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Covariant Functional Schroedinger Formalism and Application to the Hawking Effect

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Abstract

We develop a manifestly, generally coordinate covariant functional Schroedinger formalism. We study the usual problem of Hawking radiation in the Rindler coordinate system. The Hawking effect appears as a shift in the width of the ground state wavefunctional relative to a true vacuum state in the low momentum components. It can be a coherent rather than a thermal effect depending upon the choice of boundary conditions. We analyse the general d+1 massive solution and discuss its strange features. For example, the d+1 particle number distribution is thermal but not isotropic due to a peculiar energy momentum dispersion relation. We address a number of other conceptual issues.

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I. Introduction

In the present paper we construct a manifestly generally coordinate covariant functional differential Schroedinger equation which we take to be the basic definition of a quantum field theory. The Feynman path integral appears then as the Green´s functional of this equation, but the general solution of the functional differential equation defines the complete set of states of the theory. The equation defines the evolution of the states on arbitrary spacelike surfaces and will be supplied with some boundary condition appropos a given physical environment. The ambiguity in boundary conditions is the precise origin of coordinate system ambiguities such as "how many particles does the state contain as viewed by observer x?" We apply this formalism to the study of the Hawking effect(1), i.e., the general appearance of a thermal excitation of the vacuum as seen by accelerated observers.

The specification of the boundary conditions of a quantum field theory must make reference to particular global surfaces. While Lorentz invariant initial conditions can be prescribed in flat space (though they need not be), general coordinate invariant initial conditions do not exist. Furthermore, the quantum mechanical state of the system, which for field theory may be viewed as an amplitude to find some field configuration on a global surface, is itself a global object. The problem of formulating a generally covariant field theory is then to choose an arbitrary family of spacelike hypersurfaces (perhaps non-overlapping and contiguous, though certain singularities occur e.g. in the Rindler or blackhole cases, in which we encounter overlapping of the surfaces in the family at the horizon) continuously parameterized by a variable, "t", and to provide a manifestly covariant evolution equation for the state wavefunction, $\Psi(\varphi,t)$. Manifest covariance is then the statement that all observers agree upon the uniqueness of the evolution prescription of $\Psi(\varphi,t)$ with t on the chosen surfaces, though they may not agree upon the naturalness of the choice of evolution surfaces and initial conditions.

There is a strict necessity of introducing certain global constructions into the definition of a field theory. For example, in Heisenberg picture the time evolution of a local operator is generated by a commutator with a global Hamiltonian. We may adopt the strong principle of equivalence in demanding that the Hamiltonian density be a covariant local tensor density, but the global Hamiltonian involves an integral of this object over space and is not uniquely specified. Alternatively, we may demand local covariance in the leading short distance behavior of the operator product structure of the theory, but then the extension to the large distance behavior is ambiguous.

In a Feynman path integral the global breaking of general coordinate covariance resides in the implicit boundary conditions of the functional integrals. We are accustomed to taking limits of these surfaces, e.g. $t \to \pm\infty$, in flat spacetime, but these limits may not be taken in an unambiguous way in curved spacetime and will generally influence the infrared components of operator expectation values. Other than in these boundary conditions, the Feynman path integral is manifestly invariant.

It must be emphasized that the global breaking of general coordinate covariance by initial conditions is essentially an infrared effect. The leading UV behavior of the theory, e.g. trace anomalies of the stress tensor and the infinite counterterms, provided that the spacetime dimensionality is sufficiently small, should respect the covariance of the underlying local quantities that define the theory. A direct consequence of the infrared breaking of general coordinate covariance is the occurrence of Hawking radiation[1].

The Hawking effect arises as a consequence of the boundary condition ambiguity. Operationally we may define a vacuum state wave functional, e.g. properly defined as the ground state of the Hamiltonian, on a certain initial surface. This ground state may or may not be annihilated by particle destruction operators (it may contain a condensate). For the case of an "eternal" black hole we may take the family of surfaces to represent an "inertial" coordinate system i.e., Kruskal coordinate system in which comoving observers defined by fixed spatial coordinates fall along geodesics (the case of gravitational collapse is somewhat different as the metric is effectively time dependent as the star collapses, but one must still compare the evolution in an inertial system to that in a Schwarzschild system).

We may then examine the inertial vacuum state from the point of view of an accelerating ensemble of observers, e.g. for a black hole we choose comoving observers in a Schwarzschild coordinate system. The observer's coordinate system generally fails to cover the entire manifold and possesses horizons. We may construct a definition of "Hamiltonian" and "ground state" in the accelerating system which now differs from that defined in the inertial system since the evolution along constant time surfaces in the accelerated system is not equivalent to the evolution along the inertial surfaces. The defining inertial vacuum can be compared to that defined by the accelerating observers on some common surface and is found to be an excited state, e.g. it may be full of particles as the particle number expectation is computed by the accelerating observer. Indeed, other attributes of excitation may be manifested (e.g. perhaps restoration of broken symmetries or deconfinement?). The spectrum of particles is generally cut-off in the UV, e.g. it is typically a thermal spectrum with exponential suppression of high momentum modes and thus contributes corrections only to the finite parts of operator matrix elements.

In actuality, this system of states described by the accelerated observer is completely fictitious: one should ask only physical questions of the true global state but in the restricted space of the accelerated observer. That is, the only real physics is that of experiments and detectors, e.g. Unruh-DeWitt detectors[2,3], comoving with the accelerated observers in the background inertial vacuum. However, these fictitious states "defined" by the accelerated observer may conveniently parameterize the outcome of detector experiments performed by the accelerated observers. Indeed, these fictitious states may be viewed as global Unruh-DeWitt detectors covering the range of the observer's coordinate manifold.

In our opinion, whether or not the spectrum appears to be "thermal" is somewhat secondary to the fundamental origin of the effect as a consequence of the infrared breaking of general coordinate covariance. Thus, Hawking radiation is a coordinate system dependent effect in the sense of a boundary condition, and is not something intrinsic to a particular geometry and common to all coordinate systems in that geometry.

Presently we shall analyse the familiar Rindler problem(4), the Minkowski vacuum as viewed by an ensemble of accelerating observers, but in the context of our functional Schroedinger picture. As first described by Rindler(4) and subsequently first addressed in free field theory by Fulling(5), this problem contains essentially all of the global attributes of the black-hole case, but is technically somewhat simpler.

In the context of Hawking's effect, the Rindler problem was first treated by Davies(6) and subsequently amplified by Unruh(2) in his classic paper clarifying the notion of particles in curved spacetime. Unruh treats the d-dimensional case of the Rindler problem by performing the Bogoliubov transformation on a light-plane. Though he obtains essentially the correct result, we believe that this derivation is incomplete since the transverse mass effects scale to zero on the light-plane. What is desired in this case is a direct transformation on a spacelike surface, e.g. t=0. The extremely peculiar features of the solution regarding it's anisotropy and energy-momentum dispersion relation have not been previously discussed. We find that a thermal distribution of particles is observed in the d+1 massive problem, but that there is a preferred axis of motion in the longitudinal direction. In fact, the groundstate is simply an infinite direct product of two-dimensional vacua in the absence of interactions. A considerable body of literature on the 1+1 problem now exists(7-12), but relatively few discussions of the full d+1 massive problem have been previously given, none of which seem to address the physical implications and nature of the energy momentum dispersion relation.

We also study the issue of whether or not the Hawking effect is truly thermal. The measurement of the number operator expectation is insufficient to determine this. In fact, if Dirichlet or Neumann boundary conditions are applied at the origin of Rindler space the density matrix is not thermal, but is that of a coherent state although the number operator expectation value is that of a thermal Bose gas. The presence of a horizon is necessary to obtain a thermal density matrix if one chooses to define the state by integrating out the modes beyond the horizon (If the world beyond the horizon is identified with the world at infinity this is not the correct procedure and the density matrix is again that of a coherent state. Here we consider the ambiguity recently discussed by ´t Hooft in the value of the Hawking temperature[13]. We claim this ambiguity arises in the nonthermal nature of the state. The number operator expectation involves the usual value of the temperature).

