Fermi National Accelerator Laboratory

FERMILAB-Pub.-84/89-A
September, 1984

COSMOLOGY WITH DECAYING PARTICLES

Michael S. Turner

Theoretical Astrophysics
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, IL 60510

and

Enrico Fermi Institute
The University of Chicago
Chicago, IL 60637

ABSTRACT

We consider a cosmological model in which an unstable massive relic particle species (denoted by 'X') has an initial mass density relative to baryons $\beta^{-1} \equiv \rho_X/\rho_B >> 1$, and then decays recently (redshift $z \lesssim 1000$) into particles which are still relativistic today (denoted by 'R'). We write down and solve the coupled equations for the cosmic scale factor a(t), the energy density in the various components (ρ_X, ρ_R, ρ_B), and the growth of linear density perturbations ($\delta\rho/\rho$). The solutions form a one parameter (β) family of solutions; physically $\beta^{-1} \simeq (\Omega_R/\Omega_{NR}) \times (1 + z_D)$ = (ratio today of energy density of relativistic to nonrelativistic particles) $\times$ (1 + redshift of (decay). We discuss the observational implications of such a cosmological model and compare our results to earlier results computed in 'the simultaneous decay approximation'. In an appendix we briefly consider the case where one of the decay products of the X is massive and becomes nonrelativistic by the present epoch.

I. INTRODUCTION

Theoretical prejudice, specifically the naturalness of the flat, Einstein-deSitter cosmology, argues strongly that Ω should be 1 (more precisely that the curvature term, k/a^2, should be negligible).[1,2] The inflationary Universe paradigm provides a very attractive way of implementing this prejudice.[2-5] To date, observations have not supported the view that $\Omega = 1$. The observational data together with the highly non-trivial assumption that light (i.e., visible galaxies) provides a good tracer of the mass in the Universe seem to suggest that $\Omega \simeq 0.2 \pm$ '0.1', where the '$\pm$0.1' is not might to represent a formal uncertainty, but rather indicates the spread in the determinations of Ω reported in the literature.[6] [It is already well known that the mass associated with the observed light cannot be the whole story as $\Omega_{\text{luminous}} \simeq 0.01$; so at best we can hope that light traces mass.] Very recently, Bardeen[7] and Kaiser[8] (among others) have begun to explore the possibility that the discrepancy between theory and observation, the so-called 'Ω-problem', could be resolved in scenarios where, for astrophysical reasons, light does not trace mass (specifically because visible galaxies form only at 3-σ peaks in the density contrast $\delta\rho/\rho$).

Another possible solution to the Ω-problem which has been recently suggested is that most of the energy density in the Universe resides in a smooth component.[9,10] [All of the dynamical techniques for measuring Ω are insensitive to a smooth, unclustered (on scales >> 30 Mpc) component[11].] That smooth component could be 'hot' particles (i.e., particles with a very large internal velocity dispersion, $\langle v^2\rangle^{1/2} \gg 10^{-3}c$), which by virtue of their high speeds could not cluster (on scales $\lesssim$ 30 Mpc), and would thus be smoothly distributed. The most frequently mentioned origin for these 'hot' particles is the recent decay of a massive relic particle species. If the decay products of this parent species (denoted by 'X') are very light, then they will still be relativistic today ($v \simeq c$). On the other hand if at least one of the decay products has a mass not too different from that of the X, then today some of the decay products might be 'hot', but not relativistic.

The other possibility for the smooth component is a relic cosmological constant (of unexplained origin!).[10] In either smooth component scenario, the 'Ω-problem' is not resolved in a totally satisfactory manner. The 'Ω-problem' is in fact a timescale problem -- in an $\Omega \neq 1$ (i.e., $k \neq 0$) cosmology there is an uncomfortably large timescale (relative to the fundamental gravitational timescale $t_{pl} \simeq 10^{-43}$sec): the time at which the curvature term (k/a^2) becomes comparable to the energy density term. In both the decaying particle scenario and the $\Lambda \neq 0$ scenario there are also timescales -- the lifetime of the unstable particle species X and for $\Lambda \neq 0$ the epoch at which the energy density in particles is comparable to $\Lambda/8\pi G$. The only consolation is that one of the latter timescales might be more easily explained in terms of fundamental microphysics.

