b_2 is the usual multipole error coefficient for a dipole with bend angle Θ. To be consistent $S = (B''l)/2(B\rho)$ for a sextupole magnet. The kicks are:

$$\Delta X' = -S(X^2 - Y^2)$$
$$\Delta Y' = 2SXY.$$

The conversion to circular form is best done by defining ***two*** values for ***each*** sextupole:

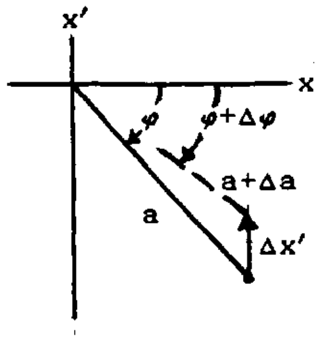
$$s = (\beta_x{}^3/\beta_0)^{1/2}\, S$$
$$\bar{s} = (\beta_x \beta_y{}^2/\beta_0)^{1/2}\, S = (\beta_y/\beta_x)\, s$$

then
$$\Delta x' = -sx^2 + \bar{s}y^2$$
$$\Delta y' = 2\bar{s}xy.$$

Our first step is to remove the amplitude-dependent orbit distortion, thus

$$\Delta x' = -s(a\cos\varphi)^2 + \bar{s}(b\cos\vartheta)^2$$
$$= -\tfrac{1}{2}sa^2\cos 2\varphi + \tfrac{1}{2}\bar{s}b^2\cos 2\vartheta - [\tfrac{1}{2}sa^2 - \tfrac{1}{2}\bar{s}b^2]$$

where the last two terms are to be set aside. We can now express these kicks as Δa, Δb, $\Delta\varphi$, $\Delta\vartheta$ (first order):



$$\Delta\varphi = -(\Delta x'/a)\cos\varphi$$
$$= \tfrac{1}{2}sa\cos 2\varphi\cos\varphi - \tfrac{1}{2}\bar{s}(b^2/a)\cos 2\vartheta\cos\varphi$$

$$\Delta a = -\Delta x'\sin\varphi$$
$$= \tfrac{1}{2}sa^2\cos 2\varphi\sin\varphi - \tfrac{1}{2}\bar{s}b^2\cos 2\vartheta\sin\varphi$$

$$\Delta\vartheta = -(\Delta y'/b)\cos\vartheta$$
$$= -2\bar{s}a\cos\varphi\cos^2\vartheta$$

$$\Delta b = -\Delta y'\sin\vartheta$$
$$= -2\bar{s}ab\cos\varphi\cos\vartheta\sin\vartheta.$$

$$\Delta\varphi = \tfrac{1}{4}s\,a\,(\cos 3\varphi + \cos\varphi) - \tfrac{1}{4}\bar{s}(b^2/a)(\cos\sigma + \cos\delta)$$

$$\Delta a = \tfrac{1}{4}s\,a^2(\sin 3\varphi - \sin\varphi) - \tfrac{1}{4}\bar{s}b^2(\sin\sigma - \sin\delta)$$

$$\Delta\vartheta = -\tfrac{1}{4}\bar{s}\,2a\,(2\cos\varphi + \cos\sigma + \cos\delta) \qquad \sigma = 2\vartheta + \varphi$$
$$\delta = 2\vartheta - \varphi$$

$$\Delta b = -\tfrac{1}{4}\bar{s}\,2ab\,(\sin\sigma + \sin\delta)$$

The angles are phases at the sextupole. They should read as $\varphi + \varphi_s$ and $\vartheta + \vartheta_s$, an initial phase plus the lattice phase advance from the reference point to the sextupole. For one turn simply sum the individual Δ's.

The Δ's for a single sextupole are now in the form where we can apply the algorithms. Five B,A pairs are needed (fifth one from the first term of $\Delta\vartheta$). The various meanings for α are shown in the following table.

name B_α	angle α_s	m_s	$\mu_\alpha/2\pi$	$F(\alpha)$ $G(\alpha)$
B_1 A_1	φ_s	$s/4$	ν_x	$F_1=A_1\cos\varphi+B_1\sin\varphi$ $G_1=A_1\sin\varphi-B_1\cos\varphi$
B_3 A_3	$3\varphi_s$	$s/4$	$3\nu_x$	$F_3=A_3\cos3\varphi+B_3\sin3\varphi$ $G_3=A_3\sin3\varphi-B_3\cos3\varphi$
B_s A_s	$\sigma_s=2\vartheta_s+\varphi_s$	$\bar{s}/4$	$2\nu_y+\nu_x$	$F_s=A_s\cos\sigma+B_s\sin\sigma$ $G_s=A_s\sin\sigma-B_s\cos\sigma$
B_d A_d	$\delta_s=2\vartheta_s-\varphi_s$	$\bar{s}/4$	$2\nu_y-\nu_x$	$F_d=A_d\cos\delta+B_d\sin\delta$ $G_d=A_d\sin\delta-B_d\cos\delta$
$\bar{B}$ $\bar{A}$	φ_s	$\bar{s}/4$	ν_x	$\bar{F}=\bar{A}\cos\varphi+\bar{B}\sin\varphi$

The B,A's at $\alpha_s=0$ are evaluated by the closed orbit algorithm using α_s and m_s. Note that they depend ***only*** on the multipole distribution. We can now write the distortion expressions, using f(a,b) from the Δ's:

$$\delta\varphi = a(F_3+F_1) - (b^2/a)(F_s+F_d)$$

$$\delta a = a^2(G_3-G_1) - b^2(G_s-G_d)$$

$$\delta\vartheta = -2a(2\bar{F}+F_s+F_d)$$

$$\delta b = -2ab(G_s+G_d)$$

$$x_0 = -2a^2B_1+2b^2\bar{B}, \qquad x'_0=-2a^2A_1+2b^2\bar{A} \qquad \text{(set-aside terms)}$$

These expressions are to be evaluated for each point of interest on the hyper-sphere (a,b,φ,ϑ) to determine the local distortion to get to the correct hyper-egg. In "circular" space

$$x = x_0+(a+\delta a)\cos(\varphi+\delta\varphi) \qquad y = (b+\delta b)\cos(\vartheta+\delta\vartheta)$$
$$x' = x'_0-(a+\delta a)\sin(\varphi+\delta\varphi) \qquad y' = -(b+\delta b)\sin(\vartheta+\delta\vartheta)$$

The figures traced by a beam of particles, all with the same amplitudes, or by many turns of a single particle (which is the same thing) are no longer two circles but are distorted broadened shapes. As we shall see, most of this distorted, two dimensional motion is described by the expressions above.

Let $a_1=a+\delta a$, a_1 can be found from x,x', and δa calculated using it in place of a should differ only in second order, however this ***inverse distortion*** calculation is much improved by iteration: $a_1\rightarrow\delta a\rightarrow a\rightarrow\delta a\rightarrow a\rightarrow\rightarrow$ (also include b,φ,ϑ).

Figures 12 and 13 show what we have accomplished. This example uses 384 simple 80^0 FODO cells, with 768 random sextupoles. (The rms $\langle s \rangle$ is chosen to simulate the SSC reference design sextupole error at 5mm amplitudes). First one calculates the B,A values for the observation point. They are shown at the lower right in units where a,b would be 1 (½cm. units). The particle starts with a=b=1 and $\varphi=\vartheta=0$ and the starting values x,x′ and y,y′ are calculated using the B,A's. This ensures that we start with a consistent emittance.

Figure 12 shows the distorted "circular" shape created by plotting many turns, with tunes near 85.4 . The shapes are typical of sextupole distortion. About one third of random arrays would show equal or greater distortion. What we must now show is that the distortion functions predict most of this distortion.

One can ***directly*** plot these same figures by choosing an array of values for φ and ϑ. The result is plots that are not different from figure 12; however a better way is to follow one particle as before but this time to perform the ***inverse*** distortion ***before plotting.*** This is shown in figure 13 (indicated by the label <u>cor</u>rected in place of <u>dis</u>torted). A perfect prediction would produce perfect circles, any remaining distortion would be from higher order effects. Clearly our distortion functions do an excellent job.

A note on computation. The following algorithm, borrowed from normal orbit distortion calculation, is used for large arrays. One needs from normal linear theory β_x and β_y at each sextupole, to calculate s and $\bar{s}$; also $\Delta\varphi$ and $\Delta\vartheta$, the phase advance between sextupoles. In the usual long strings of matched standard cells these are repetitive values.

One starts a vector $B_\alpha, A_\alpha=0$. At each sextupole one adds $\Delta A=\frac{1}{4}s$ (or $\frac{1}{4}\bar{s}$). Between sextupoles, rotate

$$\begin{pmatrix} B \\ A \end{pmatrix}_2 = \begin{pmatrix} \cos\Delta\alpha & \sin\Delta\alpha \\ -\sin\Delta\alpha & \cos\Delta\alpha \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}_1 \qquad \alpha=\text{combination}(\varphi,\vartheta)$$

obtaining, say, B_t and A_t after one turn. The closed values are

$$B_{\alpha}o = \tfrac{1}{2}B_t + \tfrac{1}{2}A_t/\tan(\tfrac{1}{2}\mu_\alpha), \quad A_\alpha o = \tfrac{1}{2}A_t - \tfrac{1}{2}B_t/\tan(\tfrac{1}{2}\mu_\alpha).$$

One repeats the same calculation but starting with the ***closed*** values, then all intermediate values are also closed and one collects them as needed (for a plot, or just the maximum, etc). One can also collect the sums $\sum Bs$ required for the tune-shift calculation below.

In the example just examined, the computation of the distortion functions took about the same time as tracking two turns. The section of code was shorter than this explanation!

Second-order sextupole terms. There are many second-order terms but we are particularly interested in tune shift, that is terms in $\Delta\varphi$ and $\Delta\vartheta$ that do not depend on φ or ϑ. They will be generated by expanding $\cos^2$ or $\sin^2$, or by x_0. It is now important that we did allow for a shift of the center of the motion, x_0 and x'_0, but we must be precise about it. Thus

$$\begin{aligned}\Delta x' &= -sx^2 + \bar{s}y^2 - (-\tfrac{1}{2}sa^2 + \tfrac{1}{2}\bar{s}b^2)\\ &= -s(x_0 + (a+\delta a)\cos(\varphi+\delta\varphi))^2 + \bar{s}((b+\delta b)\cos(\vartheta+\delta\vartheta))^2 + \tfrac{1}{2}sa^2 - \tfrac{1}{2}\bar{s}b^2\end{aligned}$$

and $\Delta\varphi = -\Delta x' \cos(\varphi+\delta\varphi)/(a+\delta a)$.

We expand and collect terms with δ's and x_0. On they left is that portion of the δ's which we need to get tune shift.

$$\begin{array}{rl}
\tfrac{1}{4}s(\cos 3\varphi + 5\cos\varphi)\ \delta a & \delta a = -a^2(B_3\cos 3\varphi - B_1\cos\varphi \ldots\\
-\tfrac{1}{4}s(3\sin 3\varphi + \sin\varphi)\ a\delta\varphi & \delta\varphi = a\ (B_3\sin 3\varphi + B_1\sin\varphi \ldots\\
s\ x_0 & x_0 = -2a^2 B_1 \ldots\\
\\
-\tfrac{1}{2}\bar{s}(\cos 2\varphi + \cos\sigma + \cos\delta)\ (b/a)\delta b & \delta b = 2ab(B_s\cos\sigma + B_d\cos\delta \ldots\\
\tfrac{1}{2}\bar{s}(\sin\sigma - \sin\delta)(b^2/a)\delta\vartheta & \delta\vartheta = -2a(B_s\sin\sigma + B_d\sin\delta \ldots\\
s\ x_0 & x_0 = 2b^2\ \bar{B} \ldots\\
\\
\tfrac{1}{4}\bar{s}(\cos\sigma + \cos\delta)\ (b^2/a^2)\delta a & \delta a = \ b^2\ (B_s\cos\sigma - B_d\cos\delta \ldots\\
\tfrac{1}{4}\bar{s}(\sin\sigma - \sin\delta)\ (b^2/a)\delta\varphi & \delta\varphi = -(b^2/a)\ (B_s\sin\sigma + B_d\sin\delta \ldots
\end{array}$$

We now turn to the $\Delta\vartheta$ expressions, which are easier

$$\Delta\vartheta = -2\bar{s}(x_0 + (a+\delta a)\cos(\varphi+\delta s))\cos^2(\vartheta+\delta\vartheta)$$

$$\begin{array}{rl}
\tfrac{1}{2}\bar{s}(\sin\sigma + \sin\delta)\ a\delta\vartheta & \delta\vartheta = -2a(B_s\sin\sigma + B_d\sin\delta \ldots\\
-\bar{s}\cos\varphi\ \ \delta a & \delta a = a^2\ B_1\cos\varphi \ldots\\
\bar{s}\sin\varphi\ \ a\delta\varphi & \delta\varphi = a\ \ B_1\sin\varphi \ldots\\
-\bar{s}\ x_0 & x_0 = -2a^2 B_1 \ldots\\
\\
-\tfrac{1}{2}\bar{s}(\cos\sigma + \cos\delta)\ \ \delta a & \delta a = \ b^2\ (B_s\cos\sigma - B_d\cos\delta \ldots\\
\tfrac{1}{2}\bar{s}(\sin\sigma - \sin\delta)\ a\delta\varphi & \delta\varphi = -(b^2/a)\ (B_s\cos\sigma + B_d\cos\delta \ldots\\
-\bar{s}\ x_0 & x_0 = \ 2b^2\ \ \bar{B} \ldots
\end{array}$$

Collecting the terms and summing the products of B s at each sextupole, we get the most important expressions

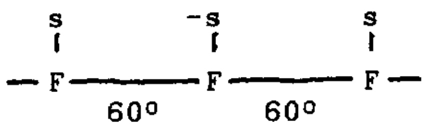
$$2\pi\Delta\nu_x = \quad -\tfrac{1}{2}(\sum B_3 s + 3\sum B_1 s) \quad - \quad \underbrace{(\sum B_s\bar{s} + \sum B_d\bar{s} - 2\sum B_1\bar{s})}_{\text{same}}$$

$$2\pi\Delta\nu_y = -(\sum B_s\bar{s} + \sum B_d\bar{s} - 2\sum B_1\bar{s}) - \tfrac{1}{2}(\sum B_s\bar{s} - \sum B_d\bar{s} + 4\sum\bar{B}\,\bar{s})$$

These expressions only use B's because the symmetry of A-term distortion does not give tune shift. Most second order terms use both. The expressions are exact for amplitude-squared tune shift. The next terms are fourth order. In numerical verification by tracking, there is often a substantial $2\varphi-2\vartheta$ term which requires a careful choice of the number of turns to be sure that it averages out when $\nu_x \sim \nu_y$.

The Sextupole DISTORTION BUMP as an example.

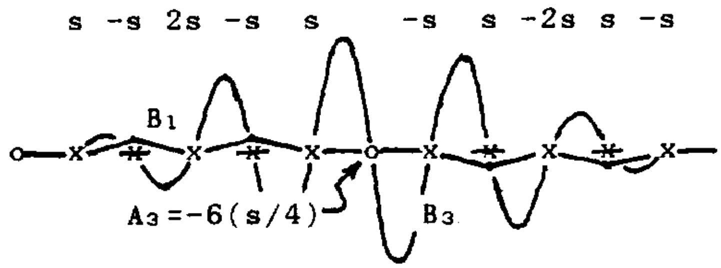
In the introduction we used a special sextupole structure as a dramatic demonstration that distortion varies around the ring. This structure is very simple and instructive. (Figures 4 and 5 correspond to 2 and 3 except that the tune is exactly .4,.4 . Figure 6 is a corrected version of 4.)



three bump

Consider simple cells, close to 60° in both planes, with sextupoles as shown. For three functions with $\alpha \sim 60^\circ$ this is a "bump" with no effect outside. For the functions with $\alpha \sim 180^\circ$ (B_3, A_3 and B_s, A_s) the kicks add. Because $\beta_y/\beta_x = 1/3$ for 60° cells, $\Delta A_s = \Delta A_3/3$.