In d+1 dimensional Minkowski spacetime we may construct a quantum field theory for a real scalar field following the procedure of (1) obtaining the local stress-tensor and canonical momentum densities by functional differentiation of the action (2) imposing on a spacelike hypersurface, e.g. t=constant, equal time commutation relations between the field and canonical momentum (3) implementing the e.t.c. relation by replacing in the stress-tensor density the canonical momentum by a d-dimensional functional derivative with respect to the field and adopting the convention (Schroedinger picture) that all field variables are time independent instantaneous configurations (4) contructing Schroedinger´s equation by first constructing a global Hamiltonian by integrating T_{oo} over the spacelike hypersurface and then introducing a wavefunctional, $\Psi(\varphi,t)$, satisfying the functional differential equation:

$$\int d^d x \; T_{00} \Psi(\varphi,t) \equiv H\Psi(\varphi,t) = i\frac{d}{dt}\Psi(\varphi,t) \quad (1)$$

$\Psi(\varphi,t)$ is the amplitude to find instantaneous field configuration $\varphi(x)$ at time t. The solution to eq.(1) may always be given in free-field theory in the presence of external classical gravitational fields as a gaussian wavefunctional. An explicit implementation of the above recipe is given in Minkowski space in the Appendix. One of us has applied this recipe recently to the solution of field theory in deSitter space and obtains the trace anomaly by direct evaluation of the expectation value of $T_{\mu\nu}$ in the wave-functional solution of eq.(1)[14]. This formalism has clear advantages in application to the problem of cosmological inflation[14] and the present understanding of Hawking radiation in the formalism is essential.

There are distinct advantages to adopting the present formalism. One obtains an explicit representation of the vacuum state as an amplitude to find a given field configuration at a given time t. This prescription is especially useful for contemplating non-perturbative effects for an exact solution to eq.(1) cannot be given, but a physically sensible ansatz approximation to the ground state can be studied. For example, Feynman has recently attempted to construct a model of a groundstate which contains the physics of quark confinement in an unbroken Yang-Mills field theory in 2+1 dimensions where an exact solution to eq.(1) cannot be presently be given(15). In a subsequent publication we intend to address the question: "Does the Hawking temperature restore a broken symmetry?" in which one must transform the broken vacuum state from the inertial system to the accelerated system(16). The state is defined by a variational calculation of the one-loop effective potential in an interacting theory. This has a particularly simple interpretation in Schroedinger picture, but a less obvious one in Heisenberg operator formalism, or even in the path integral approach.

II. Functional Schroedinger Picture

How is the dynamical evolution of the wave-functional for the quantum field theory described in curved spacetime? There is no single prescription for the dynamics simply because of the different possible ways of slicing spacetime into families of spacelike hypersurfaces. However, it is possible to write the canonical commutation relations and the Schroedinger equation in a manifestly generally-covariant form.

We begin by choosing a coordinate system, x^μ $(\mu=0,1,\ldots,d)$, for the spacetime manifold. Next, we choose a family of spacelike hypersurfaces, $\Sigma(s)$, labeled by the arbitrary parameter s; each value of s uniquely specifies a hypersurface. In each hypersurface we choose a system of coordinates, ξ^i $(i=1,\ldots,d)$, intrinsic to the hypersurface and independent of s. The embedding of the hypersurfaces into spacetime is specified by the equation $x^\mu = x^\mu(s,\xi^i)$. The timelike vector field $\partial x^\mu/\partial s$, is locally normal to the hypersurface $\Sigma(s)$; we do not need to assume any particular normalization for the normal vector. We may further define a differential volume element intrinsic to the d-dimensional hypersurface, Σ:

$$\frac{d^d \Sigma_\mu}{d\xi^1 \cdots d\xi^d} \equiv \frac{D\Sigma_\mu}{D\xi} = |g|^{1/2} \varepsilon_{\mu\nu_1\cdots\nu_d} \frac{\partial x^{\nu_1}}{\partial \xi^1} \cdots \frac{\partial x^{\nu_d}}{\partial \xi^d} \qquad (2)$$

where $\varepsilon_{01..d}=1$. This volume element is invariant under changes of the intrinsic coordinates ξ^i, and is related to the invariant volume element in the spacetime manifold by:

$$\frac{\partial x^\mu}{\partial s} \frac{D\Sigma_\mu}{D\xi} d^d\xi \, ds = |g|^{1/2} dx^0 \cdots dx^d \qquad (3)$$

where $g=\det(g_{\mu\nu})$, and $g_{\mu\nu}$ are the spacetime metric tensor components in the coordinate system x^μ.

We now consider the dynamics of a real scalar field in d+1 spacetime dimensions. The action, A, is given by:

$$A = \int d^{d+1}x \left\{ |g|^{1/2} \frac{1}{2} \left(g_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi - m^2 \varphi^2 \right) \right\} \tag{4}$$

The stress tensor $T_{\mu\nu}$ for the field theory (we shall not presently employ "new improved" stress tensors) is given by the usual functional derivative of the action with respect to the metric:

$$T_{\mu\nu} = -\frac{2}{|g|^{1/2}} \frac{\delta A}{\delta g^{\mu\nu}} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \left(\partial_\rho \varphi \partial^\rho \varphi - m^2 \varphi^2 \right) \tag{5}$$

The canonical momentum on a given hypersurface is defined by:

$$\pi = |g|^{-1/2} \frac{\partial x^\mu}{\partial s} \frac{\delta A}{\delta \partial^\mu \varphi} = \frac{\partial x^\mu}{\partial s} \partial_\mu \varphi \tag{6}$$

The theory is quantized by imposing a covariant commutation relation on a given hypersurface:

$$[\varphi(s,\xi),\pi(s,\xi')] = i\,\delta_{\Sigma}(\xi,\xi') \tag{7}$$

where $\delta_{\Sigma'}(\xi,\xi')$ is defined by:

$$\delta_{\Sigma'}(\xi,\xi') = \left|\frac{\partial x^{\mu}}{\partial s}\frac{\partial x_{\mu}}{\partial s}\right|^{1/2}\left|\frac{D\Sigma^{\nu}}{D\xi}\frac{D\Sigma_{\nu}}{D\xi}\right|^{-1/2}\delta(\xi^{1}-\xi'^{1})\ldots\delta(\xi^{d}-\xi'^{d}) \tag{8}$$

The factor $|\partial x^{\mu}/\partial s|$ in the normalization of the commutator delta-function is present to guarantee the reparameterization invariance of the theory under the transformation s-->s(s′). There is also a Jacobian factor which maintains the invariance under redefinitions of the intrinsic coordinate system and which reduces to the usual commutator in the flat spacetime limit. The equal-time commutator is a global object in the sense that it refers to a particular spacelike hypersurface.

In the Schroedinger picture the wave-functional evolves with s and the field operators are s-independent. The field operator $\varphi(\xi)$ plays the analogue role of the coordinate of a particle in wave mechanics. We may now implement the commutation relations by taking $\pi(\xi)$ to be a functional differential operator:

$$\pi(\xi) = -i \left|\frac{\partial x^\mu}{\partial s}\right| \left|\frac{D\Sigma^\nu}{D\xi}\right|^{-1} \frac{\delta^{(d)}}{\delta\varphi(\xi)} \tag{9}$$

where $\delta^{(d)}\varphi(\xi)/\delta\varphi(\xi') = \delta^{(d)}(\xi - \xi')$.

The Hamiltonian H(s) is defined on the hypersurface $\Sigma(s)$ by:

$$H(s) = \int_\Sigma \frac{D\Sigma^\mu}{D\xi} T_{\mu\nu} \frac{\partial x^\nu}{\partial s} d^d\xi \tag{10}$$

where $T_{\mu\nu}$ is the stress-tensor of eq.(5). If the spacetime admits a timelike Killing vector field the hypersurfaces can be chosen so that $\partial x^\mu/\partial s$ is the timelike Killing vector field and then H becomes independent of s. In general H(s) is s-dependent, e.g. on certain choices of hypersurfaces in deSitter space.

The functional Schroedinger equation determines the evolution of the wave-functional $\Psi(\varphi, s)$:

$$H(s)\Psi(\varphi, s) = i\frac{d}{ds}\Psi(\varphi, s) \tag{11}$$

Here one may solve for $\partial_0\varphi$ in terms of π as defined in eq.(6) and substitute the operator expression into eq.(11) to obtain the functional differential equation. The Schroedinger equation is manifestly invariant in form under changes in x^μ or ξ and the reparameterization of the time variable s.