In this paper the cosmology of a model Universe with an unstable relic particle species which decays in the recent past (redshift $\lesssim$ 1000) is explored in detail: the evolution of the cosmic scale factor a(t), the energy density in the various components, the growth of linear density perturbations, and the kinematics of the model (the age of the Universe, the look back distance, the comoving proper volume at various redshifts, etc.). The brief introduction presented above was meant to provide the motivation for considering such cosmological models. In the next section the simultaneous decay approximation (i.e., that all the X particles decay simultaneously at a time $t \simeq \tau_X \equiv \Gamma^{-1}$) which has been used previously to discuss decaying particle cosmologies will be reviewed. The exact equations for $a(t)$, ρ_i, and $\delta\rho/\rho$ will be derived (for the case in which all the decay products are still relativistic today), and the one parameter family of solutions will be discussed. In Section III, the cosmological implications of the solutions will be described. Section IV contains a summary and concluding remarks. In the appendix the case where one of the decay products becomes nonrelativistic by the present epoch is briefly considered. Here the solutions form a two parameter family of solutions.

II. EQUATIONS FOR THE DECAYING PARTICLE COSMOLOGY

A. Review of the Simultaneous Decay Approximation (SDA)

To begin let us review the basic scenario in the SDA. Consider a massive particle species X with mass m_X, decay width Γ, and relic abundance (relative to 3K photons) before it decays $r \equiv n_X/n_\gamma$. For now assume that its decay products are so light that they are still relativistic at the present epoch; in the appendix the case where one of the decay products is sufficiently massive that it is nonrelativistic today is considered. We are interested in the case where the mass density contributed by Xs (before they decay) is significantly larger than that of the baryons and other stable nonrelativistic (NR) particles. Define the ratio of their mass densities (before the Xs decay, i.e., $t \ll \Gamma^{-1}$) to be β:

$$\beta \equiv \rho_{NR}/\rho_X \quad , \tag{1a}$$

$$= 0.263(\Omega_{NR}h^2/\theta^3 m_{100} r) \quad , \tag{1b}$$

$$= 0.965(\Omega_{NR}h^2/\theta^3 \tilde{m}_{100}), \tag{1c}$$

where Ω_{NR} is the fraction of critical density contributed by stable, NR particles today ($\equiv \rho_{NR}/\rho_c$, $\rho_c = 3H_o^2/8\pi G$), $H_o = 100$ h kms^{-1} Mpc^{-1} is the present value of the Hubble parameter, 2.7 θ K is the present photon temperature, $m_{100} = m_X/100$ eV, and $\tilde{m}_{100}$100eV is the mass of a relic eutrino species which would contribute the same mass density as the X does [$\tilde{m}_{100} = (m_X/100\text{eV})(r/(3/11))$]. Note that if the stable NR particles are all baryons, then

$$\beta = 3.44 \times 10^{-3} \; (\eta_{10}/\tilde{m}_{100}), \tag{2}$$

where $\eta_{10} \equiv 10^{10} \; \eta \equiv 10^{10} \; (n_b/n_\gamma)$; for reference big bang nucleosynthesis constrains η_{10} to the interval (4,7) (ref. 12).

At very early times the Universe is radiation-dominated; at a photon temperature T_{eq}, time t_{eq} and cosmic scale factor a_{eq} the Universe becomes matter-dominated (by Xs and the stable NR particles; specifically: $\rho_X + \rho_{NR} = \rho_\gamma + \rho_{\nu\bar{\nu}}$) where

$$a_{eq}/a_o = 4\times10^{-5}(\beta^{-1}+1)^{-1}(\Omega_{NR}h^2/\theta^4)^{-1}, \tag{3}$$

$$= 4\times10^{-5}\ \beta\ (\Omega_{NR}h^2/\theta^4)^{-1} \quad ,$$

$$t_{eq} \simeq 3.8\times10^{10}\sec(1+\beta^{-1})^{-2}(\Omega_{NR}h^2/\theta^3)^{-2} \quad , \tag{4}$$
$$\simeq 3.8\times10^{10}\sec\ \beta^2(\Omega_{NR}h^2/\theta^3)^{-2} \quad ,$$