We use four of these bumps to construct a big bump shown on the right. It has no effect outside and a large A_3 and A_s at the center. The angle is not exactly 60°, but by adding many empty cells we have 60.12° for a tune of 33.4 . One finds the actual distortion functions using the simple algorithm. The values are $A_3=-5.84(s/4)$ and $A_s=-1.95(s/4)$. In the computation s=.15 so $A_3=-.219$, $A_s=-.073$, which for a=b=1 means strong effects. We can write



$$\delta\varphi = aA_3 \cos 3\varphi - (b^2/a)A_s \cos\sigma$$
$$\delta a = a^2 A_3 \sin 3\varphi - b^2 A_s \sin\sigma$$
$$\delta\vartheta = -2aA_s \cos\sigma$$
$$\delta b = -2abA_s \sin\sigma$$

for $a=b=1$, $\varphi=\vartheta=0$

$x = \cos(A_3 - A_s)$ $\quad y = \cos(2A_s)$

$x' = -\sin(A_3 - A_s)$ $\quad y' = \sin(2A_s)$

(both angles are -.146)

If one started $x=y=1, x'=y'=0$ then $a=1.14$, $b=1.03$ - a bigger beam. One must iterate several times to perform the inverse distortion (hyper-egg to hypersphere) which was used to plot figure 6. This not suprising in view of the large δ's. The problem is, why does it work so well? Of course there are many small higher order resonances but the tune shifts, which only use B, are small:

peaks	$B_1=.866(s/4)$	$\sum B_1 s=-.866s^2$	
	$\bar{B}=.866(s/4)/3$	$\sum B_1\bar{s}=\sum\bar{B}s=-.866s^2/3$	$\Delta\nu_x=.0029$
	$B_d=.866(s/4)/3$	$\sum B_d\bar{s}=\sum\bar{B}\bar{s}=-.866s^2/9$	$\Delta\nu_y=-.0012$

SKEW SEXTUPOLE DISTORTION

Call ***x*** (and a, φ, etc.) ***vertical*** and ***y horizontal*** and use all of the above equations! A large skew sextupole is just as bad as the normal kind, it just does it standing up.

SKEW QUADRUPOLE DISTORTION

Skew quadrupole effects are linear, that is the amplitude factors out. Nobody talks about skew quad effects beyond the obvious resonant coupling at $\nu_x \sim \nu_y$, which is readily cured by rotating a few quads, any quads because the coupling depends on a simple sum around the ring. Obviously there remains a ***skew distortion*** but it has not been solved in a practical way. ***Because the efffect is linear and doesn't disappear at small amplitude,*** the moment we admit to its existence, then we begin to lose hold on our linear theory - not just amplitudes, but betas, phases, everything! Unfortunately, it now appears that the largest multipole error in superconducting magnets is usually skew quadrupole; so, with a sense of relief we write

$$\Delta X' = Q\,Y \qquad \Delta x' = q\,y \qquad Q = a_1\Theta$$
$$\Delta Y' = Q\,X \qquad \Delta y' = q\,x \qquad q = (\beta_x\beta_y)^{1/2}Q$$

$$\Delta\varphi = (-\tfrac{1}{2}q)(b/a)(\cos\sigma + \cos\delta) \qquad \Delta\vartheta = (-\tfrac{1}{2}q)(a/b)(\cos\sigma + \cos\delta) \qquad \sigma = \varphi + \vartheta$$
$$\Delta a = (-\tfrac{1}{2}q)\;b\;(\sin\sigma + \sin\delta) \qquad \Delta b = (-\tfrac{1}{2}q)\;a\;(\sin\sigma - \sin\delta) \qquad \delta = \varphi - \vartheta$$

Then $m_s = -\tfrac{1}{2}q$ and we have two α's, σ and δ, and we need two distortion functions, B_s, A_s and B_d, A_d. They are found by a closed orbit algorithm, as usual. Then we can write

$$\delta\varphi = (b/a)(A_s\cos\sigma + B_s\sin\sigma + A_d\cos\delta + B_d\sin\delta)$$
$$\delta a = \;b\;(A_s\sin\sigma - B_s\cos\sigma + A_d\sin\delta - B_d\cos\delta)$$
$$\delta\vartheta = (a/b)(A_s\cos\sigma + B_s\sin\sigma + A_d\cos\delta + B_d\sin\delta)$$
$$\delta b = \;a\;(A_s\sin\sigma - B_s\cos\sigma - A_d\sin\delta + B_d\cos\delta)$$

In this case only we will use a different procedure: let

distorted $x = a\cos\varphi + \delta x$, where $\delta x = \delta a\cos\varphi - a\delta\varphi\sin\varphi$
$x' = -a\sin\varphi + \delta x'$ $\quad \delta x' = -\delta a\sin\varphi - a\delta\varphi\cos\varphi$, etc. then

$$\delta x = -(B_s + B_d)y - (A_s - A_d)y'$$
$$\delta x' = (B_s - B_d)y' - (A_s + A_d)y$$
$$\delta y = -(B_s - B_d)x - (A_s - A_d)x'$$
$$\delta y' = (B_s + B_d)x' - (A_s + A_d)x$$

where x, x' y, y' are points on the hypersphere, and the δ's are distortion displacements to reach the hyper-egg. The ***inverse*** distortion is easy.

The nature of the second order terms is easy to see. Substitute $y+\delta y$ in the expression above for $\Delta x'$, the added terms depend only on x, x' and are still linear, so there is only a small contribution to normal quadrupole error (beta distortion).

This theory works very well indeed as is shown in the figures. It makes ***paper*** machines calculable, but it does not make ***real*** rings any easier to operate. Skew quadrupole does not, by itself, cause beam loss, but it can make a ring very difficult or even impossible to adjust in a reasonable time. Our distortion functions provide a rational basis for effecting a cure, such as "shuffling" measured dipoles on installation.

OCTUPOLE DISTORTION

The ***normal octupole*** equations are

$\Delta x' = -\underline{m}\, x^3 + 3m\, xy^2$ where $\underline{m} = O(\beta_x{}^2/\beta_0)$ $m = O(\beta_x \beta_y/\beta_0)$

$\Delta y' = 3m\, x^2 y - \overline{m}\, y^3$ $\overline{m} = O(\beta_y{}^2/\beta_0)$ and $O = b_3 \Theta$

Following the usual expansion, we require eight functions

	B_α, A_α	m_s	α_s
from a closed-orbit algorithm	B_1 A_1	$\underline{m}/8$	$4\varphi_s$
	B_2 A_2	$\underline{m}/8$	$2\varphi_s$
	B_3 A_3	$m/8$	$2\varphi_s + 2\vartheta_s$
	B_4 A_4	$m/8$	$2\varphi_s - 2\vartheta_s$
	B_5 A_5	$m/8$	$2\varphi_s$
	B_6 A_6	$m/8$	$2\vartheta_s$
	B_7 A_7	$\overline{m}/8$	$4\vartheta_s$
	B_8 A_8	$\overline{m}/8$	$2\vartheta_s$

Use $F_\alpha = A_\alpha \cos\alpha + B_\alpha \sin\alpha$, $G_\alpha = A_\alpha \sin\alpha - B_\alpha \cos\alpha$, as before, and the distortion expressions are

$$\delta\varphi = a^2(F_1 + 4F_2) - 3b^2(F_3 + F_4 + 2F_5 + 2F_6)$$

$$\delta a = a^3(G_1 + 2G_2) - 3ab^2(G_3 + G_4 + 2G_5)$$

$$\delta\vartheta = -3a^2(F_3 + F_4 + 2F_5 + 2F_6) + b^2(F_7 + 4F_8)$$

$$\delta b = -3a^2 b(G_3 - G_4 + 2G_6) + b^3(G_7 + 2G_8)$$

In ***addition*** there is the usual octupole ***tune shift***

$$2\pi\Delta\nu_x = a^2\,(3/8)\textstyle\sum \underline{m} - b^2\,(3/4)\sum m$$
$$2\pi\Delta\nu_y = -a^2\,(3/4)\textstyle\sum m + b^2\,(3/8)\sum \overline{m}$$

(See discussion at end of skew octupole.)

SKEW OCTUPOLE DISTORTION

The ***skew octupole*** equations are

$$\Delta x' = 3m\, x^2 y - \bar{m}\, y^3 \qquad \text{where} \quad m = O(\beta_x{}^3 \beta_y / \beta_0{}^2)^{1/2} \quad \bar{m} = O(\beta_x \beta_y{}^3/\beta_0{}^2)^{1/2}$$
$$\Delta y' = m\, x^3 - 3\bar{m}\, xy^2 \qquad \text{and} \quad O = a_3\Theta$$

Following the usual expansion, we require eight functions

	B_α, A_α	m_s	α_s
from a	B_1 A_1	$-m/8$	$3\varphi_s + \vartheta_s$
closed-orbit	B_2 A_2	$-m/8$	$3\varphi_s - \vartheta_s$
algorithm	B_3 A_3	$-m/8$	$\varphi_s + \vartheta_s$
	B_4 A_4	$-m/8$	$\varphi_s - \vartheta_s$
	B_5 A_5	$-\bar{m}/8$	$\varphi_s + \vartheta_s$
	B_6 A_6	$-\bar{m}/8$	$\varphi_s - \vartheta_s$
	B_7 A_7	$-\bar{m}/8$	$\varphi_s + 3\vartheta_s$
	B_8 A_8	$-\bar{m}/8$	$\varphi_s - 3\vartheta_s$

Use $F_\alpha = A_\alpha \cos\alpha + B_\alpha \sin\alpha$, $G_\alpha = A_\alpha \sin\alpha - B_\alpha \cos\alpha$, as before, and the distortion expressions are

$$\delta\varphi = 3ab(F_1 + F_2 + 3F_3 + 3F_4) - b^3/a(3F_5 + 3F_6 + F_7 + F_8)$$

$$\delta a = 3a^2 b(G_1 + G_2 + G_3 + G_4) - b^3(3G_5 + 3G_6 + G_7 + G_8)$$

$$\delta\vartheta = a^3/b(F_1 + F_2 + 3F_3 + 3F_4) - 3ab(3F_5 + 3F_6 + F_7 + F_7)$$

$$\delta b = a^3(G_1 - G_2 + 3G_3 - 3G_4) - 3ab^2(G_5 - G_6 + G_7 - G_8)$$

Octupoles have a small ***second-order tune shift*** which is usually ***negative***.

Figures 18-21 show that predictions of beam distortion are good. The strengths are the same as the sextupole case but they normally would be substantially smaller; thus we probably do not need to continue to higher multipoles. There are so many terms that there is no typical octupole pattern: the normal example has particulaly small mixing terms (the middle four), and the skew example has its tune shifted to 5/13 (it is not on a resonance).

TRACKING and DISTORTION FUNCTIONS

There are two suggestions for enhancing tracking studies by the use of distortion functions. The first step in both cases is the evaluation of the functions, a modest increase in the program but not a significant increase in running time.

The first suggestion is to use the "inverse distortion" calculation, before plotting, to remove the fuzzy but predictable first order distortion from the plots in order to see underlying structure. The figures make this clear.

In this paper, the tracking plots include a third "a-b" plot in addition to x,x' and y,y'. Two-dimensional resonances, which are more numerous and stronger than one-dimensional resonances, do not show patterned behaviour in the normal plots but they do have characteristic a-b plots. A particle on an $m\nu_x+n\nu_y$ resonance (n can be negative) will move so that $mb^2-na^2=\text{const}$.

Skew quadrupole effects are shown in figures 7 - 11, all for the same random set. Figures 7 and 8 show the variation with lattice position. Figure 9 is a "corrected" version of figure 8 using the inverse distortion calculation before plotting. Almost all the distortion is calculable without tracking. The sum terms (s) generate a-b plots which slope upward, the d terms downward. In this random set, the mean is made precisely zero to avoid resonance coupling ($B_d=0$). In figures 10 and 11, 10% of the "natural" mean is restored and the a-b plot is wider in the direction expected.

The sextupole diagrams, figures 12 -17 are better examples of uncovering hidden structure. Figures 12 and 13 show the typical sextupole distortion. It is predictable. In figures 14 and 15 we have tuned the rings to $\nu_x=\nu_y$, allowing for tune shift. The a-b plot in figure 15 is simply $a^2+b^2=1$, showing that the smearing in the other plots is just a coupling - in this case a second order term $2\nu_x-2\nu_y$ which is not removed by the first order correction. Figure 16 shows an a-b plot with a slope 2/3, a small $3\nu_x+2\nu_y$ resonance (as verified by the tunes) which would never be seen in the usual plots. The following example will show how these small effects can combine in a serious manner.

Figure 17 is a carefully chosen example. It uses a different random set with more tune shift. The initial tune was set at .399,.399 which of course generates second-order coupled motion. In this case the tune shift coefficients are such that for smaller a (larger b) $\nu_x \rightarrow .4$, where there is a small third order resonance. The a-b path starts off normally towards small a but after a brush with the resonance it returns on a different path. After 2500 turns it has not returned even close to the initial point! Those "benign" difference resonances combined with amplitude dependent tune shift can use minor sum resonances to create meandering amplitudes.

The second suggestion is to make a prior selection of suitable random sets for tracking studies, based on distortion functions, and to ***include a statement of random set properties with the tracking results.***

One knows from ordinary orbit distortion that various random arrays with the same rms. can have very different effects, and this also applies to distortion functions. We must allow for "bad luck", particularly when we do not have clear diagnostic and correction proceedures. To find what is bad it will be necessary to mix a variety of extreme examples, with particular attention to tune shift.

The random arrays for our examples were selected by examining the amplitude of the distortion functions – $(B^2+A^2)^{1/2}$ – for several random seeds and choosing an interesting case. It is instructive to estimate the probability for the sextupole case, as an example.

Let there be N cells with n magnets per half cell. One can attribute n magnet errors to each quad (approximatly). The rms $\langle m\rangle$ will be $b_k\beta_0\Theta/n^{1/2}$ times a function of $\beta_x/\beta^0, \beta_y/\beta_0$. Standard scaling of lattices makes $\beta_0\Theta \sim 3$, and for the sextupole B_3, A_3 the function is 1; thus if $b_2=1.6/m^2$ for n=5 then $\langle m\rangle=2/m$. (We used $a\langle m\rangle=.01$, so a=5mm.) The rms value for B_3 (or A_3) should be $\langle m\rangle(N/2)^{1/2}/(8\sin \frac{1}{2}\mu_\alpha) \sim 3\langle m\rangle$. From ordinary orbit theory we know that 50% of the time we can expect a peak of twice the rms, or greater, and $A_3=.06$ is just twice the rms. Of course one was also selecting larger values for the other functions so the probability of a "worse" array is less than 50%, to say how much one needs to study the correlated distributions.

The distribution of ***second-order tune shift*** is particularly sensitive to correlations. All the coefficients have ***negative*** means and wide distributions, and have a tune dependence which suggests that one should avoid the lower ($0 \rightarrow \frac{1}{4}$)and upper quarters of tune space. It is necessary to evaluate the tune shift for each normal ***and skew*** sextupole array.

An all-in-together example. In figures 22-26 there is a combination of all the previous random sets plus skew sextupole. The rms values at the quads are:

	skew quadrupole	.004	(mean=0)
(~SSC	normal sextupole	.02/cm	
values)	skew sextupole	.02/cm	
	normal octupole	.01/cm^2	
	skew octupole	.01/cm^2,	

and the (equal) amplitudes are shown at the bottom left in cm. The first-order predictions are still good, however there are now many more higher combinations and hence a background of "little ones". Previously we saw a $2\nu_x-2\nu_y$ coupling. From the cross product of skew and normal sextupole there is even more at $\nu_x-\nu_y$, and this shows in the "corrected" plots as a modest contribution to the smearing. The 7mm case is included to show that a particle can be stable but totally irrational!

NON-TRACKING

The most significant results from this method may well come from simple expressions derived without tracking. Consider this example. Superconducting dipoles have a ***systematic sextupole*** at low field. It will be corrected by the chromaticity adjustment of sextupoles at the quads. Is this good enough?

In the following table, the values for $\frac{1}{2}S_F$ and $\frac{1}{2}S_D$ and the beta functions were calculated by the usual thin lens expressions. The distortion functions are calculated like off-momentum vectors (with α and s) Use A's=0 at quads for closure.

	quad F	.18	.34	.50	.66	.82	quad D	
			dipole	position				
β_x	1	.785	.621	.483	.371	.284	.217	(β_0)
β_y	.217	.284	.371	.483	.621	.785	1	(β_0)
φ_s	0	3.48	7.43	12.45	18.96	27.46	40	deg.
ϑ_s	0	12.54	21.04	27.55	32.57	36.51	40	deg.
S	-.325	.2	.2	.2	.2	.2	-.537	$(b_2\theta)$
s	-.325	.139	.098	.067	.045	.034	-.054	$(\beta_0 b_2\theta)$
$\bar{s}$	-.071	.050	.058	.067	.076	.084	-.250	$(\beta_0 b_2\theta)$
B_3	.0293	.0141	.0034	-.0039	-.0074	-.0057	.0032	$(\beta_0 b_2\theta)$
B_1	.0089	.0035	.0003	-.0017	-.0023	-.0014	.0015	
B_s	-.0045	-.0124	-.0119	-.0056	.0054	.0219	.0447	
B_d	.0105	.0033	.0014	.0023	.0037	.0033	-.0028	
$\bar{B}$	-.0011	-.0022	-.0025	-.0017	.0014	.0081	.0222	

$\sum B_3 s = -.0164$ $\quad\sum B_s \bar{s} = -.0205$ $(\beta_0 b_2\theta)^2$ for a full cell
$\sum B_1 s = -.0052$ $\quad\sum B_d \bar{s} = .0018$
$\sum B_1 \bar{s} = -.0024$ $\quad\sum \bar{B}\bar{s} = -.0101$ $\qquad\theta = \frac{1}{2}$cell bend

In a normal proton ring $\beta_0\theta \sim 3$m. The tune shift is ***systematic***, for N cells

$$\Delta\nu_x = (.0025\, a^2 + .0022\, b^2)\, N\, (\beta_0 b_2\theta)^2$$
$$\Delta\nu_y = (.0022\, a^2 + .0050\, b^2)\, N\, (\beta_0 b_2\theta)^2$$

Assume $N=100$, $b_2 \sim 1.3/\mathrm{m}^2$ then $\Delta\nu_y = .0012$ (doubler)
$N=400$, $b_2 \sim 33/\mathrm{m}^2$ at 1cm $\Delta\nu_y > 2$ units! (an SSC design)

The distortion and tune shift in each cell is very small. The distortion repeats from cell to cell, which keeps it small, but the tune shift accumulates. Tracking adds nothing new.