The complete set of solutions of eq.(11) defines the Hilbert space of the field theory and can generally by explicitly constructed in free field theory in terms of Gaussian wave-functionals, as in the Minkowski case as described in the Appendix. Moreover, the Green's functional, $G(\varphi_1, s_1; \varphi_2, s_2)$, satisfying:

$$\left(H(s_1) - i\frac{d}{ds_1}\right) G(\varphi_1, s_1; \varphi_2, s_2) = \delta(s_1 - s_2)\,\Delta(\varphi_1 - \varphi_2), \qquad (12)$$

can be expressed as the Feynman functional integral with the boundary conditions of field configuration φ_1 (φ_2) on hypersurface $\Sigma(s_1)$ ($\Sigma(s_2)$). Here $\Delta(\varphi_1 - \varphi_2)$ denotes a functional delta-function.

It is useful to consider the reduction of the general formulation to the special case of metrics satisfying the gauge constraint $g_{oi}=g^{oi}=0$, which can always be imposed without loss of generality. In this case we may directly identify s with the time variable, x^o, and the canonical momentum, canonical commutator and its implementation become:

$$\left.\begin{aligned} &\pi = \partial_0 \varphi, \\ &[\varphi(\vec{x}), \pi(\vec{x})] = i g_{00} |g|^{-1/2} \delta^d(\vec{x}-\vec{y}) \end{aligned}\right\} \pi \to -i g_{00} |g|^{-1/2} \frac{\delta^{(d)}}{\delta\varphi(\vec{x})}. \qquad (13a,b,c)$$

We construct the global Hamiltonian over the spacelike hypersurface of constant s:

$$H = \int_{\Sigma} T_{00}\, d\Sigma^{0} = \int T_{00}\, g^{00} |g|^{1/2} dx^{1} \ldots dx^{d}. \qquad (14)$$

Thus, for a real scalar field theory in this metric gauge we arrive at the functional Schroedinger equation:

$$H\Psi =$$

$$\frac{1}{2}\int d^{d}x\, g^{00}|g|^{1/2}\left\{ -\left(\frac{g_{00}}{|g|^{1/2}}\right)^{2} \frac{\delta^{2}}{\delta\varphi(x)^{2}} \right.$$

$$\left. - g_{00}\left(g^{ij}\nabla_{i}\varphi\nabla_{j}\varphi - m^{2}\varphi^{2}\right)\right\} \Psi(\varphi, s)$$

$$= i\frac{d}{ds}\Psi(\varphi, s). \qquad (15)$$

III. Applications to Accelerated Observers

(A) Massless Scalar Field in 1+1 Dimensions

We now apply the above formalism to the simplest case, a massless real scalar field theory in 1+1 dimensional Rindler space[4]. Rindler space is the region of Minkowski space defined by $x > |t|$ where t and x are the usual spacetime coordinates in flat Minkowski space. This region is covered by the Rindler coordinates (η, ξ) which are related to the Minkowski coordinates (t,x) by (we follow the textbook conventions of Birrell and Davies (17)):

$$t = a^{-1} e^{a\xi} \sinh(a\eta); \qquad x = a^{-1} e^{a\xi} \cosh(a\eta);$$

$$(-\infty < \eta ; \xi < \infty), \tag{16}$$

where we shall presently restrict our attention to the "right-hand wedge" corresponding to x > 0.

Along a world line defined by $\xi = \xi_0$=constant, an observer would experience a constant proper acceleration in the positive x-direction of magnitude $ae^{-a\xi_0} \sim 1/|x|$, and measure the elapsed proper time $\eta e^{a\xi_0}$. The metric in the Rindler coordinates is:

$$ds^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right) \tag{17}$$

and the vector $\partial/\partial\eta$ is a global timelike Killing vector.

We shall presently assume that the scalar field obeys Dirichlet boundary conditions, $\varphi = 0$, on the boundary of the Rindler wedge (this is equivalent to demanding in the Rindler system that $\varphi = 0$ for a fixed ξ_0 and then letting ξ_0 tend to minus infinity). At $t = \eta = 0$, φ may be expressed in terms of the appropriate Minkowski modes or Rindler modes. The Minkowski space analysis is essentially equivalent to the discussion given in the Appendix, except for the boundary conditions. In the present problem the expansion of $\varphi(x)$ in the Minkowski modes takes the form:

$$\varphi(x) = \int_0^\infty dK \sqrt{\frac{2}{\pi}}\, a(K) \sin(Kx); \qquad a(k) = \overline{a(K)}. \tag{18}$$

The functional derivative can likewise be written:

$$\frac{\delta}{\delta\varphi(x)} = \int_0^\infty dk\sqrt{\frac{2}{\pi}}\,\frac{\delta}{\delta a(k)}\,\sin(kx) \tag{19}$$

where:

$$\frac{\delta a(k)}{\delta a(p)} \equiv \delta(k-p). \tag{20}$$

Then the Minkowski Hamiltonian becomes:

$$H_M = \frac{1}{2}\int_0^\infty dx\left[-\frac{\delta^2}{\delta\varphi(x)^2} + (\nabla\varphi)^2\right] = \frac{1}{2}\int_0^\infty dk\left[-\frac{\delta^2}{\delta a(k)^2} + k^2 a(k)^2\right]. \tag{21}$$

The corresponding Schroedinger equation is:

$$\frac{1}{2}\int_0^\infty dk\left[-\frac{\delta^2}{\delta a(k)^2} + k^2 a(k)^2\right]\Psi_m(a(k),t) = i\frac{d}{dt}\Psi_m(a(k),t) \tag{22}$$

which has the ground state solution at t=0:

$$\Psi_M^0(\varphi, t=0) = \exp\left[-\frac{1}{2}\int_0^\infty dk\, K\, a(k)^2\right] = \exp\left[-\frac{1}{2}\int_0^\infty dx\, \varphi\sqrt{\overleftrightarrow{\nabla}\cdot\overleftrightarrow{\nabla}}\,\varphi\right]. \tag{23}$$

In Rindler space we can apply the formalism of Section II. In this system the invariant action is:

$$A = \frac{1}{2}\int_{-\infty}^{\infty} d\xi\, d\eta \left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2 - \left(\frac{\partial\varphi}{\partial\xi}\right)^2\right] \tag{24}$$

and we shall consider evolution of the vacuum wavefunctional along the surfaces parameterized by η.

Thus:

$$x^\mu = (\eta, \xi)\,; \quad \frac{\partial x^\mu}{\partial\eta} = (1, 0)\,; \quad \frac{D\Sigma^\mu}{D\xi} = (1, 0), \tag{25}$$

and ξ is the intrinsic surface coordinate. The canonical momentum and equal-η commutators are:

$$\pi = |g|^{-1/2} \frac{\partial x^\mu}{\partial s} \frac{\delta A}{\delta \partial^\mu \varphi} = \partial_\eta \varphi$$
$$[\varphi(\xi), \pi(\xi')] = i g_{00} |g|^{-1/2} \delta(\xi - \xi') = i \delta(\xi - \xi'). \tag{26}$$

Thus we implement these by the replacement:

$$\pi = -i g_{00} |g|^{-1/2} \frac{\delta}{\delta \varphi(\xi)} = -i \frac{\delta}{\delta \varphi(\xi)}. \tag{27}$$

The Rindler Hamiltonian is given by eq.(15):

$$H_R = \int \frac{\partial x^\mu}{\partial s} T_{\mu\nu} d\Sigma^\nu = \frac{1}{2} \int_{-\infty}^{\infty} \left(-\frac{\delta^2}{\delta \varphi(\xi)^2} + \left(\frac{\partial \varphi}{\partial \xi} \right)^2 \right) d\xi. \tag{28}$$

An instantaneous field configuration may be written in terms of spatial Rindler modes:

$$\varphi(\xi) = \int_{-\infty}^{\infty} d\rho \frac{1}{\sqrt{2\pi}} \beta(\rho) e^{i\rho\xi}; \qquad \overline{\beta(\rho)} = \beta(-\rho). \tag{29}$$