$$T_{eq} = 5.9\text{eV}(1+\beta^{-1})(\Omega_{NR}h^2/\theta^3) \quad , \tag{5}$$
$$\simeq 5.9\text{eV}\ \beta^{-1}(\Omega_{NR}h^2/\theta^3);$$

here a_o is the value of the cosmic scale factor at the present epoch. Note in the limit $\beta \rightarrow \infty$ we recover the usual results; for $\beta < 1$ the Universe becomes matter-dominated at an earlier epoch.

From $a = a_{eq}$ to $a = a(t=\Gamma^{-1}) \equiv a_D$ the Universe is matter-dominated; $a(t) \propto t^{2/3}$ and one expects linear density perturbations in the NR particles and Xs grow as: $\delta\rho/\rho \propto t^{2/3} \propto a(t)$. In the SDA at $t \simeq \Gamma^{-1}$ all the Xs decay at once; the Universe becomes radiation-dominated again (by the relativistic decay products of the X). For $a \gtrsim a_D$, the cosmic scale factor $a(t) \propto t^{1/2}$, and one expects linear density perturbations in the NR particles to cease growing.[13] Just after the decay epoch the ratio of the energy density in relativistic (R) debris to that in NR particles is $\rho_R/\rho_{NR} \simeq \rho_X/\rho_{NR} = \beta^{-1}(>>1)$. Due to the redshifting of the energy of each light daughter particle ($E \propto a^{-1}$) this ratio decreases as $a(t)^{-1}$. Denote the energy density in relativistic debris at the present epoch by its fraction of the critical density Ω_R. [Note, in a k = 0 FRW cosmology with $\Lambda = 0$, we must have $\Omega_{NR} + \Omega_R = 1$.] Using the fact that $\rho_R/\rho_{NR} = \beta^{-1}$ $a_D/a(t)$, it follows that (in the SDA)

$$\beta^{-1} = (\Omega_R/\Omega_{NR})\ a_o/a_D, \tag{6}$$

$$= (\Omega_R/\Omega_{NR})(1+z_D) \quad ,$$

where a given epoch is also specified by the redshift z that a photon emitted at that epoch will suffer by the present epoch: $(1+z) \equiv a_o/a(z)$.

Using the fact the Universe is radiation-dominated from $a = a_D$ until the present so that $a_o/a_D \simeq [t_o/(t_D \simeq \Gamma^{-1})]^{1/2}$, it follows that

$$\tau = \Gamma^{-1} \simeq (2.1\times10^{17}\mathrm{sec})(a_D/a_o)^2(\Omega_R h^2)^{-1/2}. \tag{7}$$

Taking this Eqn. together with Eqns. (1,4) we find that

$$\Omega_R h^2/\theta^4 \simeq 0.30\ \tilde{m}_{100}^{\ 4/3}\ \tau_9^{2/3}, \tag{8a}$$

$$\tilde{m}_{100} \simeq 2.45(\Omega_R h^2/\theta^4)^{3/4}\tau_9^{-1/2}, \tag{8b}$$

where $\tau_9 = (\tau/10^9\mathrm{yr}) = (2.09 \times 10^{-41}\mathrm{GeV}/\Gamma)$.

To summarize the scenario in the SDA: from $t=t_{eq}$ to $t\simeq\Gamma^{-1}$ the Universe is matter dominated, $a(t) \propto t^{2/3}$, and $\delta\rho/\rho \propto a$; at $t \simeq \Gamma^{-1}$ all the Xs decay so that just after $t \simeq \Gamma^{-1}$, $\rho_R/\rho_{NR} \simeq \beta^{-1}$; thereafter the Universe is radiation-dominated, $a \propto t^{1/2}$, $\rho_R/\rho_{NR} = \beta^{-1}\ a_D/a(t)$, and $\delta\rho/\rho \simeq$ cons't. Linear density perturbations grow very little before the Universe becomes matter-dominated again (when $a/a_o \simeq \Omega_R/\Omega_{NR}$); thus in the SDA the total growth factor for a linear density perturbation is predicted to be:

$$\begin{aligned} \gamma &\simeq a_D/a_{eq}\ , \\ &\simeq 2.5 \times 10^4\ (\Omega_R h^2/\theta^4)(1+\beta), \end{aligned} \tag{9}$$

$$\simeq 2.5 \times 10^4 \ (\Omega_R h^2/\theta^4).$$

As noted by Turner etal.[9] the growth factor is independent of β (or alternatively $1 + z_D$).