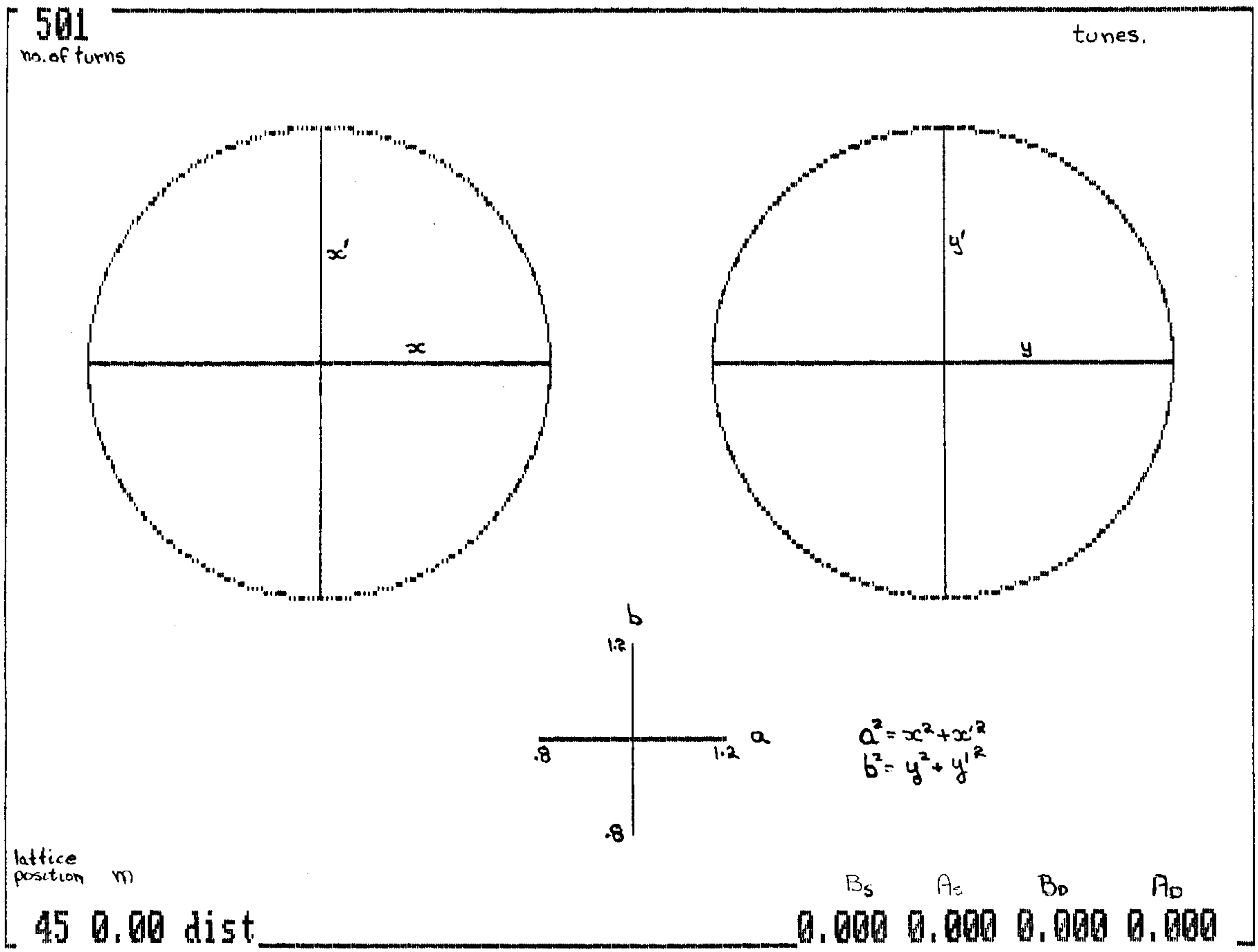
We have "derived" ***an important, second-order non-linear expression***, using a hand calculator!

CONCLUSIONS

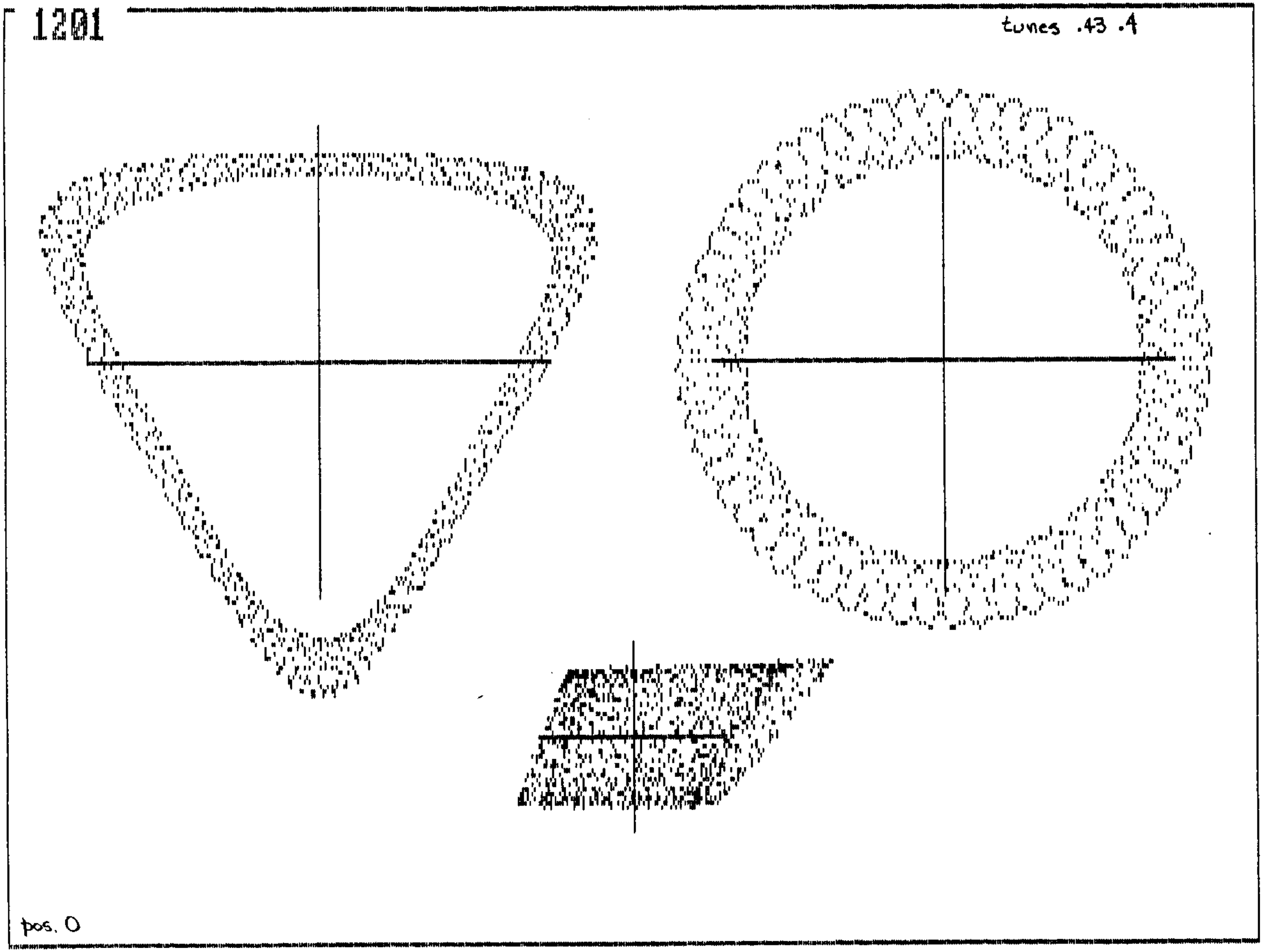
We have a set of ***distortion functions*** which are a new form of solution to the old problem of describing the ***non-linear beam shape that repeats***. They have an exact formal similarity to our usual orbit distortion calculation, and therefore have the nice property that a modest arithmetic effort will yield a precise value at one point in the ring and twice that effort will give values at all points. This latter property pemits an easy evaluation of the second-order tune shift from sextupoles - both types - which is very important.

The distortion is expanded in "orders" which here means the power of the multipole. There is no reason why one cannot extend the calculations to higher orders but from the numerical examples one can see that practically the first order is enough (except for tune shift). The first order does describe most of the distortion for marginally acceptable beams. Higher order effects do appear as small, closed resonances but there is often not much distinction between second and still higher orders. In any case the nature of the expansion (and the examples) makes clear that ***small first order is a necessary and sufficient conditon for good beams.*** One possible exception are higher-order coupling terms, $n\nu_x - n\nu_y$, from sextupoles.

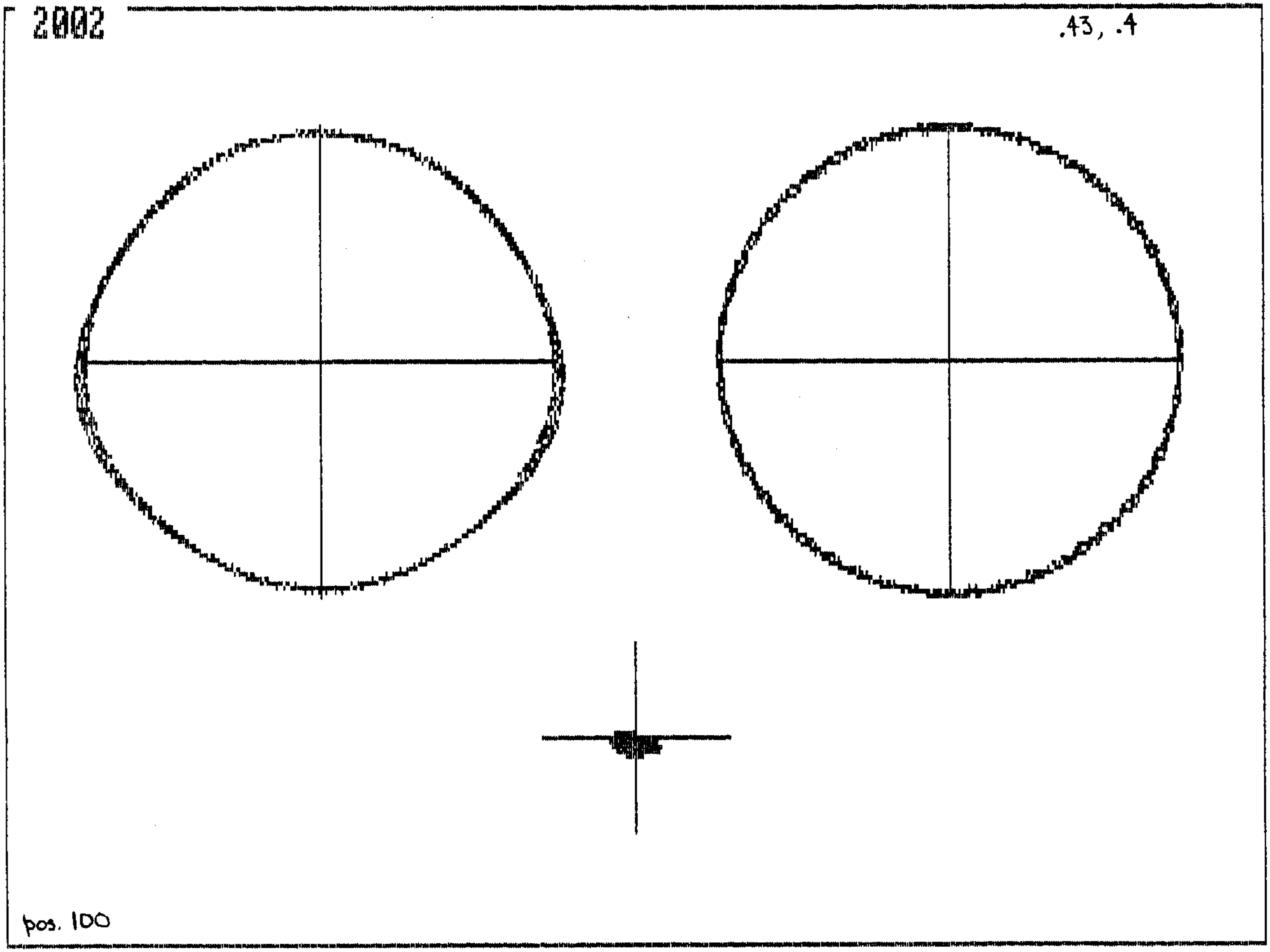
This paper is avowedly pedagogical. The author has absolutely no desire to be the custodian of a non-linear calculation code. In fact the primary purpose of this theory is to diminish our dependence on extensive calculation by providing a design tool. The examples in this paper were calculated in BASIC on a PC, and no program listing exceeded two pages, so anyone can join in.



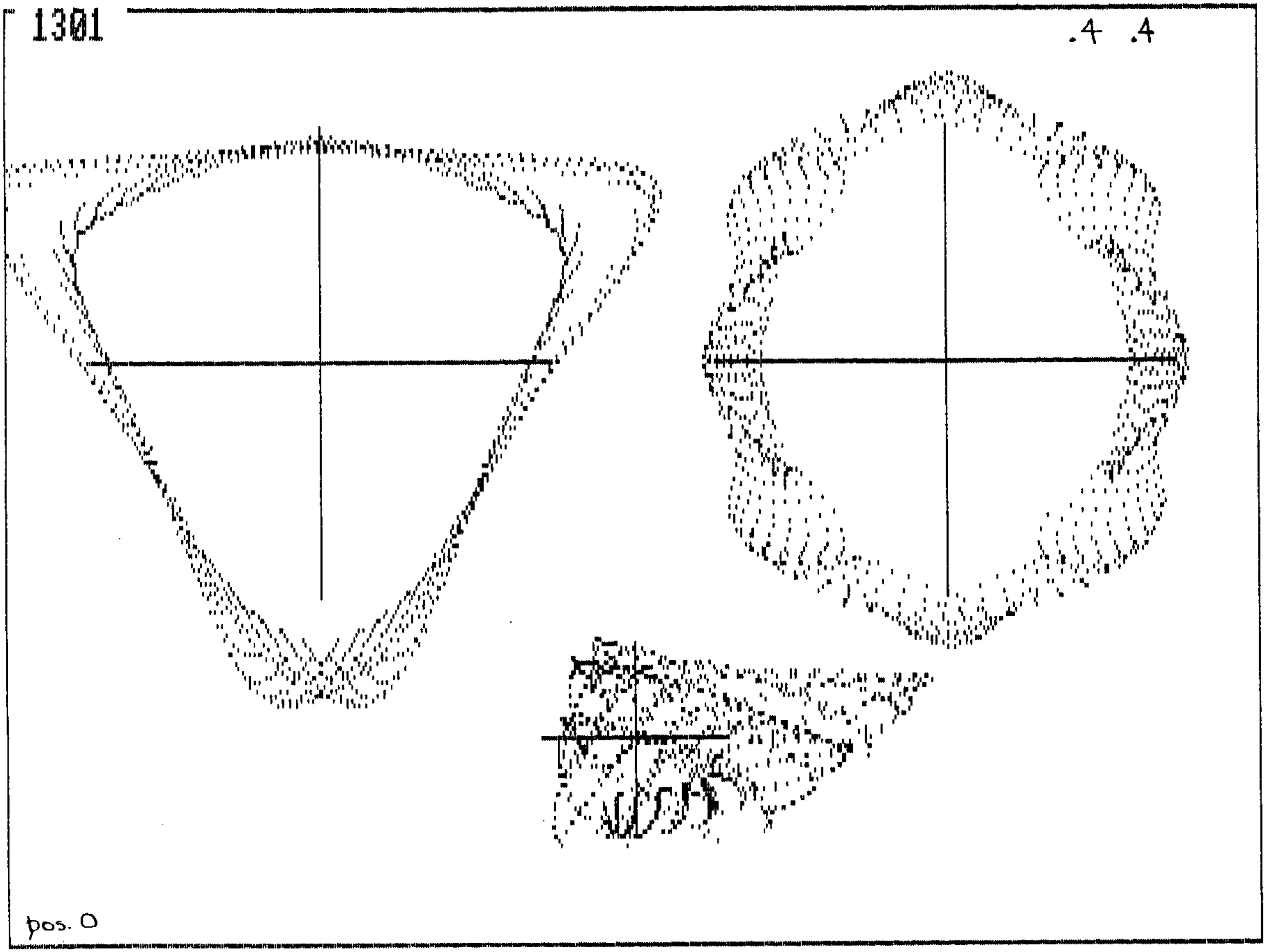
1. Plots are circles if undistorted.