The Rindler modes are formally identical to Minkowski modes without the boundary condition. This result is special to the massless 1+1 dimensional problem and results because 1+1 dimensional Rindler space is conformal to 1+1 dimensional Minkowski space and the equation of motion of the massless scalar field is conformally invariant. The functional derivative is:

$$\frac{\delta}{\delta\varphi(\xi)} = \int_{-\infty}^{\infty} d\rho \, \frac{1}{\sqrt{2\pi}} \, e^{-i\rho\xi} \, \frac{\delta}{\delta\beta(\rho)} \, ; \qquad \frac{\delta\beta(\rho)}{\delta\beta(k)} = \delta(\rho-k). \tag{30}$$

Hence the Schroedinger equation becomes:

$$\frac{1}{2}\int_{-\infty}^{\infty} d\rho \left[-\frac{\delta^2}{\delta\bar{\beta}\,\delta\beta(\rho)} + \rho^2 |\beta(\rho)|^2 \right] \Psi_R(\beta,\eta) = i\frac{d}{d\eta}\Psi_R(\beta,\eta), \tag{31}$$

and the groundstate solution in this system is:

$$\Psi_R^{\circ}(\varphi,0) = \exp\int_{-\infty}^{\infty} -\left[\frac{1}{2}\, d\rho \, |\rho| \, \beta(\rho)\overline{\beta(\rho)}\right]. \tag{32}$$

The states Ψ_M° and Ψ_R° are different because for equal values of their arguments, which are configurations of equal amplitude in their respective systems, we have different field configurations. At $t=\eta=0$ both states are defined on a common spacelike hypersurface and they can then be compared. The Bogoliubov transformation connects the two sets of amplitudes which define field configurations in the Minkowski and Rindler spaces:

$$a(\kappa) = \int_{-\infty}^{\infty} d\rho \, A(\kappa,\rho)\beta(\rho) \qquad (\kappa>0). \tag{33}$$

In the present formalism the Bogoliubov transformation is simply a Fourier transform:

$$A(\kappa,\rho) = \frac{1}{\pi}\int_0^{\infty} dx \, \sin(\kappa x) e^{i\rho\xi(x)} = \frac{1}{a\pi}\Gamma\left(1+\frac{i\rho}{a}\right)\cosh\left(\frac{\pi\rho}{2a}\right)\left|\frac{\kappa}{a}\right|^{-1-\frac{i\rho}{a}}. \tag{34}$$

We may now substitute the relation of eq.(34) into the Minkowski state of eq.(23) at t=0. We obtain:

$$\Psi_M^0 = \exp\left[-\frac{1}{2}\int_0^{\infty} dk \int_{-\infty}^{\infty} d\rho \, d\rho' \, \kappa \, A(\kappa,\rho)A(\kappa,\rho')\beta(\rho)\beta(\rho')\right], \tag{35}$$

or (using $\bar{\beta}(p) = \beta(-p)$):

$$\Psi_M^0 = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} dp \; p\coth\left(\frac{\pi p}{2a}\right)\beta(p)\overline{\beta(p)}\right], \qquad (36)$$

where the positive k integration has become a delta-function in p-p′ and we use the familiar gamma-function identities, $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

Thus we obtain the Minkowski groundstate wavefunctional in terms of the $\beta(p)$ of the Rindler field configuration. We see that Ψ_M^0 has a different structure for low momenta, $p \to a$, though for $p \to \infty$ it approaches the Rindler groundstate exponentially (note that $p\coth(\pi p/2a)$ is positive for all p as is necessary for the stability of the state). This is a reflection of the fact that the difference between Minkowski and Rindler descriptions is irrelevant for the short-distance properties of the theory.

It is interesting to compute the expectation value in the Minkowski state of the Rindler number operator for the pth momentum mode, $n_R(p)$ (we could have expressed the Rindler Hamiltonian in eq.(31) in terms of this operator in the usual way). This may be obtained directly, or more easily by considering a single simple harmonic oscillator in one degree of freedom, q. If we specify that the groundstate wavefunction is the gaussian $\exp(-\omega q^2/2)$, then the expectation value of the number operator in the state $\exp(-\hat{\omega} q^2/2)$ is given by:

$$\langle n \rangle_{\hat{\omega}} = (\omega - \hat{\omega})^2 / 4\omega\hat{\omega} . \tag{37}$$

Thus we take this result over immediately to the evaluation of the pth Rindler mode number operator expectation:

$$\langle n_R \rangle = \frac{\left(|\rho| - \rho \coth\left(\frac{\pi\rho}{2a}\right)\right)^2}{4|\rho|\rho\coth\left(\frac{\pi\rho}{2a}\right)} = \left(\exp\left(\frac{2\pi|\rho|}{a}\right) - 1\right)^{-1}. \tag{38}$$

We thus see that the Rindler modes in the Minkowski vacuum are populated in a thermal distribution (Bose gas) with a temperature $T=a/2\pi$. This is the familiar result of the Hawking effect.

If we had applied Neumann boundary conditions at the origin rather than the Dirichlet conditions (i.e. demand the vanishing of the derivative of the field configurations at x=0) we could use a Fourier cosine series rather than the sine series of eq.(18,19) and we would have obtained the state:

$$\Psi^{\circ}_{M(N)} = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\rho\ \rho \tanh\left(\frac{\pi\rho}{2a}\right)|\beta(\rho)|^2\right], \qquad (39)$$

where the coth() of eq.(36) has been replaced by the tanh(). It is easy to verify that the number operator in this state is also that of a Bose gas with $T=a/2\pi$, but we see that the states are distinctly different and clearly there will be distinguishing matrix elements.

But do these states really describe thermal systems? Not necessarily. For a thermal system the density matrix in our formalism is a straightforward generalization of that of the simple harmonic oscillator and is discussed by Feynman[18]:

$$\rho_T(\alpha_k, \alpha'_k) = \exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty} d\ell \left[\ell \coth\left(\frac{\ell}{T}\right)\left(|\alpha_\ell|^2 + |\alpha'_\ell|^2\right) - \frac{\ell}{\sinh(\ell/T)}\left(\bar{\alpha}_\ell \alpha'_\ell + \bar{\alpha}'_\ell \alpha_\ell\right)\right]\right\}. \qquad (40)$$

However, for the Minkowski state with Neumann boundary conditions the density matrix is just:

$$\rho_M = \Psi^*(\alpha'_k)\Psi(\alpha_k) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\ell\, \ell \tanh\left(\frac{\pi \ell}{2a}\right)\left(|\alpha_\ell|^2 + |\alpha'_\ell|^2\right)\right] \tag{41}$$

which is not equivalent to that of a thermal system by the absence of the cross-terms which lead to thermal mode-mixing.

Recently 't Hooft has discussed the Hawking effect where he has exploited the ambiguity in the definition of the theory due to the structure of the vacuum outside of the accelerating observer's horizon(13). In particular, 't Hooft has considered a definition of the density matrix in which he identifies the physics of the left-hand wedge with that of the right. In our language, we should equate a field configuration on the right to one on the left which is equivalent to the use of Neumann boundary conditions (since the cosine expansion is even under $x \rightarrow -x$). We might be tempted to compare the diagonal of a thermal density matrix to the density matrix obtained by such a prescription, i.e. compare eq.(40) to eq.(41) with $\alpha_k = \alpha'_k$. In this case the thermal density matrix, eq.(40), becomes formally identical to eq.(41) if we identify the temperature to be $T=a/\pi$, which is exactly twice the Hawking result and is the result obtained by 't Hooft. However, this procedure is evidently erroneous because the two density matrices are clearly inequivalent and indeed that of the pure state does produce a Bose gas distribution of the correct temperature, $a/2\pi$.

The important point here is that the Minkowski state specifies a density matrix which is that of a coherent state (factorizeable) and is not that of a thermal state. In fact it is straightforward to show that there are matrix elements of particular operators that differentiate between the thermal and coherent cases. For example, the matrix element of the operator $\partial^4/\partial\alpha_\ell^4$ is different when evaluated in eq.(40) than when evaluated in eq.(41). Thus, a physical probe coupled to π_ℓ^4 would register a nonthermal excitation.