B. The Exact Equations

Now let's examine this scenario without making the SDA. To begin we are considering a flat (i.e., $k/a^2 \ll 8\pi G\rho/3$) Friedmann-Robertson-Walker (FRW) cosmology with line element,

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{10}$$

The evolution of a is governed by the usual Friedmann equation

$$H^2 \equiv (\dot{a}/a)^2 = 8\pi G\rho/3, \tag{11}$$

where overdot signifies a time derivative and throughout we work in units where $h = c = k_B = 1$.

Write the total energy density ρ as:

$$\rho = \rho_X + \rho_{NR} + \rho_R \quad , \tag{12}$$

where ρ_i is the energy density contributed by X particles ($i = X$), stable, NR particles ($i = NR$), and the relativistic decay products of the X ($i = R$). During the epochs of interest ($a > a_{eq}$), the energy density contributed by the photons and massless neutrino species can be neglected. The equations governing the evolution of the energy density are:

$$\dot{\rho}_X = -3H\rho_X - \Gamma\rho_X \quad , \tag{13a}$$

or $$d(a^3\rho_X)/dt = -\Gamma(a^3\rho_X) \quad , \tag{13b}$$

$$\dot{\rho}_{NR} = -3H\rho_{NR} \quad , \tag{14}$$

$$\dot{\rho}_R = -4H\rho_R + \Gamma\rho_X \quad . \tag{15}$$

The first term on the rhs of Eqn. (13a) is just the dilution effect of the expansion, while the second term is due to the decays. When Eqn. (13a) is rewritten as Eqn. (13b) the physics is manifest: the number of Xs per comoving volume ($\propto a^3\rho_X/m_X$) is decreasing according to the usual exponential decay law. The energy density in stable, NR particles only decreases due to the expansion, the solution to Eqn. (14) being the familiar $\rho_{NR} \propto a^{-3}$. The two terms on the rhs of Eqn. (15) represent the dilution and redshift of the energy of the R particles ($-4H\dot{\rho}_R$), and the energy density being 'pumped in' by the decays of X particles. The solutions to Eqns. (13-15) are:

$$\rho_X = \rho_{Xi}(a/a_i)^{-3}e^{-\Gamma t}, \tag{16}$$

$$\rho_{NR} = \rho_{NRi}(a/a_i)^{-3} \quad , \tag{17}$$

$$\rho_R = (a/a_i)^{-4}\rho_{Xi}\int_0^t (a(t')/a_i)e^{-\Gamma t'}d\Gamma t' \quad , \tag{18}$$

where the initial epoch $t = t_i$ is chosen such that $t_{eq} \ll t_i \ll \Gamma^{-1}$, and $a_i = a(t_i)$, $\rho_{Xi} = \rho_X(t_i)$, and $\rho_{NRi} = \rho_{NR}(t_i)$.

In the linear regime the evolution of the density contrasts in the various components, $\delta_i \equiv \delta\rho_i/\rho_i$, are determined by:

$$\ddot{\delta}_i + 2H\dot{\delta}_i + v^2_{si}k^2\delta_i/a^2 = 4\pi G\delta\rho \equiv 4\pi G\rho\sum_i(\rho_i/\rho)\delta_i \ , \tag{19}$$

where k is the comoving wavenumber of the perturbation (physical wavelength $\lambda_{ph} = 2\pi a(t)/k$), and v_{si} is the sound speed in component $i[\equiv (dp_i/d\rho_i)^{1/2}]$. Eqn. (19) is only valid for perturbations with physical wavelengths much smaller than the horizon ($\simeq H^{-1}$). The wavelength $\lambda_J^2 = \pi v^2_{si}\delta_i/G\delta\rho$ is the Jeans wavelength; for $\lambda_{ph} \lesssim \lambda_J$ the perturbation will oscillate like a sound wave, while for $\lambda_{ph} \gtrsim \lambda_J$ the perturbation is Jeans unstable and will grow. [For a more detailed discussion of Eqn. (19) see ref. 14.]