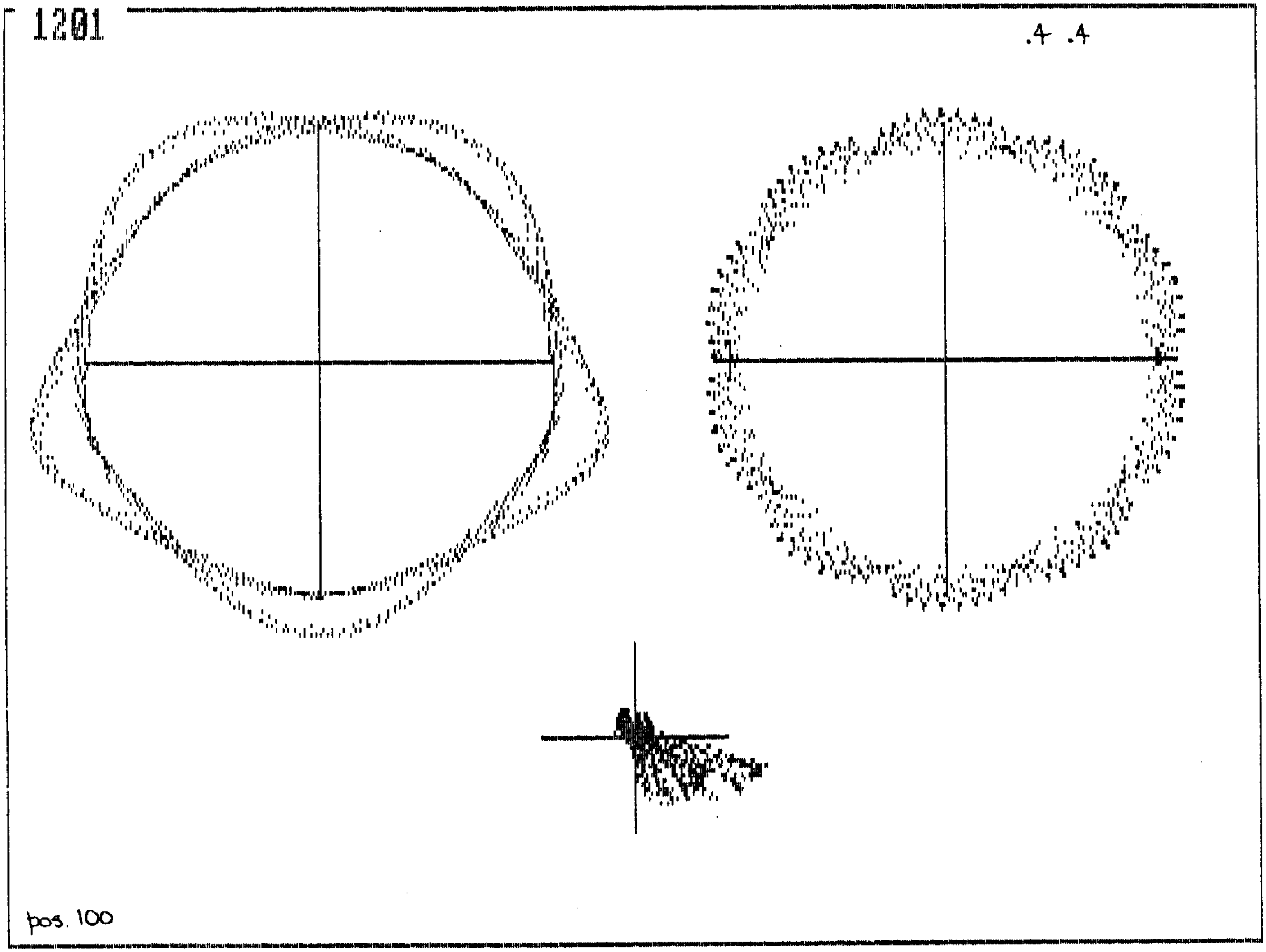
2. Sextupole distortion bump.



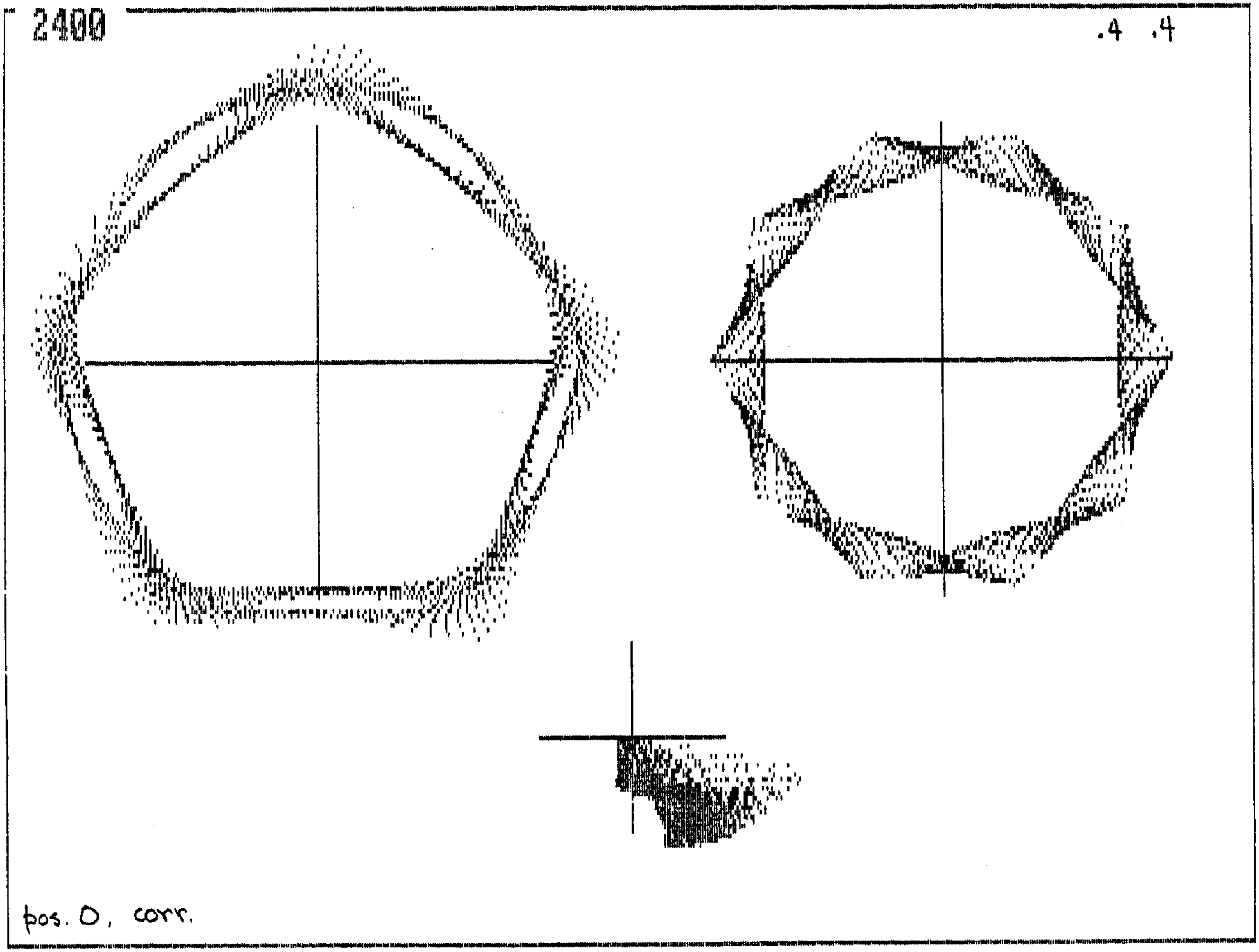
3. Sextupole distortion bump.



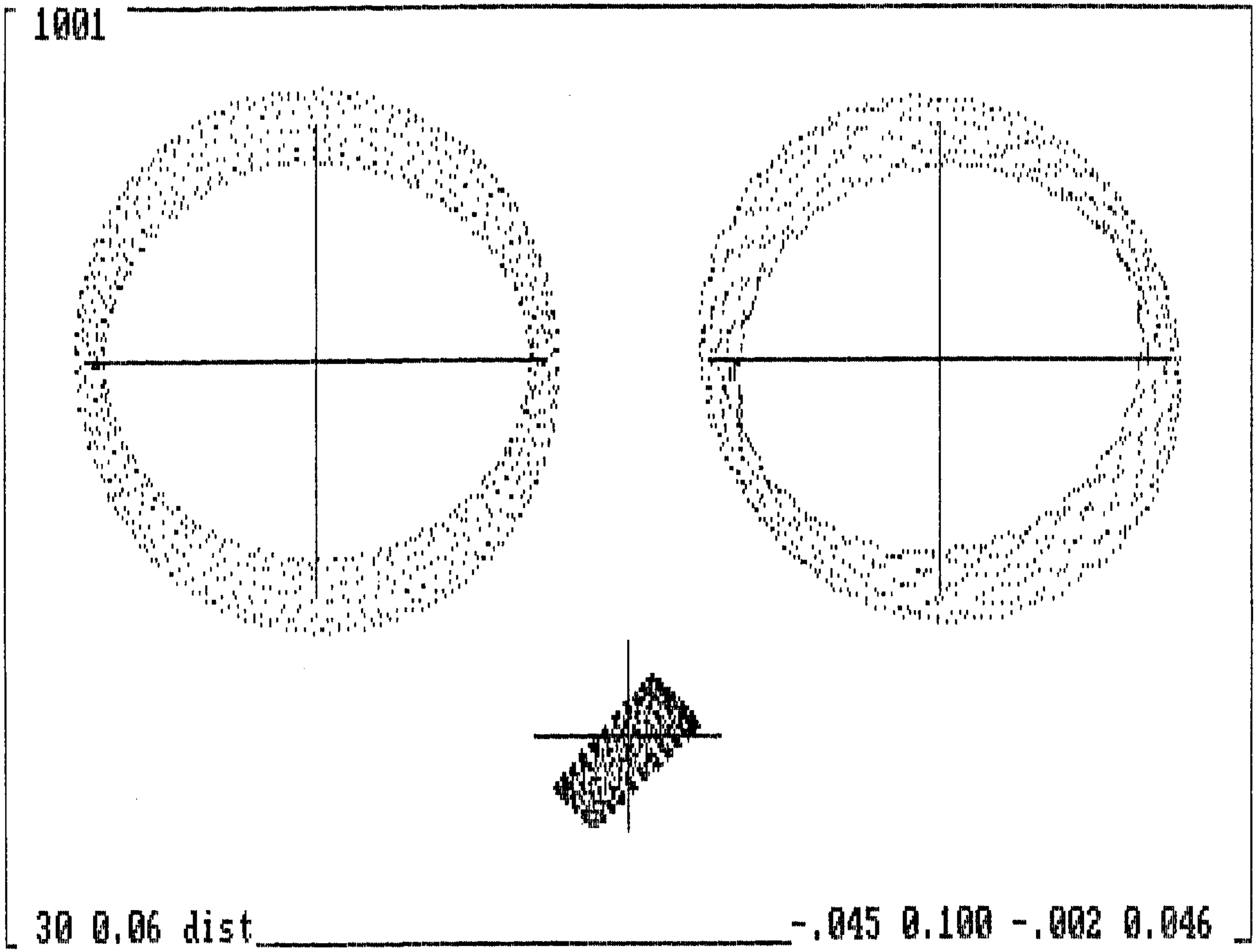
4. Sex. dust. bump



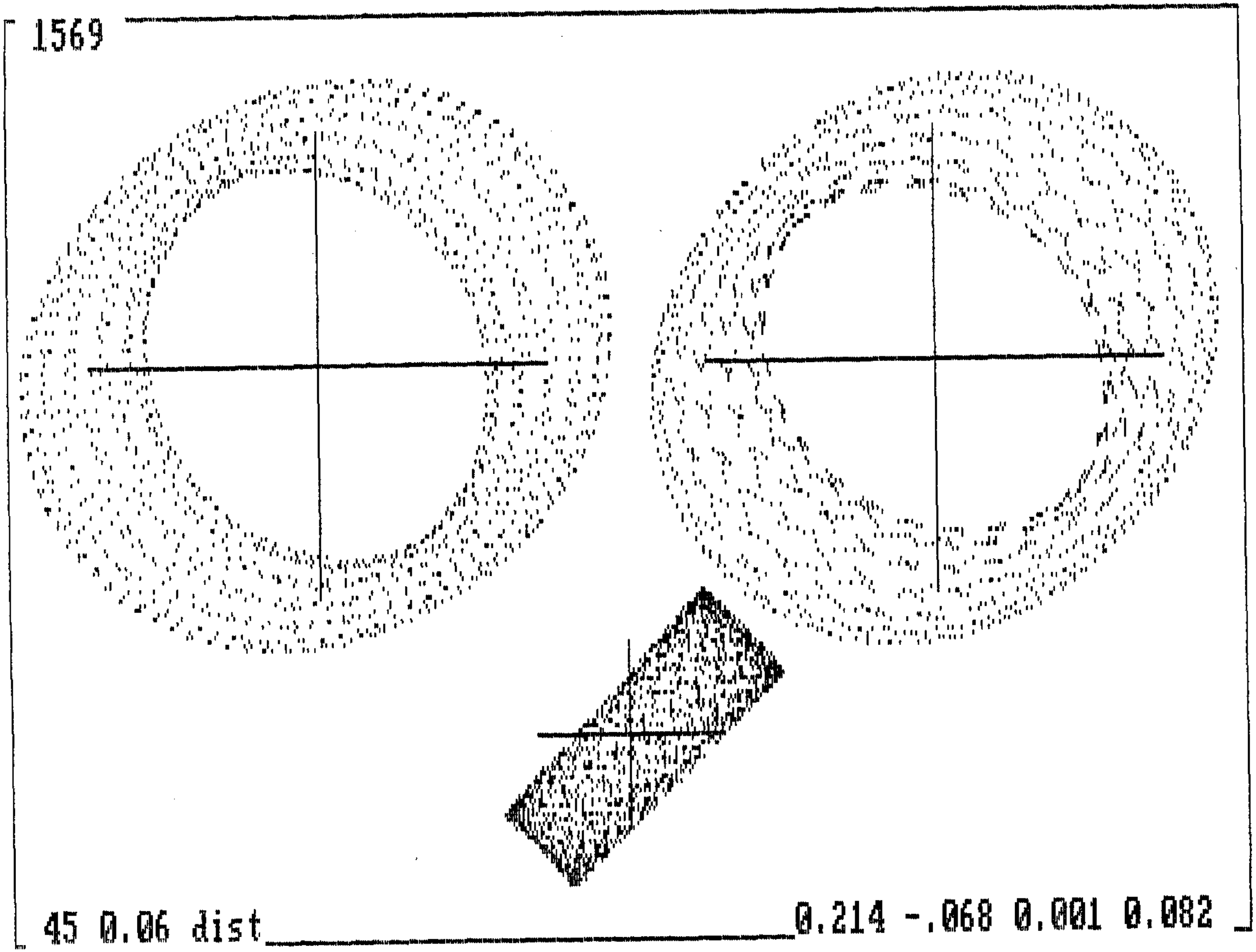
5. Sex. dist. bump.