What is the subsequent evolution of the state with η as viewed by the accelerated observer? Consider a general parameterization of the state:

$$\Psi_m(\varphi,\eta) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\rho\ \rho A(\rho,\eta)|\beta(\rho)|^2 + i\Omega(\eta)\right]. \quad (42)$$

The differential equation satisfied by $A(p,\eta)$ follows upon substitution into the Schroedinger equation:

$$-i\rho^{-1}\frac{\partial A(\rho,\eta)}{\partial\eta} = 1 - A(\rho,\eta)^2. \quad (43)$$

The boundary condition upon $A(p,\eta)$ is specified by the Minkowski state, e.g. for Dirichlet boundary conditions at $\eta=0$ we have $A(p,0)=\coth(\pi p/2a)$. We thus obtain the simple result:

$$A(p,\eta) = \coth\left(\frac{\pi p}{2a} + ip\eta\right), \tag{44}$$

which determines the evolution of the state in eq.(42) for all η. Notice that $A(p,\eta)$ is complex, reflecting the phase shift of the state with increasing η, a consequence of its coherence. It is readily seen explicitly that the number operator is conserved. Its expectation is:

$$\langle n_R(p)\rangle = \frac{\left|\,|p| - pA(p,\eta)\right|^2}{4|p|\,\mathrm{Re}(pA(p,\eta))} = \left(\exp\left(\frac{2\pi|p|}{a}\right) - 1\right)^{-1} \tag{45}$$

which is independent of time. Thus the accelerated observer detects particles in a Bose gas distribution which is constant throughout proper time. Obviously this result is a consequence of the fact that the number operator commutes with the Rindler Hamiltonian.

In the present construction one might argue that the boundary conditions are boost invariant about the origin and the Minkowski state is thus annihilated by boost generators. Thus the comparison for any η may be made to the same (boosted) state, and thus the accelerated observer will always see this thermal distribution(8,9). If the wall is located to the right of the coordinate singularity boost invariance is lost and the state has a more complicated behavior(16).

(B) The d+1 Dimensional, Massive Problem

The right hand wedge of the d+1 Rindler space is the region of d+1 Minkowski space covered by coordinates $(t,x,\vec{x}_\perp)$ with $x>|t|$. The right-hand wedge can be covered by coordinates $(\eta,\xi,\vec{x}_\perp)$ with (η,ξ) related to (t,x) as in eq.(16). We thus have $\vec{x}_\perp$ representing d-1 transverse coordinates. The metric is:

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) - d\vec{x}_\perp^{\,2} \tag{46}$$

and $\partial/\partial\eta$ is a timelike Killing vector on the manifold.

The instantaneous field configuration in Minkowski space which vanishes at x=0 can be expressed in the appropriate modes at $t=\eta=0$. We have for the field configuration and functional derivative:

$$\begin{aligned}\varphi(x,\vec{x}_\perp) &= \int_0^\infty dK \int d^{d-1}K_\perp \sqrt{\frac{2}{\pi}}\,(2\pi)^{\frac{1-d}{2}} \sin(Kx)\, e^{i\vec{K}_\perp\cdot\vec{x}_\perp}\, \alpha(K_\perp,K);\\ \frac{\delta}{\delta\varphi(x,x_\perp)} &= \int_0^\infty dK \int d^{d-1}K_\perp \sqrt{\frac{2}{\pi}}\,(2\pi)^{\frac{1-d}{2}} \sin(Kx)\, e^{-i\vec{K}_\perp\cdot\vec{x}_\perp}\, \frac{\delta}{\delta\alpha(K_\perp,K)};\end{aligned} \tag{47}$$

where $\bar{\alpha}(k_\perp,k)=\alpha(-k_\perp,k)$ and $\delta\alpha(k_\perp,k)/\delta\alpha(p_\perp,p) = \delta(k-p)\,\delta(k_\perp-p_\perp)$.

The Minkowski Hamiltonian takes the form:

$$H_M = \frac{1}{2}\int_0^\infty dk \int d^{d-1}k_\perp \left[-\frac{\delta^2}{\delta\bar{\alpha}\,\delta\alpha(k_\perp,k)} + \omega_M^2(k,k_\perp)\,|\alpha(k_\perp,k)|^2 \right] \tag{48}$$

where the energy of a given mode $(k_\perp,k)$ is $\omega(k_\perp,k) = (k_\perp^2+k^2+m^2)^{1/2}$. Thus the groundstate solution to the Minkowski space Schroedinger equation takes the form:

$$\Psi_M^0 = \exp\left[-\frac{1}{2}\int_0^\infty dk \int d^{d-1}k_\perp\, \omega_M(k_\perp,k)\,|\alpha(k_\perp,k)|^2\right]. \tag{49}$$

The Rindler Hamiltonian can be constructed with the formalism of eq.(15) and we obtain:

$$H_R = \frac{1}{2}\int_{-\infty}^{\infty} d\xi \int d^{d-1}x_\perp \left[-\frac{\delta^2}{\delta\varphi(\xi,x_\perp)^2} + \left(\frac{\partial\varphi}{\partial\xi}\right)^2 + e^{2a\xi}\left(\left(\frac{\partial\varphi}{\partial x_\perp}\right)^2 + m^2\varphi^2\right)\right] \tag{50}$$

and the action of the functional derivative is:

$$\frac{\delta\varphi(\xi,x_\perp)}{\delta\varphi(\xi',x'_\perp)} = \delta(\xi-\xi')\,\delta^{(d-1)}(\vec{x}_\perp-\vec{x}'_\perp). \qquad (51)$$

Our problem is essentially that of finding a complete set of spatial mode functions which diagonalize the Rindler Hamiltonian of eq.(50) in the appropriate momentum space. Let us assume that an instantaneous field configuration may be written:

$$\varphi(\xi,x_\perp) = \int \frac{d^{d-1}k_\perp}{(2\pi)^{(d-1)/2}} \int_0^\infty d\rho\; e^{i\vec{k}_\perp\cdot\vec{x}_\perp} R_\rho^{k_\perp}(\xi)\,\beta(k_\perp,\rho) \qquad (52)$$

where the $R^{k_\perp}_p(\xi)$ are the "longitudinal" Rindler modes. We shall explicitly construct the $R^{k_\perp}_p(\xi)$ below. They will be shown below to be orthonormal and complete:

$$\int_0^\infty d\rho\, R_\rho^{k_\perp}(\xi)\, R_\rho^{k_\perp}(\xi') = \delta(\xi - \xi'),$$
$$\int_{-\infty}^{\infty} d\xi\, R_\rho^{k_\perp}(\xi)\, R_{\rho'}^{k_\perp}(\xi) = \delta(\rho - \rho'), \tag{53}$$

and they are also real with the implication:

$$R_\rho^{k_\perp}(\xi) = \bar{R}_\rho^{k_\perp}(\xi); \qquad \bar{\beta}(k_\perp,\rho) = \beta(-k_\perp,\rho). \tag{54}$$

Thus, the functional derivative can be written in a Rindler representation:

$$\frac{\delta}{\delta\varphi(\xi,x_\perp)} = \int \frac{d^{d-1}k_\perp}{(2\pi)^{(d-1)/2}} \int_0^\infty d\rho\, e^{-i\vec{k}_\perp\cdot\vec{x}_\perp}\, R_\rho^{k_\perp}(\xi)\, \frac{\delta}{\delta\beta(k_\perp,\rho)};$$

$$\frac{\delta\beta(k_\perp,\rho)}{\delta\beta(k'_\perp,\rho')} = \delta^{d-1}(\vec{k}_\perp - \vec{k}'_\perp)\, \delta(\rho-\rho'). \tag{55}$$

Modulo surface terms at the origin the $R_k^{k_\perp}(\xi)$ diagonalize the Hamiltonian provided it can be written in a form:

$$H_R = \frac{1}{2}\int d^{d-1}K_\perp \int_0^\infty d\rho \left[-\frac{\delta^2}{\delta\bar{\beta}\,\delta\beta(K_\perp,\rho)} + \omega_R^2(K_\perp,\rho)|\beta(K_\perp,\rho)|^2 \right]. \tag{56}$$

Here $\omega_R(k_\perp,p)$ is the energy of a given mode determined by the energy-momentum dispersion relation. The sufficient condition that eq.(56) obtain follows by substituting the representations for the field configuration and functional derivative, eq.(52,55), into the Hamiltonian, eq.(50), and demanding:

$$0 = \int_{-\infty}^{\infty} d\xi \left[\left(\frac{\partial R_\rho^{K_\perp}(\xi)}{\partial\xi}\right)^2 + \left(e^{2a\xi}(K_\perp^2+m^2) - \omega_R^2\right) R_\rho^{K_\perp}(\omega)^2 \right]. \tag{57}$$