For the R particles $v_s^2 = c^2/3$ and only perturbations on scales larger than the horizon will be Jeans unstable; perturbations on the scales of interest will at best oscillate with constant amplitude. In fact, if the R particles are collisionless (the most likely case) perturbations on scales smaller than the horizon will be damped due to free-streaming of the particles (see, e.g., refs. 15). For this reason we will take $\delta_R = 0$ always. For the NR and X components we will only consider perturbations with wavelengths greater than the Jeans wavelength so that the pressure terms can be neglected. [Again the Xs are likely to be collisionless and very NR, implying that pressure effects and free streaming effects will be unimportant on the scales of interest. If the NR component is baryons then after decoupling the

Jeans length corresponds to a mass which is always $\lesssim 10^5 M_\odot$.]

If initially $\delta_X = \delta_{NR}$ and $\dot{\delta}_X = \dot{\delta}_{NR}$, then $\delta_X(t) = \delta_{NR}(t)$ for all time. On the other hand if initially $\delta_X \neq \delta_{NR}$, then within a few expansion times they will become equal. Thus it suffices to follow the evolution of δ_{NR} alone, supplemented by $\delta_X = \delta_{NR}$:

$$\ddot{\delta}_{NR} + 2H\dot{\delta}_{NR} - 3/2\ H^2\delta_{NR}(\rho_X+\rho_{NR})/\rho = 0\ . \tag{20}$$

Eqns. (16-18,20) are the 'master equations' for the decaying particle cosmology. By introducing some dimensionless variables they can be recast into a more useful set of coupled differential equations. These variables are:

$$x = \Gamma t,$$

$$f_X = \rho_X/\rho_{Xi}$$

$$f_R = \rho_R/\rho_{Xi}\ ,$$

$$f_{NR} = \rho_{NR}/\rho_{Xi}\ ,$$

$$H_i^2 = 8\pi G\rho_{Xi}/3\ ,$$

$$x_H = \Gamma/H_i\ ;$$

the dimensionless set of equations corresponding to Eqns. (16-18,20) are

$$a'/a = (f_X+f_R+f_{NR})^{1/2}x_H^{-1}, \tag{21a}$$

$$f_R = a^{-4}\int_0^x a(x')\ e^{-x'}\ dx', \tag{21b}$$

$$\delta_{NR}'' + 2(a'/a)\delta_{NR}' - 3/2\ (a'/a)^2\delta = 0\ , \tag{21c}$$

$$f_X = a^{-3}\ e^{-x}\ , \tag{21d}$$

$$f_{NR} = \beta\ a^{-3}\ , \tag{21e}$$

where as before $\delta \equiv \delta_{NR}(f_X+f_{NR})/(f_X+f_{NR}+f_R)$, prime indicates derivative with respect to x, and a(t) has been normalized such that $a_i = a(t_i) = 1$. Note that early on (i.e., $t \ll \Gamma^{-1}$ or $x \ll 1$), the Universe is strictly matter-dominated so that $a(x) = (x/x_i)^{2/3}$; thus Eqn. (21a) implies that

$$x_i = 2/3\ (1+\beta)^{-1/2}x_H\ .$$

It should be clear that (subject to specifying $\delta_{NR}(x_i)$ and $\delta_{NR}'(x_i)$) the solutions to this set of equations are a one parameter (namely β) family of solutions. Recall that in the language of the SDA $\beta^{-1} = (1+z_D)(\Omega_R/\Omega_{NR})$. How does one exploit this nice feature of the set of equations which govern the decaying particle cosmology? This should become very clear in the next section, but briefly the idea is: (1) select the value of β which is of interest; (2) integrate Eqns. (21a-e); (3) the present epoch is then specified by when the value of f_R/f_{NR} is equal to the desired value of Ω_R/Ω_{NR}; (4) the values of x and a(x) for the present epoch (x_o and a_o), along with the present value of the Hubble parameter H_o can then be used to convert all dimensionless quantities into dimensional quantities. In the next section we will use numerical solutions to these equations to describe the decaying particle cosmology in some detail.