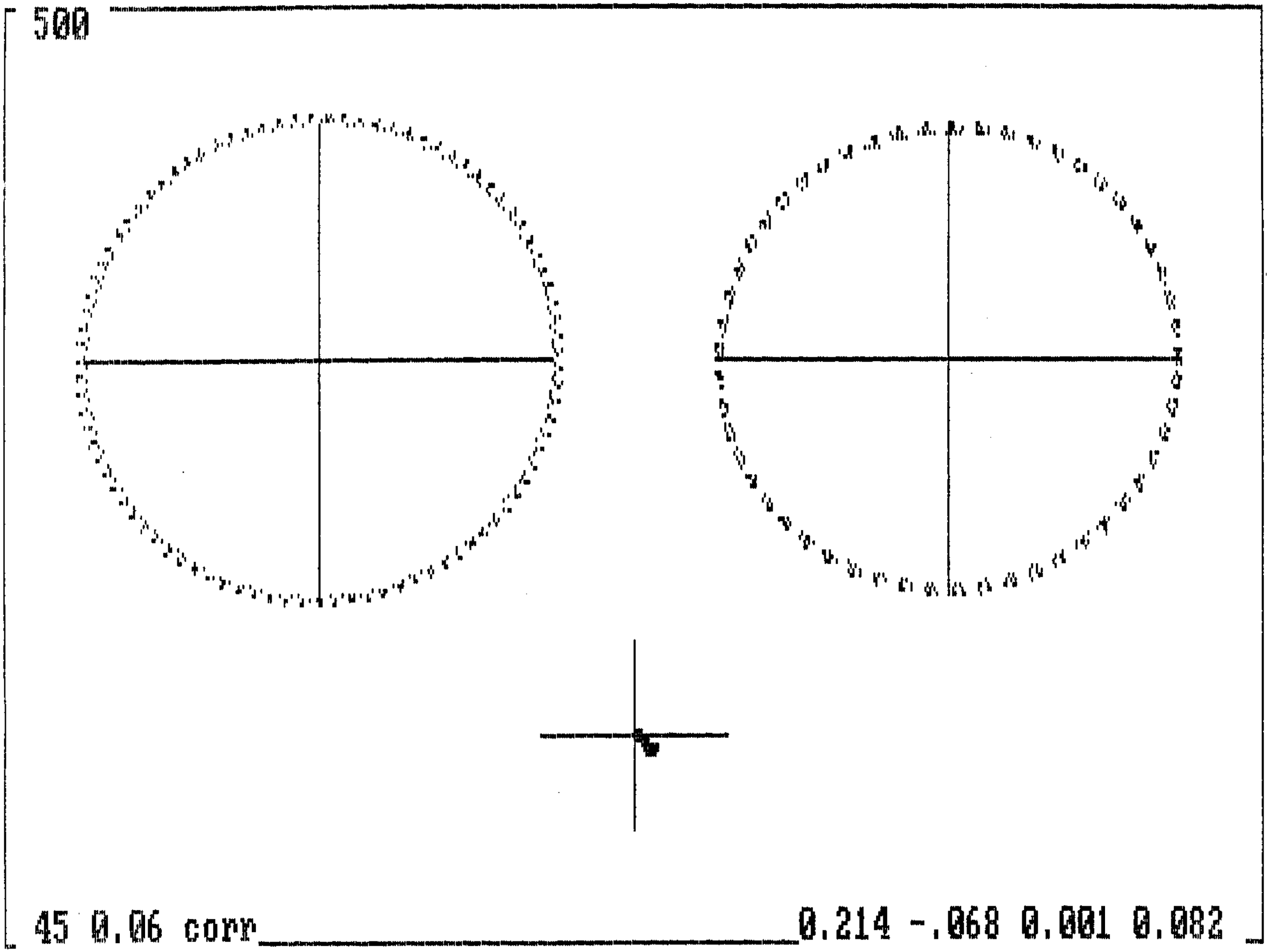
6. Sex. dist. bump, with inverse dist. calculation



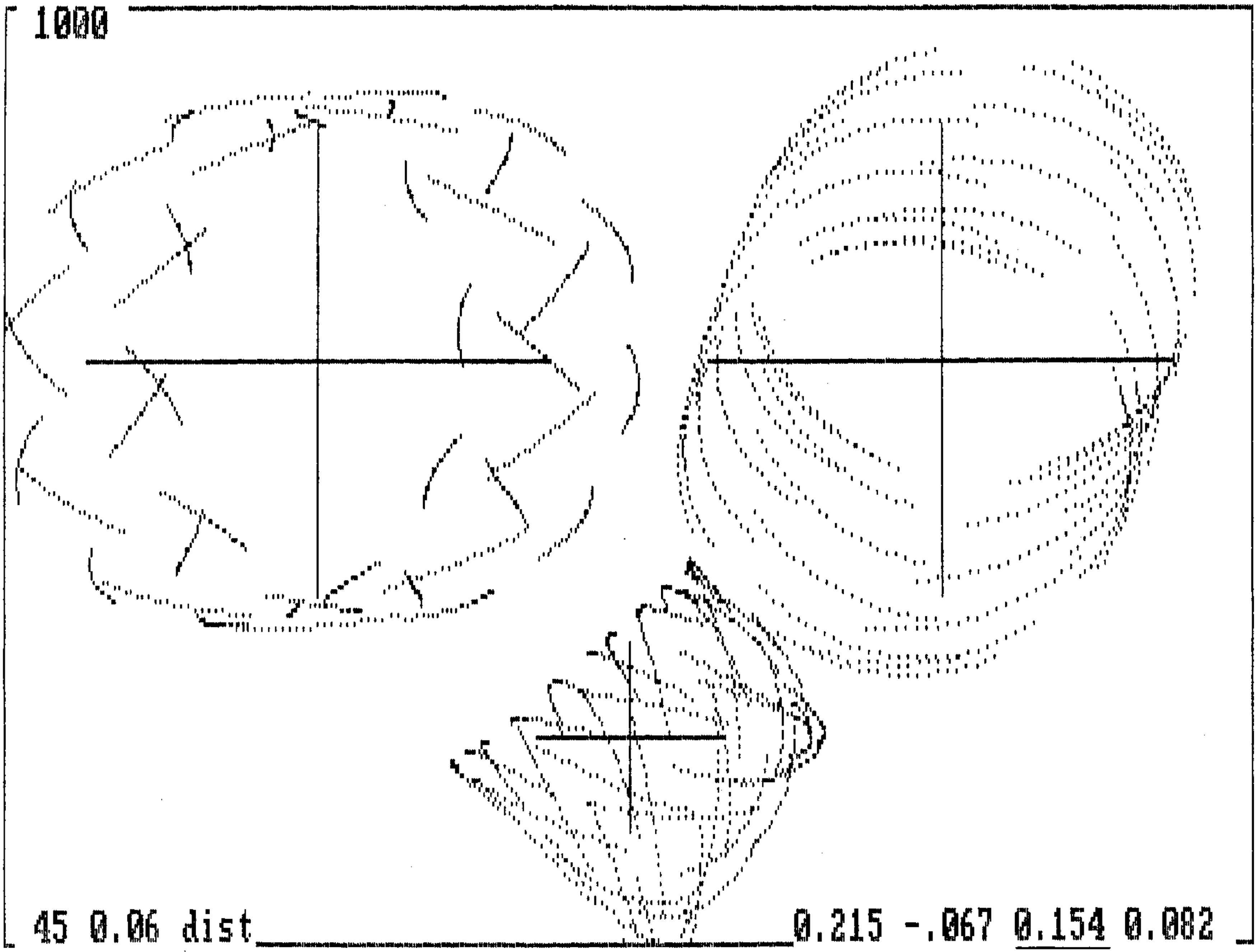
7. Rnd. Skew Quad.



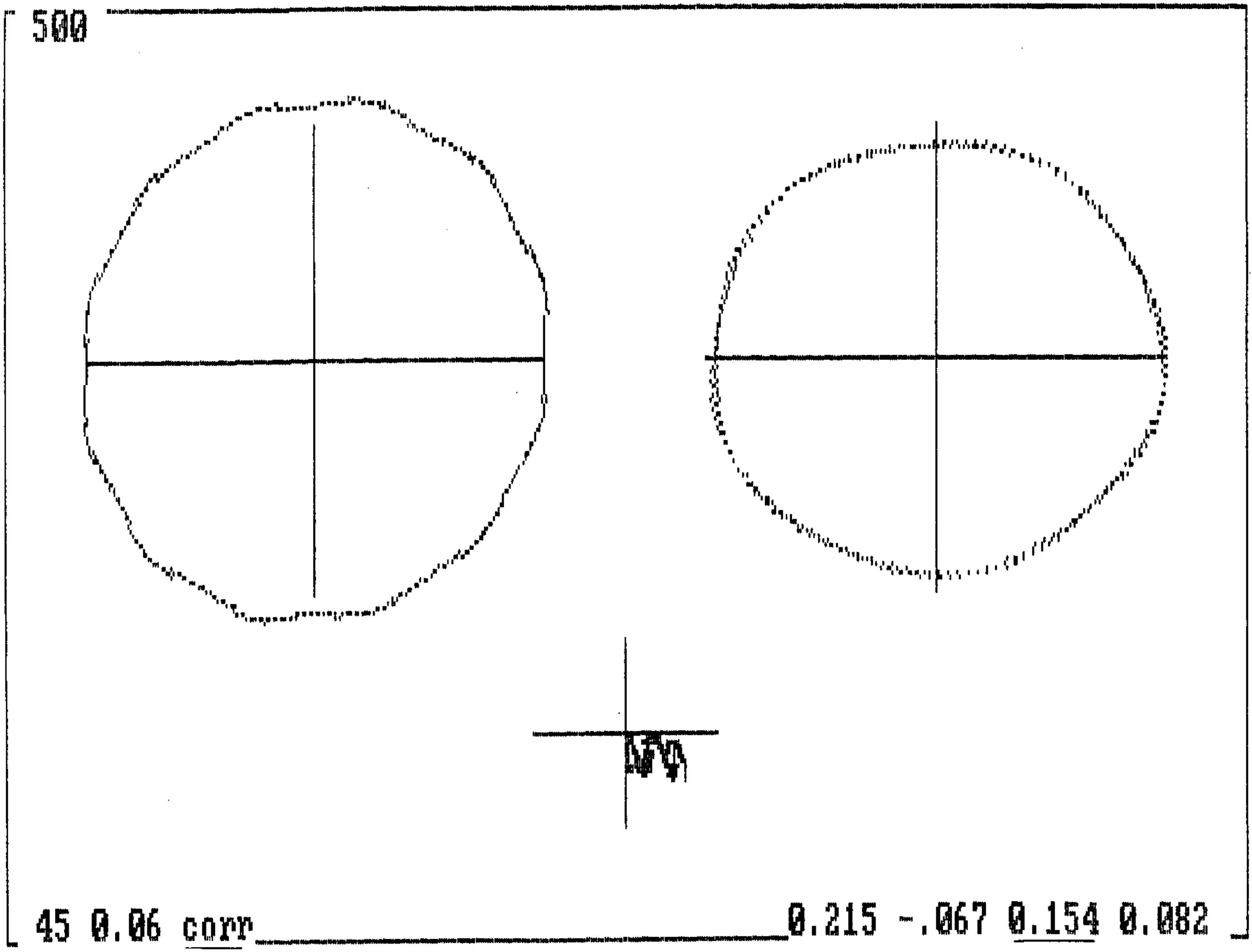
8. Rnd. Skew Quad.



9. RND SKEW QUAD.

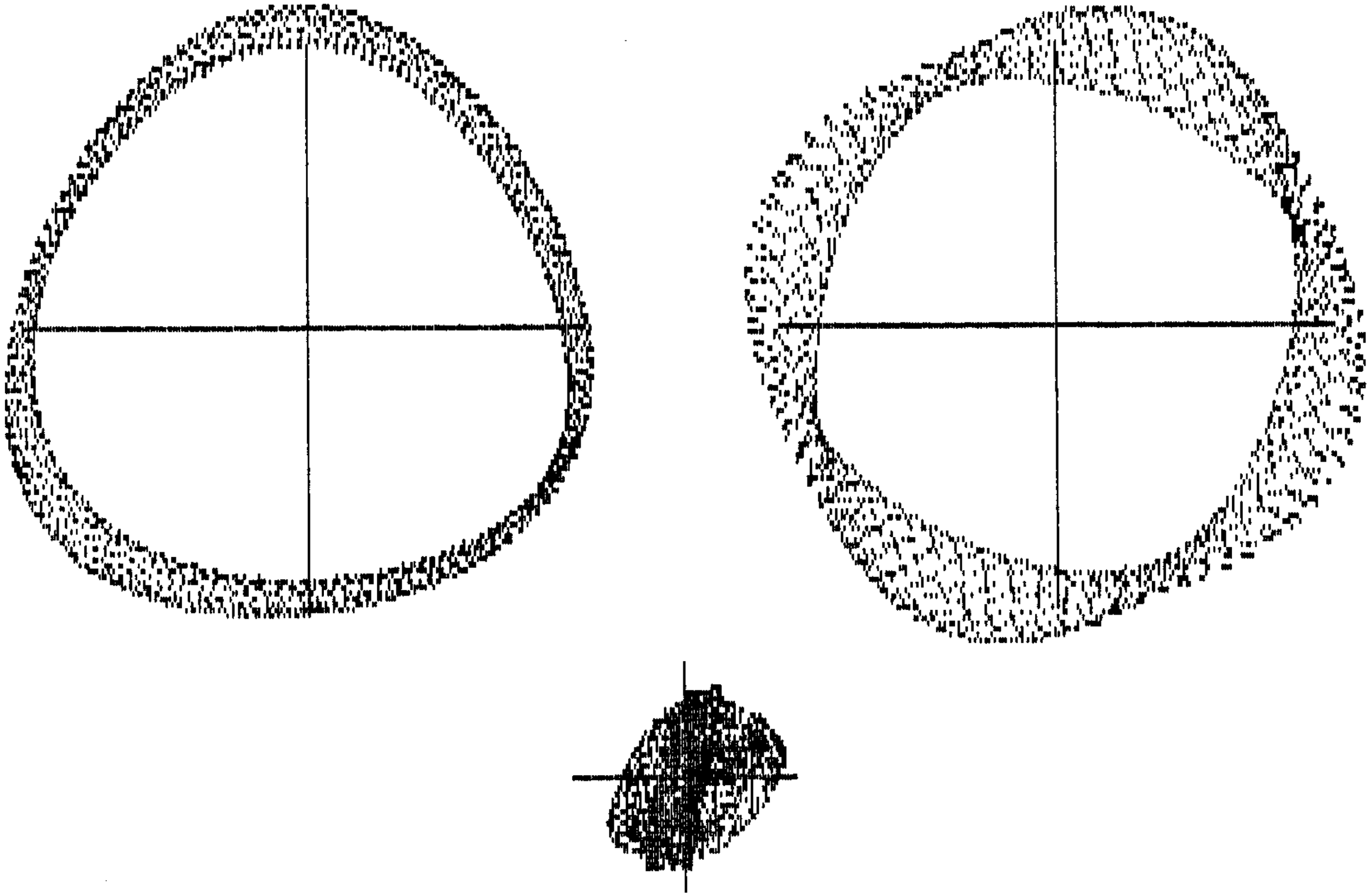


10. RND. SKEW QUAD.



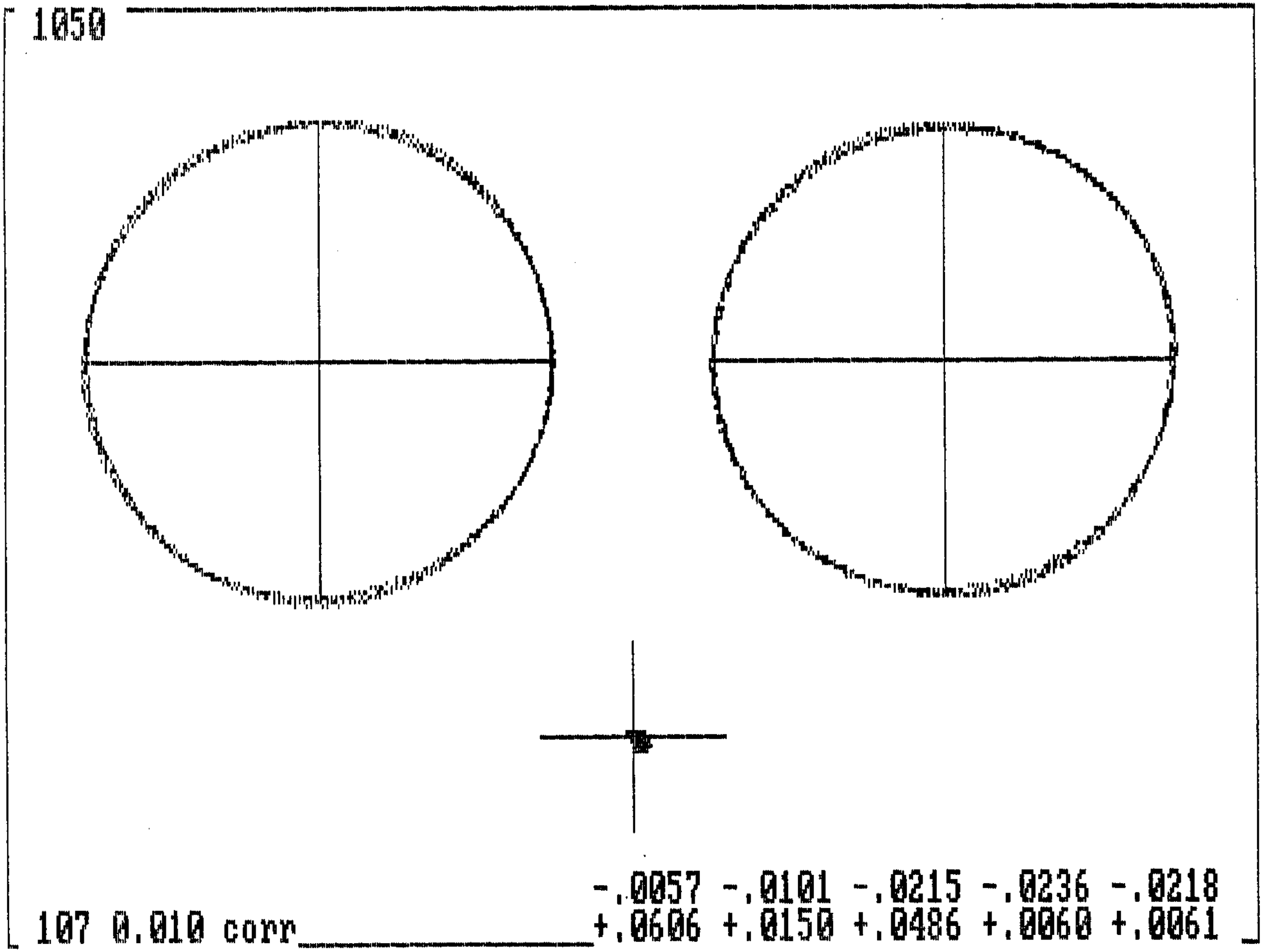
11. RND SKEW QUAD.