Integrating by parts we obtain:

$$\int d\xi \left[\left(-\frac{\partial^2 R_\rho^{K_\perp}(\xi)}{\partial\xi^2} + \left(e^{2a\xi}(K_\perp^2+m^2) - \omega_R^2\right) R_\rho^{K_\perp}(\xi) \right) R_\rho^{K_\perp}(\xi) \right] + R_\rho^{K_\perp} \frac{\partial R_\rho^{K_\perp}(\xi)}{\partial\xi} \Bigg|_{\xi=-\infty}^{\xi=\infty} = 0. \tag{58}$$

In the double wedge problem as discussed in Sect.IV the surface term produced at $\xi_R=-\infty$ ($\xi_L=\infty$) by the RH (LH) integration need not vanish because continuity of the field configuration across the origin will ensure that the upper LH surface term cancels against the lower RH term. We require an expansion, however, which produces no such terms as ξ_R --> ∞ (ξ_L --> $-\infty$). In the present problem we must demand either Dirichlet or Neumann boundary conditions as ξ_R --> $-\infty$ and the exponentially vanishing solution as $\xi_R \to \infty$. The Dirichlet boundary conditions may be implemented by demanding that $R^{k_\perp}_p(\xi)$ vanish for $\xi = \xi_o$ and then taking the limit as $\xi_o \to -\infty$, which results in a discretization of the spectrum which approaches a continuum in the limit $\xi_o \to -\infty$. Such an effect occurs in the Green's function analysis of Candelas and Deutsch[19]. If we consider only momenta small compared to ξ_o^{-1} the continuum spectrum obtains and we may neglect the effects of these surface terms.

Thus, our basis functions satisfy the modified Bessel equation of imaginary index:

$$\frac{\partial^2 R^{k_\perp}_\rho(\xi)}{\partial \xi^2} - e^{2a\xi} m_\perp^2 R^{k_\perp}_\rho(\xi) + \omega_R^2(k_\perp, \rho) R^{k_\perp}_\rho(\xi) = 0;$$

$$m_\perp^2 \equiv \vec{k}_\perp^{\,2} + m^2. \tag{59}$$

These, of course, are just the longitudinal modes obtained by separation of the Klein-Gordon equation in this coordinate system for a mode of time dependence $\exp(i\omega\eta)$. Henceforth we shall write the "transverse mass", $m_\perp^2 = \vec{k}_\perp^{\,2} + m^2$.

The correct expansion is essentially that of the massive d=1 case first studied by Fulling[5] and is based upon the standard analysis given in Titchmarsh[20]. Candelas and Deutsch[19] have constructed the Green's functions for the problems of accelerating walls in d+1 massless scalar field theories and also encounter these basis functions, though there are distinct physical differences between the problem studied in ref.(19) and the present. Recently the full d+1 massive problem has been treated by Haag, Narnhofer and Stein[7], though their analysis was unknown to us until the completion of this work. These ideas have also been previously discussed extensively by Boulware(9).

Here we identify the $R^{k_\perp}_{p}(\xi)$ with the modified Bessel functions of imaginary order. In the range, $x > 0$ there are two solutions, one blowing up exponentially as $x \to \infty$ which we discard. It should be noted however that this is an additional constraint which was not implemented in the 1+1 dimensional problem and which effectively halves the number of degrees of freedom in the d+1 massive case.

With the appropriate normalization the solution to eq.(59) is:

$$R^{k_\perp}_{p}(\xi) = \frac{1}{\pi}\left(\frac{2p}{a}\sinh\left(\frac{\pi p}{a}\right)\right)^{1/2} K_{\frac{ip}{a}}\left(m_\perp a^{-1} e^{a\xi}\right), \tag{60}$$

where $K_{ip}(z)$ is the modified Bessel function of imaginary order, p.

The dispersion relation between energy and momentum is thus:

$$\omega(k_\perp, p) = p \tag{61}$$

What is the physical interpretation of this bizarre result? Why does the transverse mass decouple from the energy-momentum relationship? In a sense we believe it arises because an eternally accelerating observer is not sensitive to the transverse mass in an ordinary Lorentz-boosted dispersion relation but for a brief instant of his proper time when the particle is at rest relative to him. This occurs for a negligible amount of time over the entire history of his world line. For consider the particle having zero longitudinal momentum at some instant, thus $p^{\mu} = (m_{\perp}, k_{\perp}, 0)$. The accelerated observer at time η measures $p^{\mu\prime} = (m_{\perp}\cosh a\eta,\ k_{\perp},\ m_{\perp}\sinh a\eta)$, or $p_0'^2 = p_z'^2 \coth^2(a\eta)$. If we simply average this relation over a long interval of time we obtain the effective result $p_0'^2 = p_z'^2$.

The completeness and orthogonality of the modes $R^{k_{\perp}}_{p}(\xi)$ may be demonstrated in several ways. These are neatly summarized by the properties of the Kontorovich-Lebedev transforms(21). The completeness is also inherent in the discussion of Titchmarsh(20) and the orthogonality may be proved directly by the use of Weber-Schafheitlin integrals(21). We shall not presently elaborate these properties of the mode functions furth%r.

The completeness and orthonormality of the mode functions thus establishes the diagonalization of the Rindler Hamiltonian, which now takes the form of eq.(56). Thus, the Rindler Hamiltonian has a ground-state wave-functional of the form:

$$\Psi_R^{o} = \exp\left[-\frac{1}{2}\int d^{d-1}k_{\perp}\int_0^{\infty} d\rho\ \omega_R(k_{\perp},\rho)\,|\beta(k_{\perp},\rho)|^2\right]. \tag{62}$$

The static groundstate of a system reflects correlations that have acquired over the entire preceding history of the system (hence spacelike correlations are not vanishing, though causality is effective). This state can be written in a form, as a consequence of the energy-momentum dispersion relation of eq.(58), which shows that it is an infinite tensor product of independent and equivalent 1+1 dimensional model groundstates:

$$\Psi_R^0 = \prod_{K_\perp} \exp\left[-\frac{1}{2}\int_0^\infty d\rho\ \rho\ |\beta(K_\perp,\rho)|^2\right]. \qquad (63)$$

We note, however, that eq.(58-60) are strictly valid only for small longitudinal momenta (e.g. if the Dirichlet boundary conditions are applied at ξ_0 then these are valid for momenta small compared to ξ_0^{-1}). In the presence of interactions, and even free regulator fields, this decomposition will become invalidated. Free regulator fields must restore the full singularity structure of the field theory to that of the Minkowski case and they must reunite the transverse and longitudinal momenta and masses(16).

We now turn to the comparison of the Minkowski and Rindler descriptions of the vacuum state. As in the preceding section we wish to express the Minkowski ground-state, Ψ_M^0, in terms of the $\beta(p_\perp,p)$ of the Rindler expansions. To accomplish this we introduce the Bogoliubov transformation relating the two sets of field configurations:

$$\alpha(k_\perp, k) = \int_0^\infty d\rho \; A(k_\perp, k, \rho)\, \beta(k_\perp, \rho). \qquad (64)$$

The same transverse momenta appear in both sets of mode amplitudes in eq.(64). The coefficients $A(k_\perp, k, \rho)$ are given by:

$$A(k_\perp, k, \rho) = \int_0^\infty dx \sqrt{\tfrac{2}{\pi}} \sin(kx)\, R_\rho^{k_\perp}(\xi(x))$$
$$= -\frac{i}{2}\left(\frac{2\rho}{\pi a}\coth\left(\frac{\pi\rho}{2a}\right)\right)^{1/2} (k^2 + m_\perp^2)^{-1/2} \cdot$$
$$\left[\left(\frac{(k^2+m_\perp^2)+k}{m_\perp}\right)^{i\rho/a} - \left(\frac{(k^2+m_\perp^2)-k}{m_\perp}\right)^{i\rho/a}\right]. \qquad (65)$$

To express Ψ_M^0 in terms of the Rindler amplitudes we need evaluate:

$$\Psi_M^0 = \exp\left[-\frac{1}{2}\int d^{d-1}k_\perp \int_0^\infty dk\, d\rho\, d\rho' \,(k^2+m_\perp^2)^{1/2} \qquad (66)\right.$$
$$\left. \overline{A(k_\perp, k, \rho)}\, A(k_\perp, k, \rho')\, \overline{\beta(k_\perp, \rho)}\, \beta(k_\perp, \rho')\right].$$

Performing the k-integration first and then introducing the substitutions:

$$V = \frac{K}{m_\perp} \; ; \quad u = V + \left(1 + V^2\right)^{1/2} ; \tag{67}$$

we find that the integration over u produces a delta-function $\delta(p-p')$ which leads to the final simple result:

$$\Psi_M^0 = \exp\left[-\frac{1}{2}\int_0^\infty d\rho \int dK_\perp^{d-1} \, \rho \coth\left(\frac{\pi\rho}{2d}\right) |\beta(K_\perp,\rho)|^2\right] . \tag{68}$$

Thus, by the same arguments given previously for the 1+1 case we see that Ψ_M^0 is an excited state relative to the groundstate of eq.(63) and will have the thermal Bose gas distribution in energy. However, now the distribution is not rotationally invariant but rather is directed along the longitudinal momentum axis. This is a consequence of the peculiar energy momentum dispersion relation and answers the puzzling question as to how the accelerating observer in d>1 dimensions can see both a thermal distribution of particles and yet have a preferred axis of acceleration.

Of course, as in the 1+1 problem, this is a coherent state and it will evolve with η in a manner identical to that in eq.(44), but the particle number distribution will again be conserved.

IV. The Complete Manifold Problem

In the present section we shall conclude with an analysis of the standard two wedge 1+1 dimensional massless problem. We have been cavalier in our treatment of boundary effects, but will describe the subtleties as they arise. Our primary aim is to recover the prescription given recently by Unruh and Wald[23] under which the Rindler problem gives a truly thermal density matrix.

We assume in Minkowski space an expansion in plane waves (complex exponentials) for the instantaneous field configurations and thus the analysis of the Appendix is strictly applicable. Thus, we take the groundstate solution to be that of eq.(A.10), though we shall presently consider d=1. We introduce the familiar two-wedge Rindler coordinates:

$$x>0 \ (\text{RH wedge}): \quad \begin{cases} x = a^{-1} e^{a\xi_R} \cosh(a\eta) \\ t = a^{-1} e^{a\xi_R} \sinh(a\eta) \end{cases} \qquad x<0 \ (\text{LH wedge}): \quad \begin{cases} -x = a^{-1} e^{a\xi_L} \cosh a\eta \\ -t = a^{-1} e^{a\xi_L} \sinh a\eta \, . \end{cases} \tag{69}$$

An instantaneous field configuration we assume to be parameterized by:

$$\varphi(\xi) = \theta(x)\int_{-\infty}^{\infty}\beta(l)\,e^{il\xi_R(x)}\frac{dl}{2\pi} + \theta(-x)\int_{-\infty}^{\infty}\gamma(l)\,e^{il\xi_L(x)}\frac{dl}{2\pi}, \tag{70}$$

in terms of the R and L spatial Rindler coordinates. Thus, the functional derivative is:

$$\frac{\delta}{\delta\varphi(\xi)} = \theta(x)\int_{-\infty}^{\infty}e^{-il\xi_R(x)}\frac{\delta}{\delta\beta(l)}\frac{dl}{2\pi} + \theta(-x)\int_{-\infty}^{\infty}e^{-il\xi_L(x)}\frac{\delta}{\delta\gamma(l)}\frac{dl}{2\pi}. \tag{71}$$

Note that there is a potential inconsistency in using the above expansions and the Minkowski expansion into plane waves. Any regularization of the Minkowski momentum integrals severely restricts the validity of an expansion as in eq.(70) as $\xi_R \to -\infty$ (or $\xi_L \to \infty$) since the Rindler modes oscillate wildly (have infinite Minkowski momentum) in these limits. Equivalently, if we adopt a Fourier sine series in Minkowski space on the right and independently on the left wedges, the range of non-uniform convergence in these series is a range of convergence in the above series. This suggests that we should be using Fourier sine series in Rindler space for $\xi_R > (\ln a\epsilon)/a$ and $\xi_L <$ $(-\ln(a\epsilon)/a)$ where the range of validity in Minkowski space is $x > \epsilon$ and $x < -\epsilon$. A more thorough treatment of the continuity conditions is beyond the scope of the present treatment.

Proceeding, we take the coordinates within the evolution surfaces parameterized by η to be $(\eta, \xi_{R,L})$ and the normal vector in Rindler coordinates is $dx^{\mu}/d\eta = (1,0)$, and $g_{0i} = g^{0i} = 0$. Thus, we find the Rindler coordinate system to lead to the Hamiltonian:

$$H_R = \frac{1}{2}\int d\xi_L \left[-\frac{\delta^2}{\delta\varphi^2} + \left(\frac{\partial\varphi}{\partial\xi_L}\right)^2\right] + \frac{1}{2}\int d\xi_R \left[-\frac{\delta^2}{\delta\varphi^2} + \left(\frac{\partial\varphi}{\partial\xi_R}\right)^2\right]$$
$$+ \text{(surface terms)}. \qquad (72)$$

Here we've indicated another potential subtlety which is the possibility of surface terms connecting the left and right hand wedges. Such surface terms no doubt exist. If we imagine a lattice version of the field theory the mere presence of the coordinate singularity in eq.(69) does not permit us to sever the nearest neighbor interactions linking the left and right hand wedges. However, there would seem to be no obvious way for the observer intrinsic to a given wedge to decide what these terms must be. As mentioned earlier, the Minkowski vacuum dictates the physical outcome of any measurement by the observer. Thus one is free to define a Rindler state by a Hamiltonian in which these surface terms are neglected, though such states are not relevant to the physics.

The ground state solution to the Schroedinger equation with the above Hamiltonian but neglecting surface terms is:

$$\Psi_R^0 = \exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{(2\pi)}\,|k|\left(|\beta(k)|^2+|\gamma(k)|^2\right)\right\}. \qquad (73)$$

Again we wish to project the Minkowski vacuum state into this basis (in this sense we are performing a measurement in a generalized Unruh-DeWitt detector defined by the Rindler state). Equating Minkowski and Rindler field configurations over the entire space leads to the relation:

$$\alpha_K = \int_{-\infty}^{\infty} A(k,l)\,\beta(l)\,\frac{dl}{2\pi} + \int_{-\infty}^{\infty} B(k,l)\,\gamma(l)\,\frac{dl}{2\pi} \qquad (74)$$

where α_K is the Minkowski field Fourier component as defined in eq.(A.1). The transformations are straightforward and we obtain:

$$A(k,l) = -\frac{i}{a}\,\Gamma\left(1+\frac{il}{a}\right)\left|\frac{k}{a}\right|^{-1-il/a}\left[\theta(k)e^{\pi l/2a} - \theta(-k)e^{-\pi l/2a}\right];$$

$$B(k,l) = \frac{i}{a}\,\Gamma\left(1+\frac{il}{a}\right)\left|\frac{k}{a}\right|^{-1-il/a}\left[\theta(k)e^{-\pi l/2a} - \theta(-k)e^{\pi l/2a}\right]. \qquad (75)$$

Thus, inserting eq.(74) into the Minkowski vacuum state of eq.(A.10) yields its representation in terms of the Rindler modes:

$$\Psi_m^0 = \exp\Big[-\frac{1}{2}\int\frac{d\ell}{2\pi}\,\ell\coth\left(\frac{\pi\ell}{a}\right)\left(|\beta(\ell)|^2+|\gamma(\ell)|^2\right) \qquad (76)$$
$$-\,\ell\left(\sinh\left(\frac{\pi\ell}{a}\right)\right)^{-1}\left(\overline{\beta(\ell)}\gamma(\ell)+\overline{\gamma(\ell)}\beta(\ell)\right)\Big].$$

Note that the result of neglecting the surface terms in the Rindler Hamiltonian has led to the groundstate of eq.(73) which lacks cross-terms of the form $\overline{\beta(\ell)}\gamma(\ell)$ + hc. Thus, in that state the spacelike correlations across the origin are forced to vanish. The Minkowski vacuum clearly has such correlations and they show up in the cross-terms in the state of eq.(76).