2558

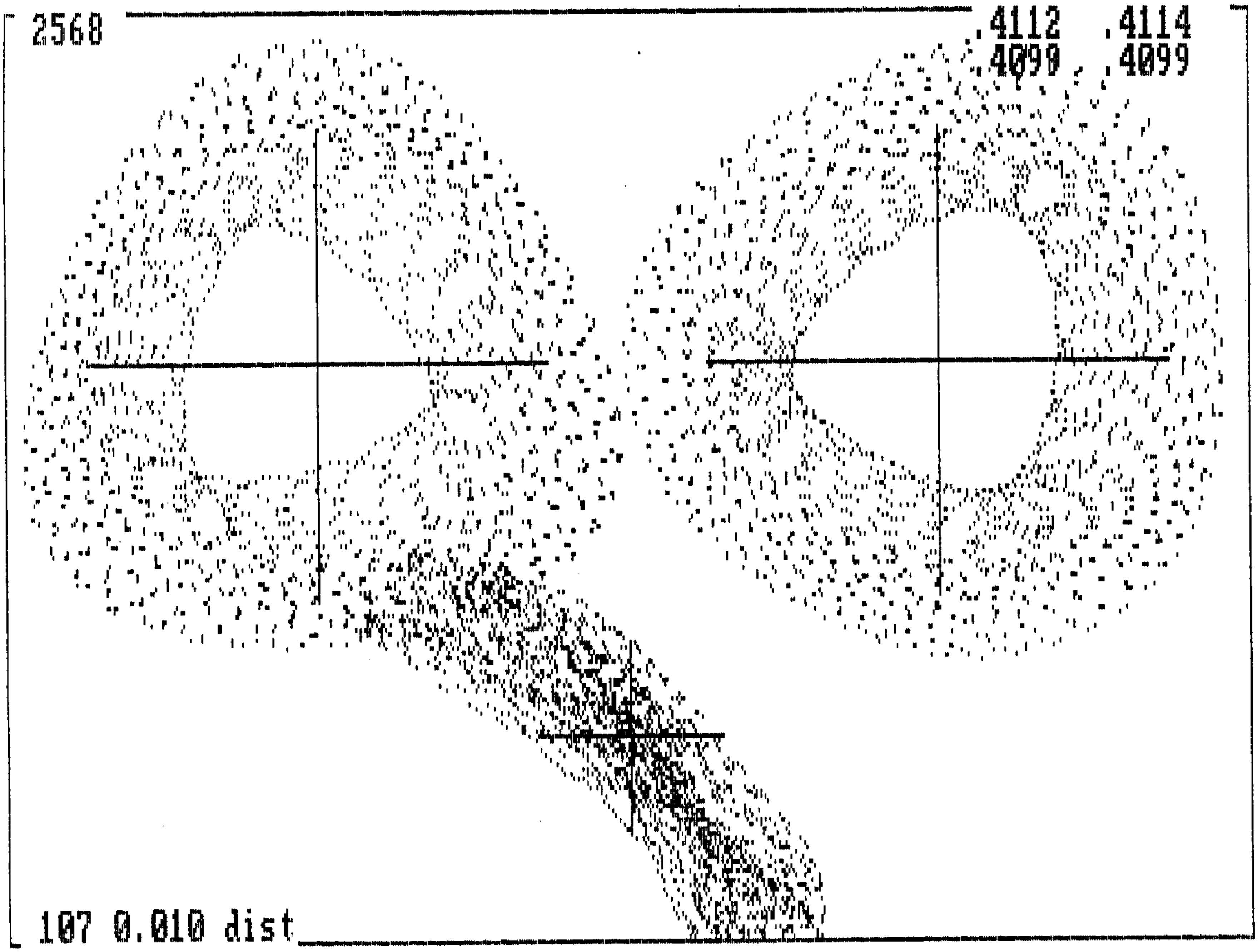


	B_a	B_1	B_s	B_D	$\bar{B}$
	-.0057	-.0101	-.0215	-.0236	-.0218
107 0.010 dist	+.0606	+.0150	+.0486	+.0060	+.0061

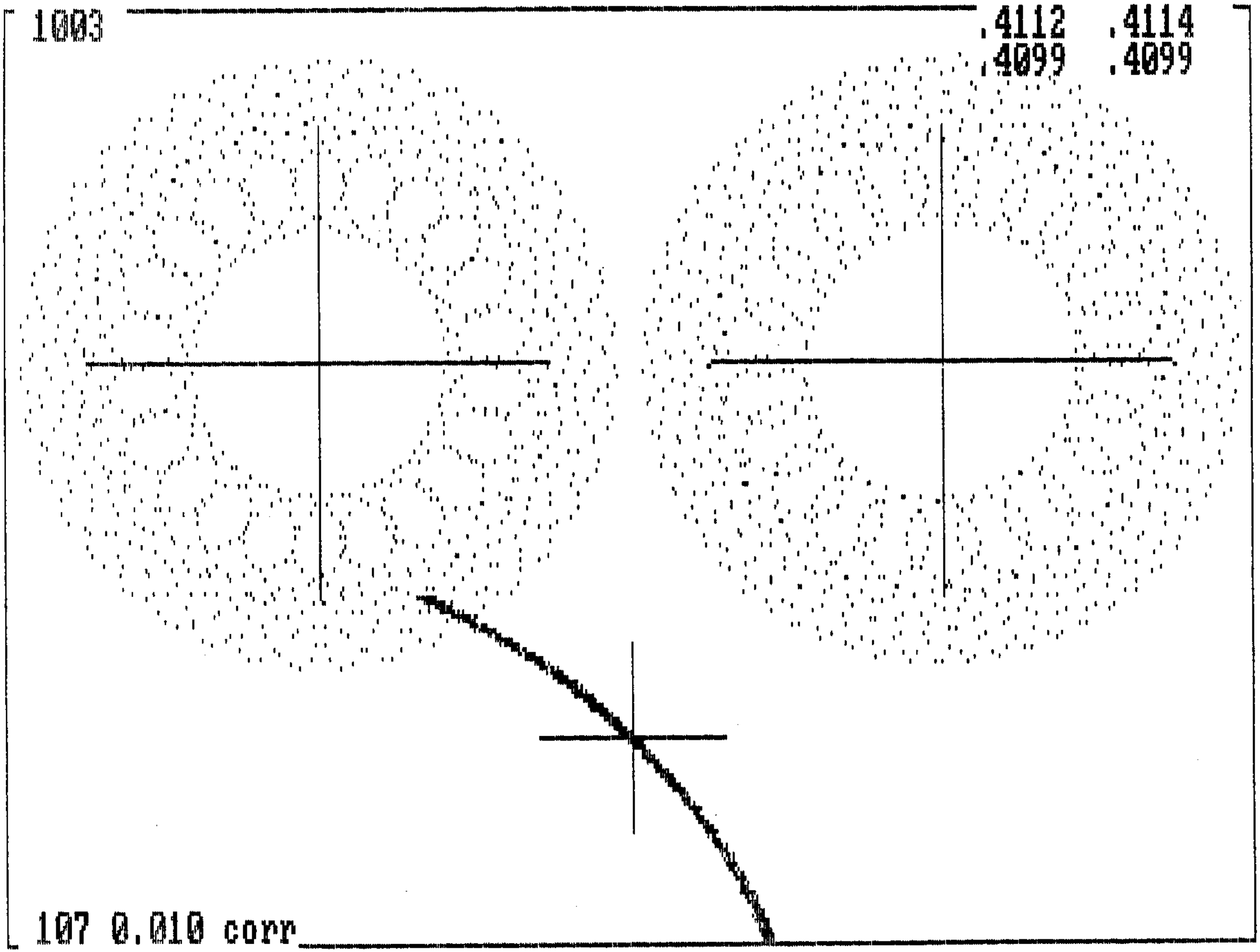
12. RANDOM SEXTUPOLE.



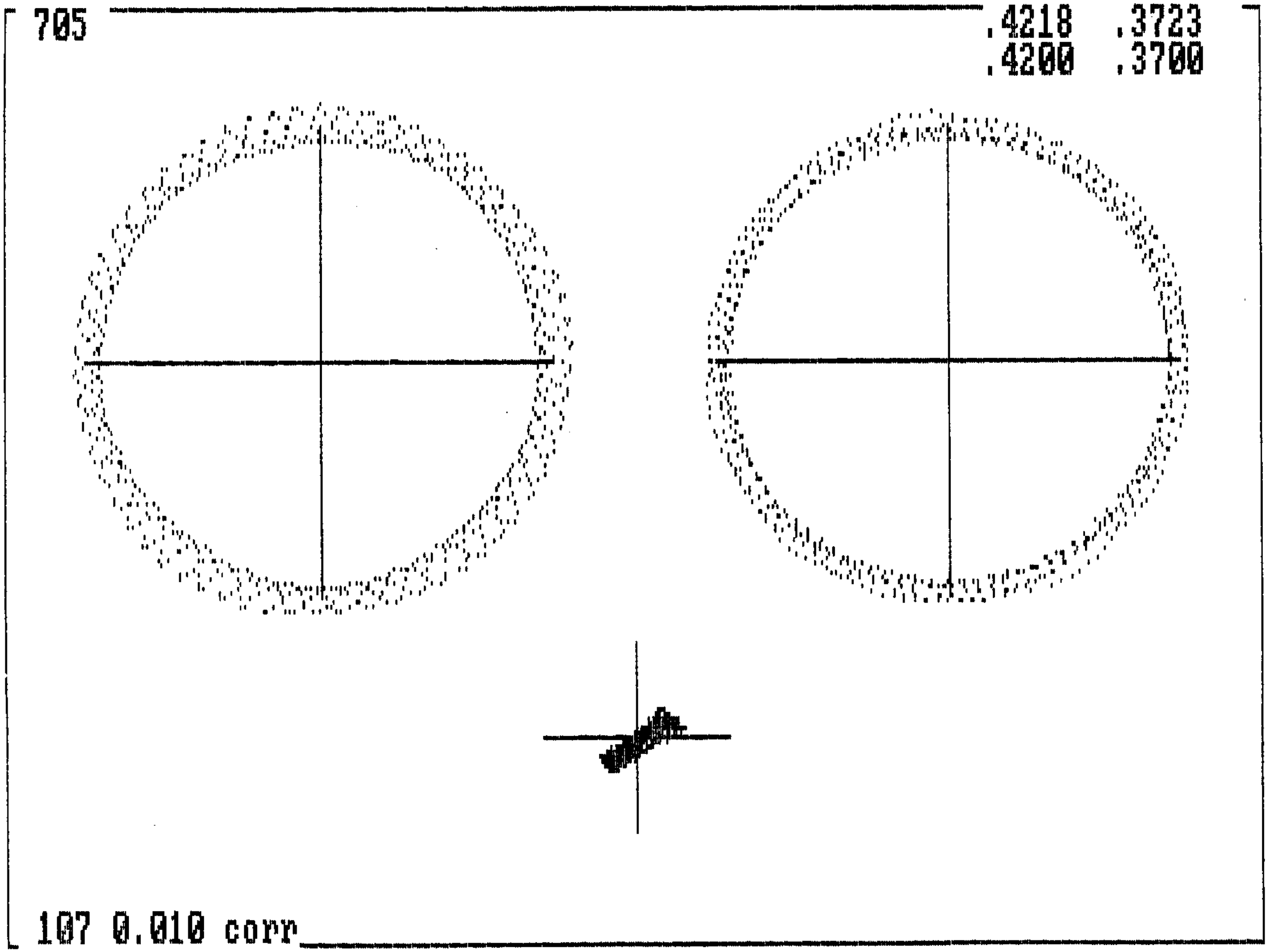
13. Random Sext.



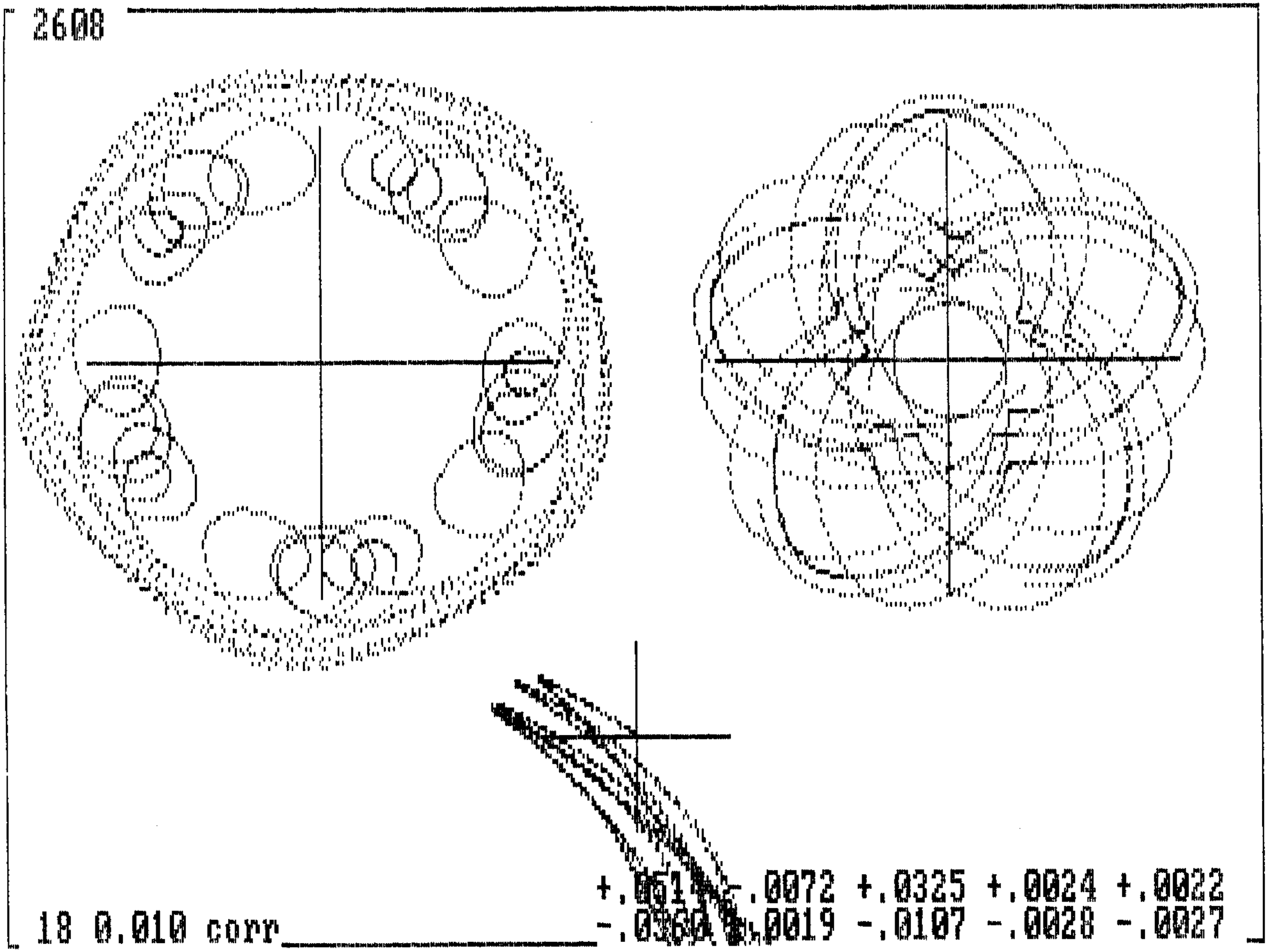
14. Rnd. Sext at $2\varphi - 2\theta = 0$