It is now possible to give a prescription by which the physics seen by an observer on the left hand wedge is truly thermal, but we emphasize that this is no more than a prescription. Construct the density matrix and functionally integrate out the modes on the left-hand wedge for the observer on the right:

$$\rho(\beta',\beta) \equiv \int_{-\infty}^{\infty}\prod_{\ell=-\infty}^{\infty} d\gamma(\ell)\,d\overline{\gamma(\ell)}\;\overline{\Psi_m^0}(\beta',\gamma)\,\Psi_m^0(\beta,\gamma)$$

$$= \exp\left[-\frac{1}{2}\int \frac{d\ell}{2\pi}\left\{\ell\coth\left(\frac{2\pi\ell}{a}\right)\left(|\beta(\ell)|^2 + |\beta'(\ell)|^2\right) - \ell\left(\sinh\left(\frac{2\pi\ell}{a}\right)\right)^{-1}\left(\overline{\beta(\ell)}\beta'(\ell) + \overline{\beta'(\ell)}\beta(\ell)\right)\right\}\right]. \quad (77)$$

This result can be compared to the density matrix of a truly thermal system as given by Feynman(18) and written in eq.(40). We see that now the systems are identical and the temperature can be inferred directly from eq.(77) to be the usual $a/2\pi$. This prescription has been previously given by Unruh and Wald(22).

The problem here is that that the system on the left wedge need not be in it's groundstate, i.e. we could equally well have integrated over the left hand wedge with some arbitrary operator function of the $\gamma(\ell)$ in the integrand. This will lead to a physical distinction between the systems and will affect the physical measurements on the right wedge. The previously considered wall at the origin is equivalent to inserting a projection operator that equates the $\gamma(\ell)$ that are odd in their momenta to the $\beta(\ell)$ on the right, while forcing the even $\gamma(\ell)$ to be zero. Nonetheless, the Unruh-Wald prescription is physically reasonable. We certainly expect that the state which lies beyond the horizon of a black-hole is, to a good approximation, a true vacuum insofar as its Casimir effects are concerned.

In conclusion, there are several remaining ambiguities to be resolved concerning (a) the continuity conditions of the field configurations which are the basic degrees of freedom of the quantum field theory, (b) the role of surface terms in the Hamiltonian of the observer in the singular coordinate system (c) the limitations on computability of the observer in the singular system owing to the ambiguity of the definition of the state beyond his horizon and the physical spatial correlations. Probably these are fundamental limitations which cannot be decided by observers intrinsic to the singular coordinate system, but they have not been faced by a fundamental approach to the construction of the field theory previously and will be discussed elsewhere[16].

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Appendix: Minkowski Space Formalism

In the present appendix we review the functional Schroedinger formalism and establish our conventions. We assume flat d+1 dimensional Minkowski spacetime.

A time independent real field configuration may be represented in continuum or finite volume mode sums:

$$\varphi(x) = \int \frac{d^d k}{(2\pi)^d} \alpha_k e^{ik\cdot x} = V^{-1} \sum a_k e^{ik\cdot x} \tag{A.1}$$

$$\alpha_k = \bar{\alpha}_{-k}; \qquad a_k = \bar{a}_{-k}.$$

The d-space functional derivative is:

$$\frac{\delta\varphi(\vec{x})}{\delta\varphi(\vec{y})} = \delta^d(\vec{x}-\vec{y}) = V^{-1}\delta_{\vec{x},\vec{y}} \tag{A.2}$$

$$\frac{\delta}{\delta\varphi(\vec{x})} = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \frac{\delta}{\delta\alpha(k)}; \qquad \frac{\delta\alpha(p)}{\delta\alpha(k)} = (2\pi)^d \delta^d(k-p).$$

Given the Hamiltonian:

$$H = \frac{1}{2}\int d^d x \left[\pi^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + m^2\varphi^2\right] \tag{A.3}$$

and the equal-time commutation (e.t.c.) relation:

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\,\delta(\vec{x}-\vec{y}), \tag{A.4}$$

we may implement the e.t.c. relation by the replacement:

$$\pi(\vec{x}) \rightarrow -i\,\delta^{(d)}/\delta\varphi(x) \tag{A.5}$$

and arrive at the Schroedinger equation:

$$\frac{1}{2}\int d^d x \left[-\frac{\delta^2}{\delta\varphi(x)^2} + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + m^2\varphi^2\right]\Psi(\varphi,t) = i\frac{d}{dt}\Psi(\varphi,t)\,. \tag{A.6}$$

This is, of course, equivalent to a system of coupled simple harmonic oscillators. $\Psi(\varphi,t)$ is the amplitude to find $\varphi(x)$ at time t. In momentum space we may write a diagonalized expression:

$$\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left[-\frac{\delta^2}{\delta\alpha_k\,\delta\bar{\alpha}_k}+(k^2+m^2)|\alpha_k|^2\right]\Psi(\alpha_k,t)=i\frac{d}{dt}\Psi(\alpha_k,t). \tag{A.7}$$

With the ansatz:

$$\Psi=\exp\left\{-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}A(k)|\alpha_k|^2-i\Omega t\right\}, \tag{A.8}$$

we obtain:

$$H\Psi=\left\{\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left[-\frac{1}{4}(A(k)+A(-k))^2|\alpha_k|^2+(k^2+m^2)|\alpha_k|^2\right.\right.$$
$$\left.\left.+\frac{1}{2}(2\pi)^d\delta^d(0)(A(k)+A(-k))\right\}\Psi\right. =\Omega\Psi, \tag{A.9}$$

and the solution for the ground state is given by:

$$\Psi=\exp\left[-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\sqrt{\vec{k}^2+m^2}\,|\alpha_k|^2-i\Omega_0 t\right] \tag{A.10}$$

$$\Omega_0=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\sqrt{k^2+m^2}\,(2\pi)^d\delta^d(0)\ ; \qquad V\equiv(2\pi)^d\delta^d(0)$$

where Ω_0 is the zero-point energy of the system.

Multiparticle states can be constructed by the application of the particle creation operators:

$$
\begin{aligned}
a_k &\equiv \frac{1}{2\sqrt{\pi}} \left\{ \omega_k^{1/2} \alpha_{-k} + \omega_k^{-1/2} \frac{\delta}{\delta \alpha_k} \right\}; \\
a_k^{+} &\equiv \frac{1}{2\sqrt{\pi}} \left\{ \omega_k^{1/2} \alpha_k - \omega_k^{1/2} \frac{\delta}{\delta \alpha_{-k}} \right\};
\end{aligned}
\tag{A.11}
$$

each labelled by a given momentum, k. Furthermore, functional integration may be defined by limit from the finite volume theory (discrete momentum space) in the usual way. The only subtlety here is avoiding the double counting by respecting the reality conditions $\alpha_k = \overline{\alpha}_{-k}$.

References

1. S.Hawking, Comm.Math.Phys. 43,199 (1975)

2. W.Unruh, Phys.Rev. D14,870 (1976)

3. B.DeWitt, Phys. Reports, 19C, 295 (1975)

4. W. Rindler, Am. J. Phys. 34, 1174 (1966)

5. S. Fulling, Phys. Rev. D7, 2850 (1973)

6. P. Davies, J. Phys. A: Math. Gen., 8, 609 (1975)

7. R.Haag, H.Narnhofer, U.Stein, Comm.Math.Phys. 94,219 (1984)

8. R. Hughes, Cern preprint, TH.3670 (1983)

9. D. Boulware, Phys. Rev. D11, 1404 (1975) and D13, 2169 (1976)

10. K. Hinton, J. Phys. A, 16 (1983)

11. M. Soffel, B. Muller, W. Greiner, Phys. Rev. D22 (1980)

12. B. Iyer, A. Kumar, J. Phys. A, 13 (1980)

13. G. ´t Hooft, Utrecht preprint 83-0419 (1983)

14. C. Hill (in preparation)

15. R.P. Feynman, Nucl. Phys. B188, 479 (1981)

16. K. Freese, C. Hill, M. Mueller (in preparation)

17. N. Birrell, P. Davies, Cambridge Univ. Press (1982)

18. R.P. Feynman, Statistical Mechanics, W. Benjamin, p.51 (1972)

19. P. Candelas, D. Deutsch, Proc.R.Soc.Lond., A 354, 79 (1977)

20. E. Titchmarsh, Oxford Clarendon Press, (1969)

21. Bateman Table of Integral Transforms, Vol.II, Chap. 12, McGraw-Hill (1954)

22. W. Unruh, R. Wald, Phys. Rev. D29, 1047 (1984)