15. Rno Sext. Coupling

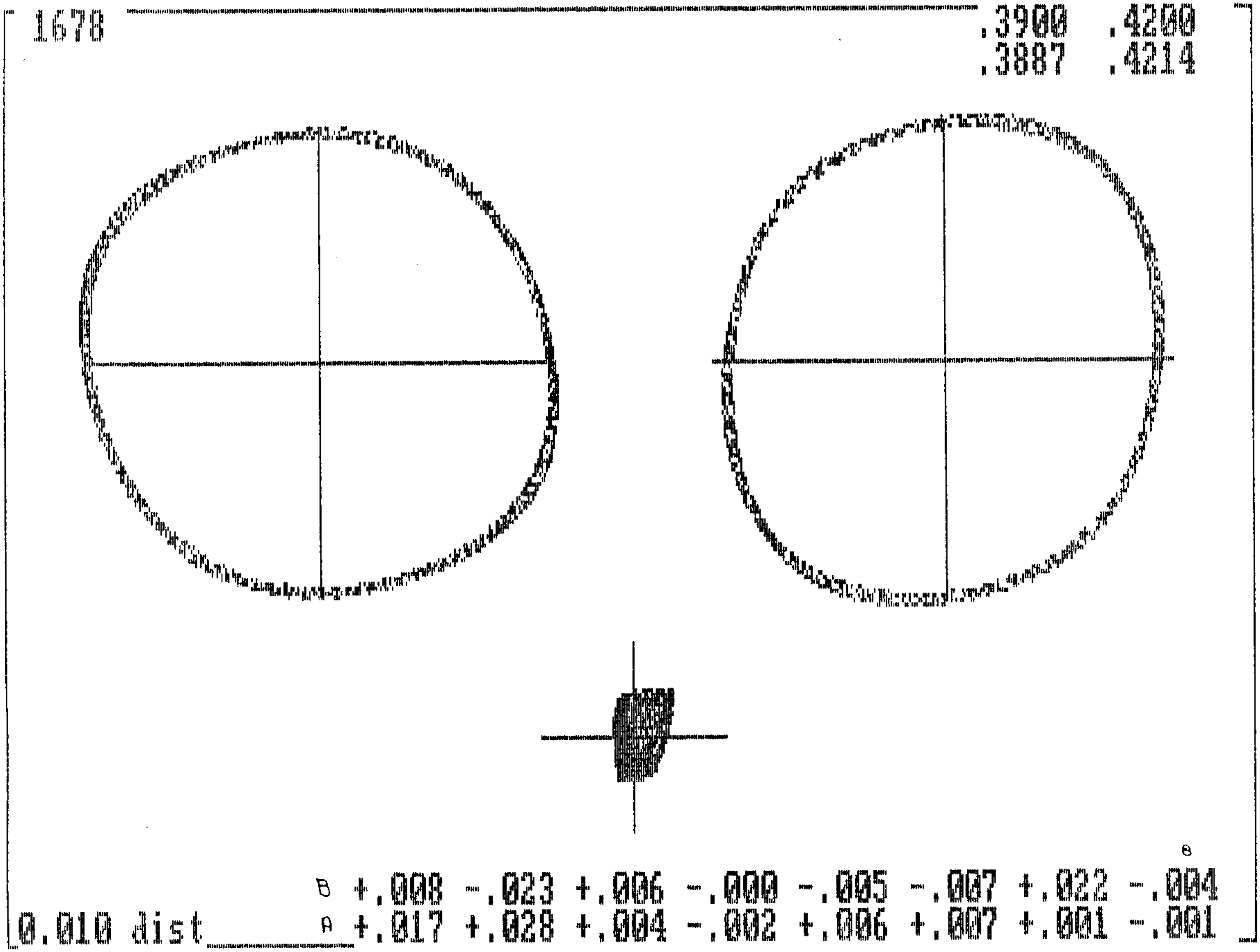


16. Sex. at $3\nu_x + 2\nu_y$

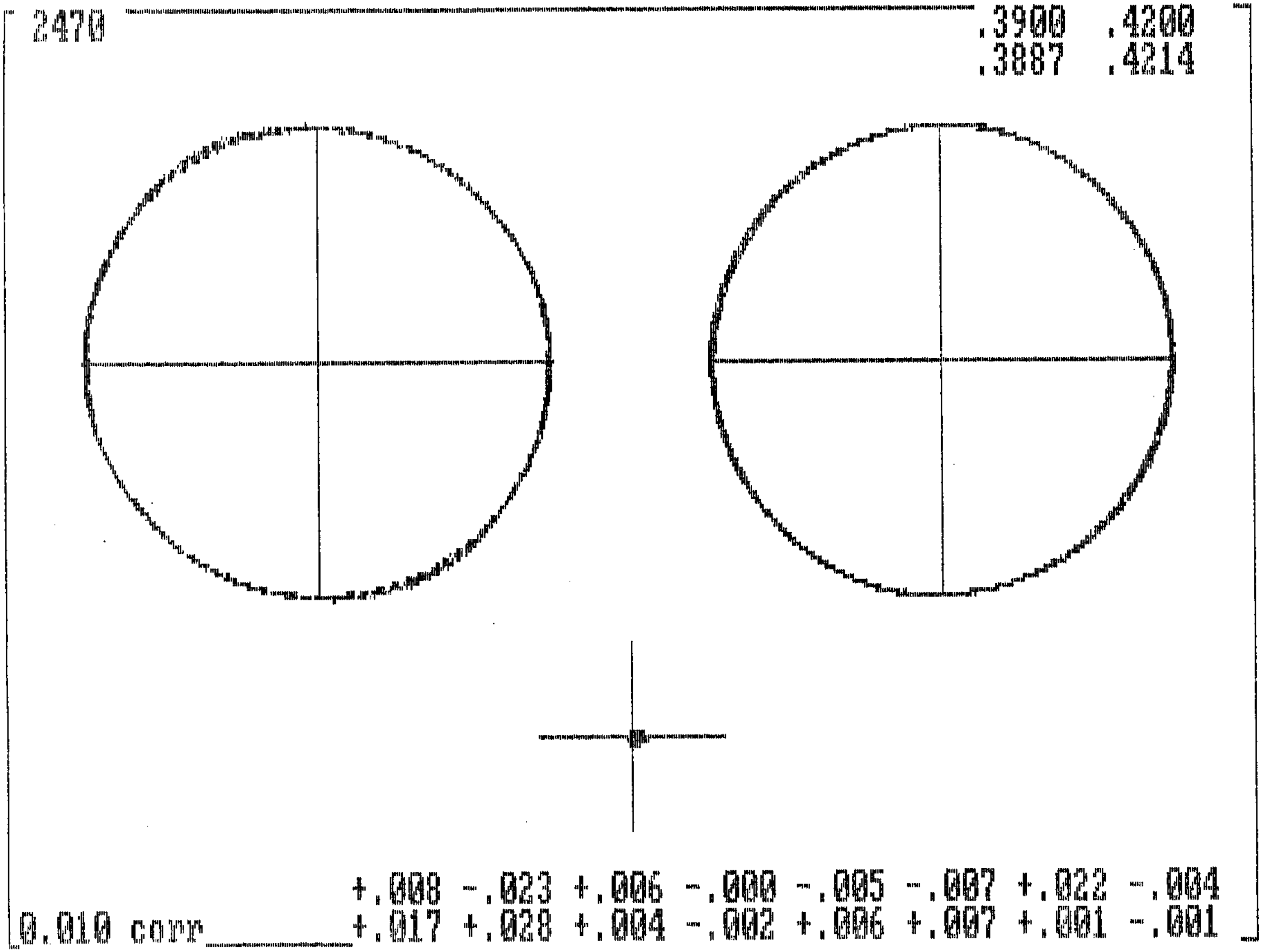


new rnd. array

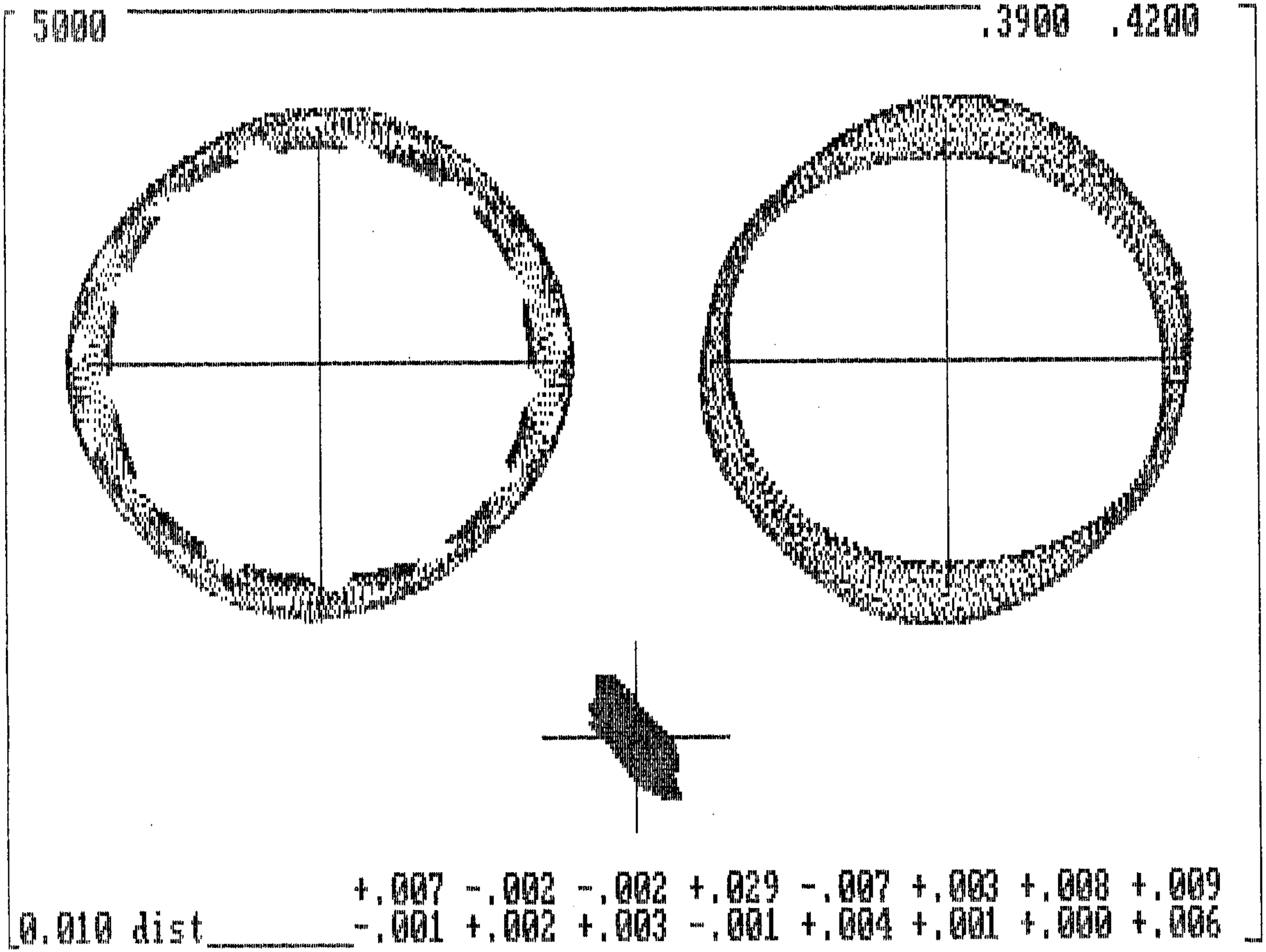
17. Rnd Sext. Not repeating.



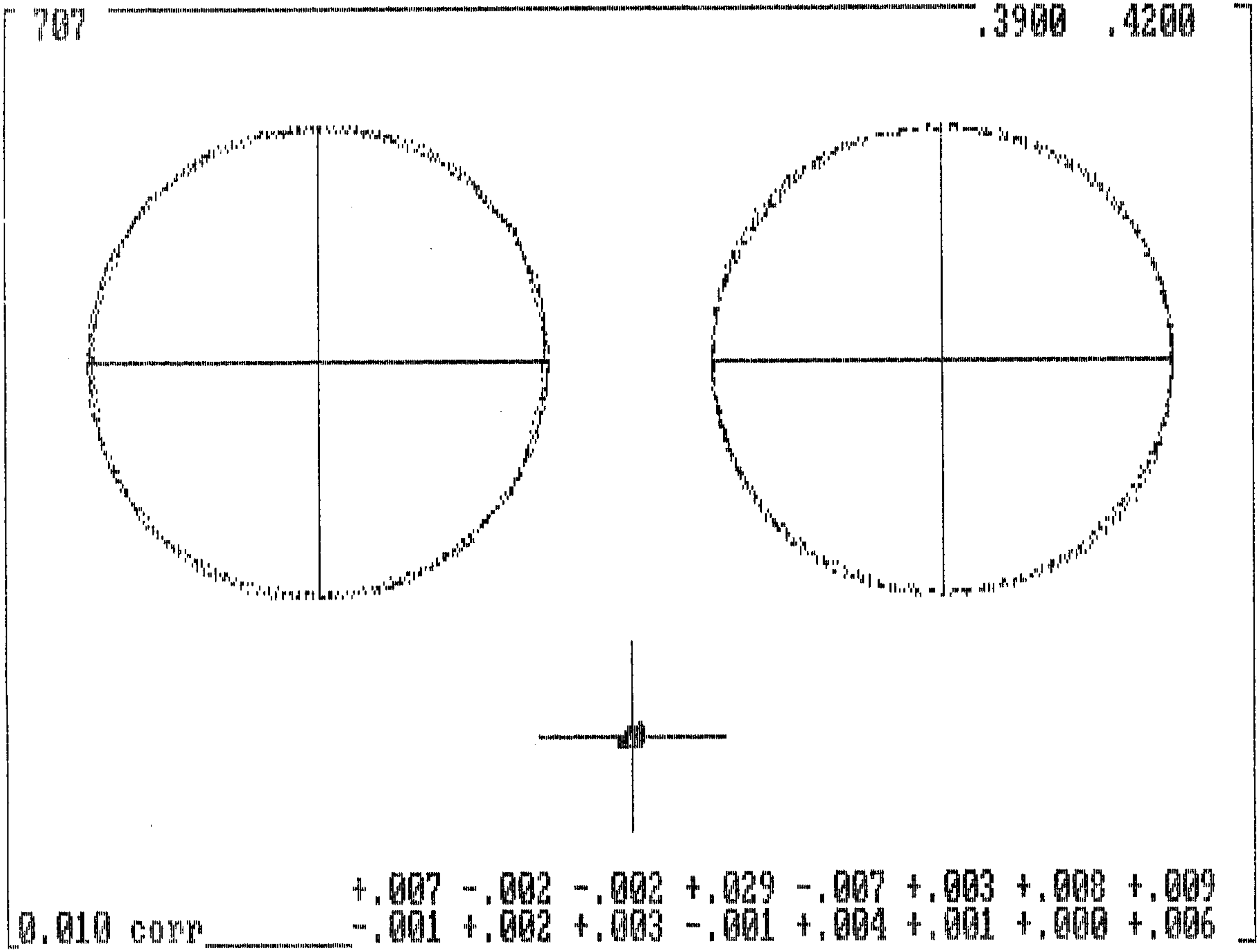
18. Normal Octupole.


2470
.3900 .4200
.3887 .4214
+.008 -.023 +.006 -.000 -.005 -.007 +.022 -.004
0.010 corr
+.017 +.028 +.004 -.002 +.006 +.007 +.001 -.001

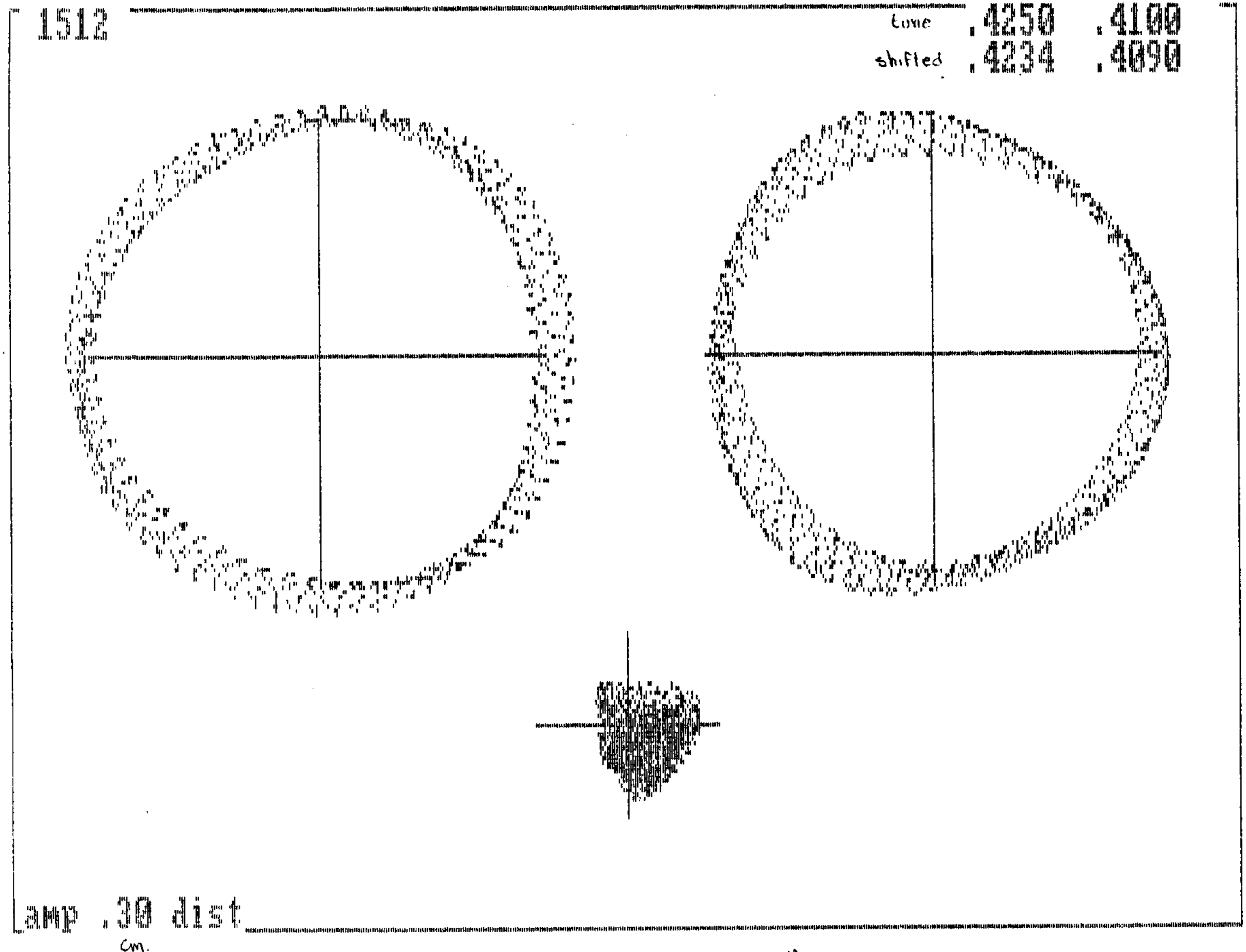
19. Norm. Oct.



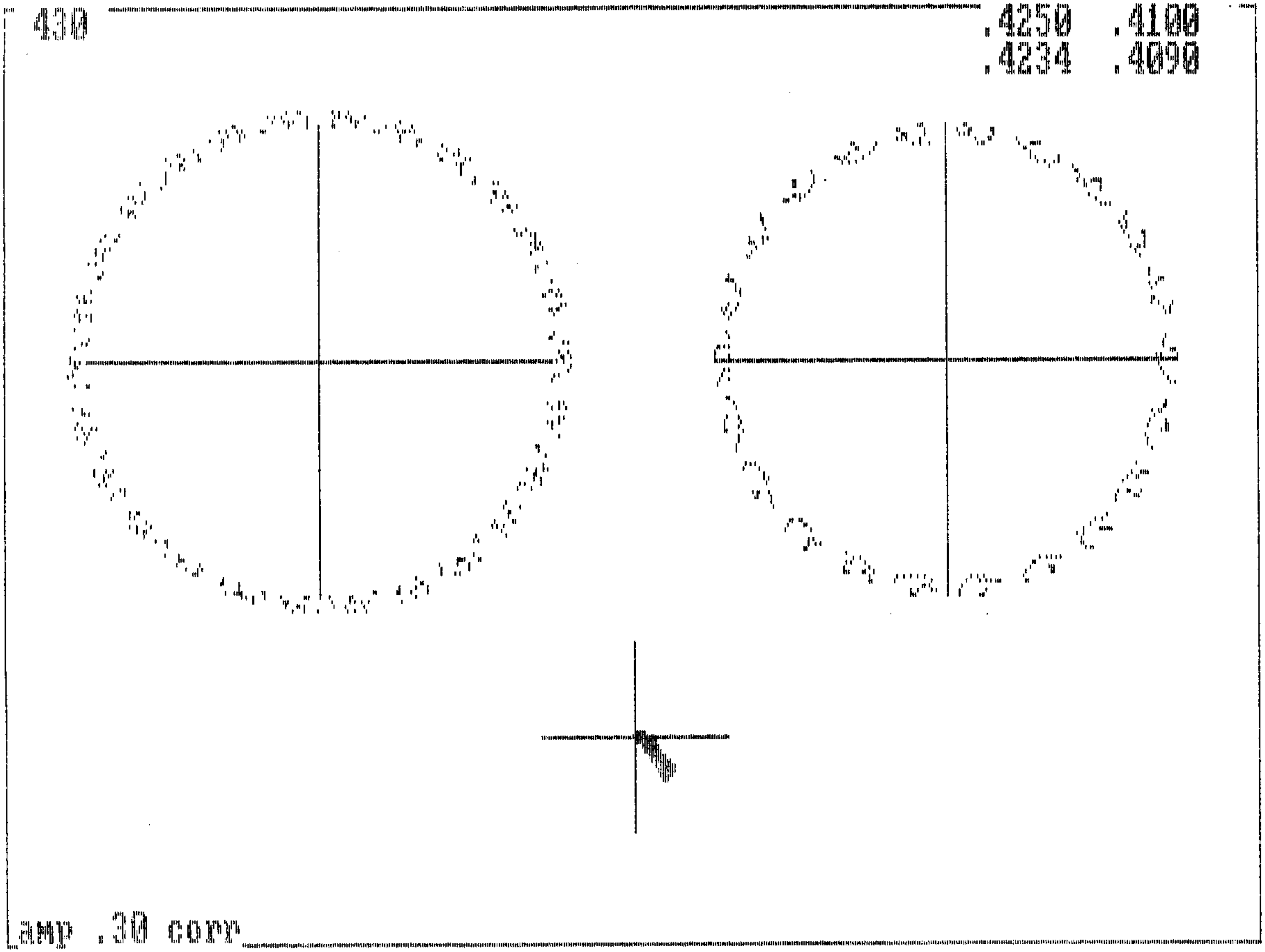
20. Skew Octupole


707
.3900 .4200
+.007 -.002 -.002 +.029 -.007 +.003 +.008 +.009
0.010 corr
-.001 +.002 +.003 -.001 +.004 +.001 +.000 +.006

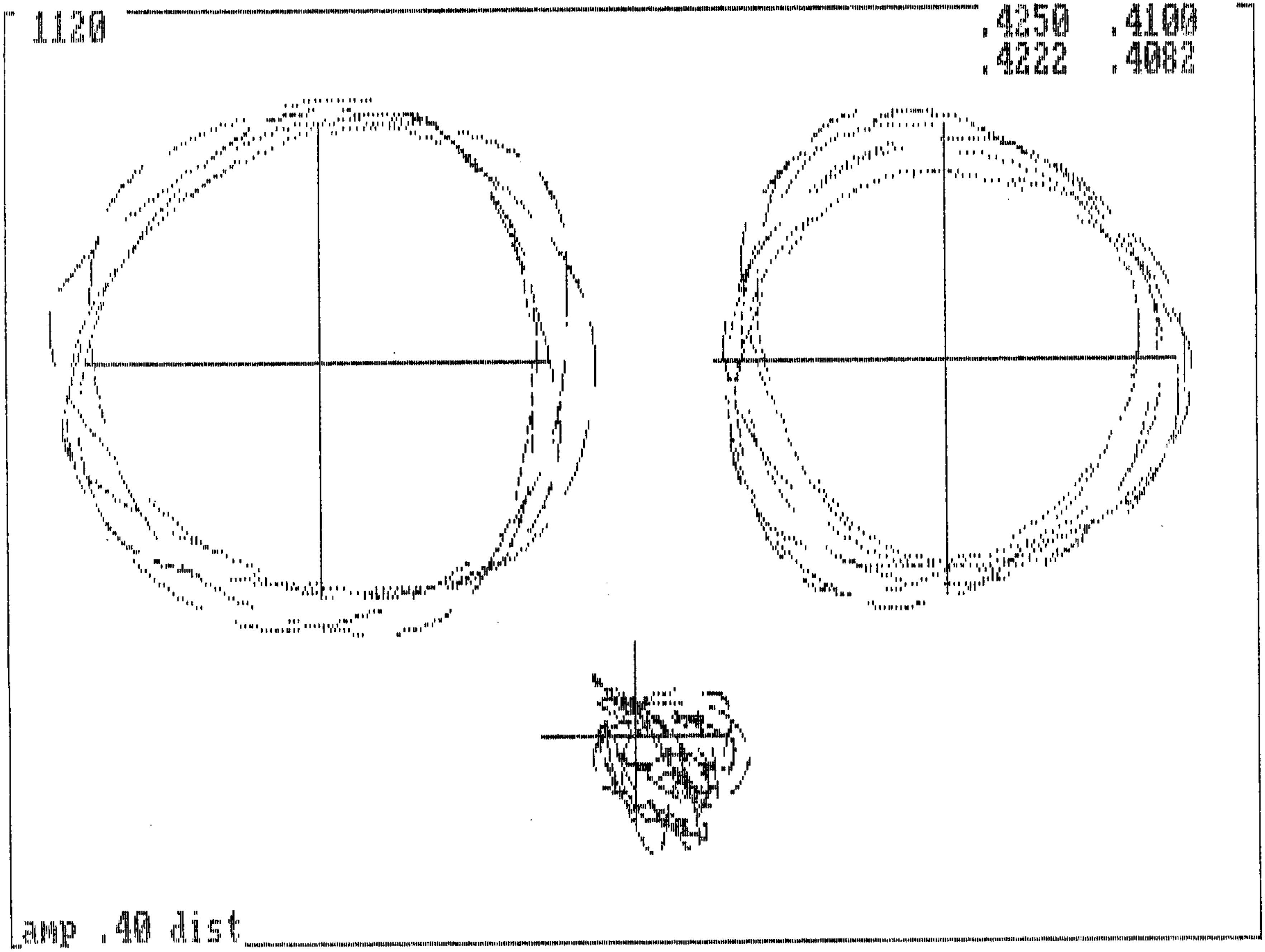
21. Skew Oct.



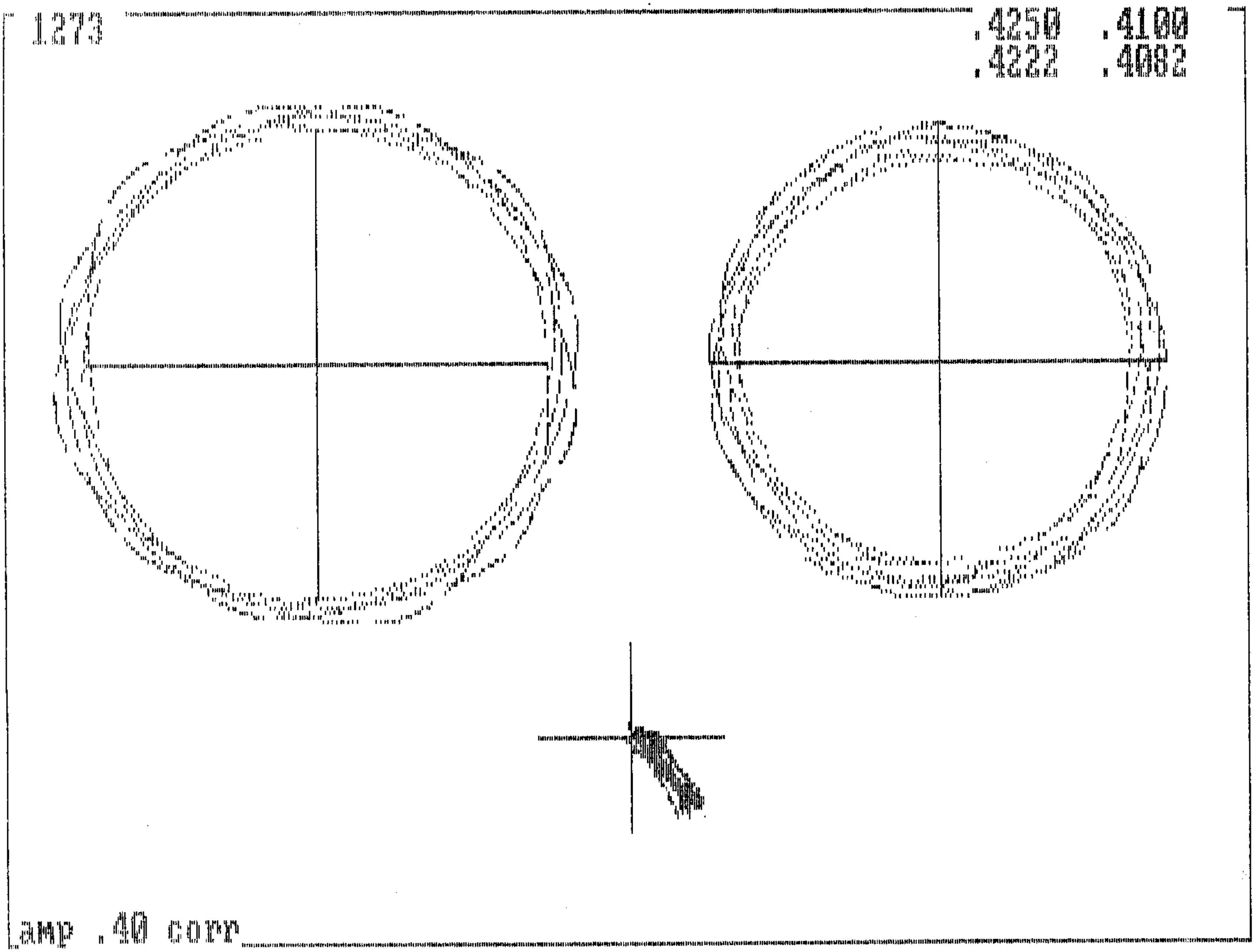
22. All-in (see text)



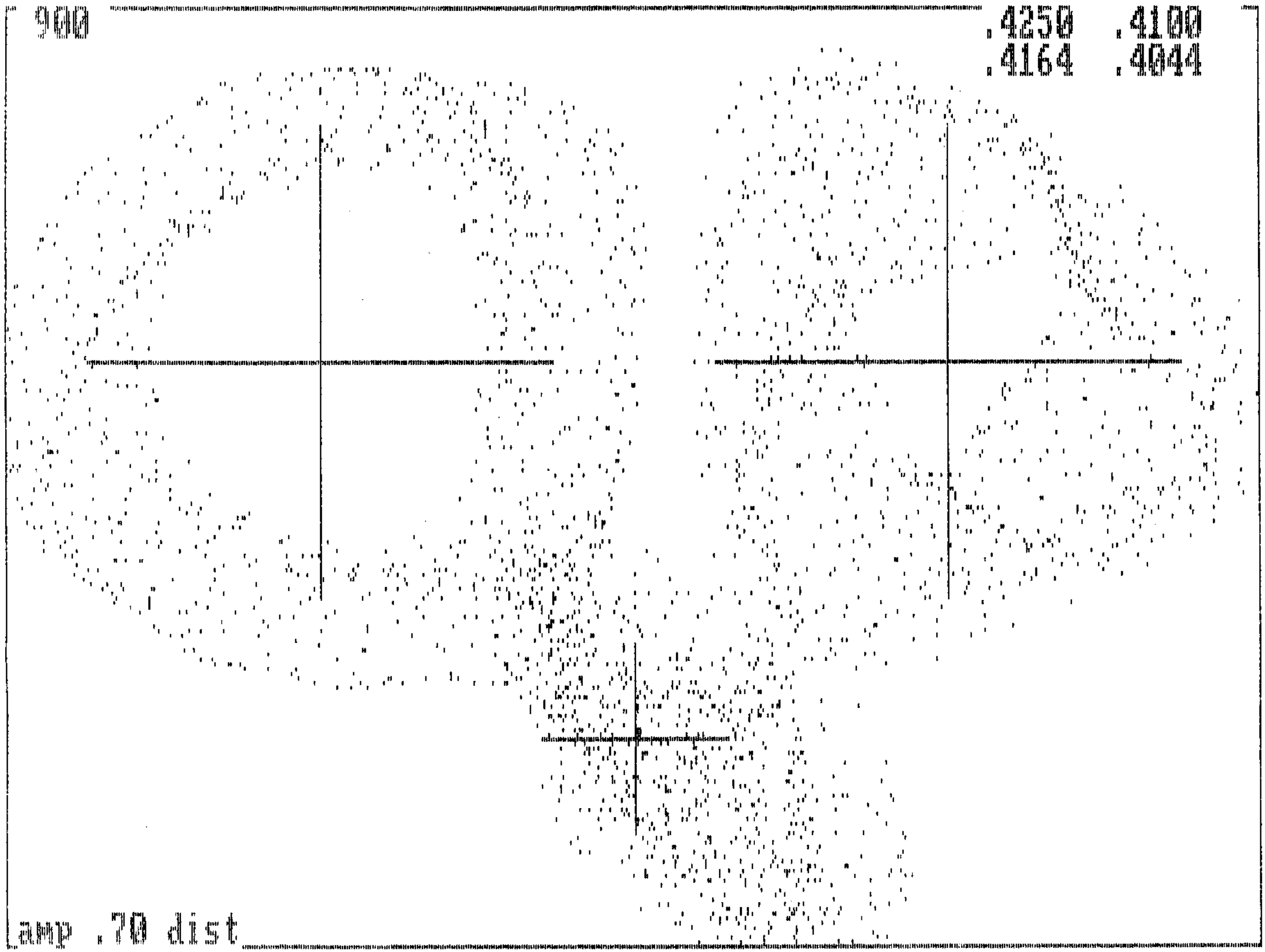
23. All-in



24. All-in



25 All-un



26. All-in