Before we go on to discuss the solutions to Eqns. (21a-e) consider the problem of calculating $\Omega_R h^2$ in terms of m_X, Γ, and θ. Because of the $\exp(-x')$ factor, the integral in the expression for f_R, cf Eqn. (21b), will converge for $x \gg 1$. In fact, in the limit of small β, it is straightforward to show that $\int_0^\infty a(x)e^{-x}dx$ is $\propto$ $x_H{}^{-2/3}$. Numerically we find that

$$\int_0^\infty a(x)e^{-x}dx = 1.09\ x_H{}^{-2/3}; \tag{22}$$

for details see ref. 16. Assuming that $t_0 \gg \Gamma^{-1}$ (i.e., $x_0 \gg 1$), so that all of the Xs have decayed by the present epoch, the present ratio of f_R to f_{NR} ($\simeq \Omega_R/\Omega_{NR}$) can be used to solve for a_0, and in turn $f_R(x_0)$:

$$\begin{aligned} a_0 &= 1.09(\Omega_R/\Omega_{NR})^{-1}\beta^{-1}x_H^{-2/3}, \\ &\simeq 1.09(1+z_D)x_H^{-2/3} \quad , \end{aligned} \tag{23}$$

$$f_R(x_0) = 0.77x_H(\Omega_R/\Omega_{NR})^4\beta^4. \tag{24}$$

In terms of $f_R(x_0)$, Ω_R is given by

$$\begin{aligned} \Omega_R &= \rho_R/(3H_0{}^2/8\pi G) \quad , \\ &= f_R(x_0)\rho_{Xi}/(3H_0^2/8\pi G); \end{aligned}$$

using this relationship and the expression for β in Eqn. (1), it follows that:

$$(\Omega_R h^2/\theta^4) = 1.41(m_{100}r)^{4/3}\tau_9^{2/3},$$
$$= 0.249\ \tilde{m}_{100}^{4/3}\tau_9^{2/3}, \tag{25a}$$

$$\tilde{m}_{100} = 2.84(\Omega_R h^2/\theta^4)^{3/4}\tau_9^{-1/2}, \tag{25b}$$

where as earlier $\tau_9 = \tau/10^9$ yrs $= 2.09 \times 10^{-41}$ GeV/Γ. Comparing these expressions with the corresponding ones which are computed in the SDA, cf Eqns. (8a,b), we see that the SDA leads to about a 20% overestimation for $\Omega_R h^2/\theta^4$, or a 14% underestimation for $\tilde{m}_{100}$. Lines of constant $\Omega_R h^2/\theta^4$ and of constant $1+z_D$ in the $\tilde{m}_{100}$ - τ plane are shown in Fig. 1.

III. THE DECAYING PARTICLE COSMOLOGY (DPC)

In this section we will discuss the solutions to the equations derived in the previous section, paying particular attention to the cosmological implications and comparing the DPC to conventional cosmological models.

A. Evolution of a(x), ρ_X/ρ_{NR}, and ρ_R/ρ_{NR}

In Fig. 2 the evolution of a, ρ_R/ρ_{NR}, and ρ_X/ρ_{NR} are shown as a function of x for β = 1/10, 1/100. So long as $(\rho_R + \rho_X) >> \rho_{NR}$ the curves displayed are universal -- with all functional dependences being determined by only the mix of R and X particles (and relatively independent of the 'slight contamination' of stable, NR particles). In this scaling limit, a(x) is universal and independent of β, while ρ_X/ρ_{NR} and ρ_R/ρ_{NR} are universal and scale with β^{-1}. This scaling behaviour can be clearly seen in Fig. 2, as well slight deviations from it as ρ_{NR} becomes comparable to $\rho_X + \rho_R$ for the $\beta = 1/10$ curves.

At early times $a(x) \propto x^{2/3}$ (as expected) and near $x \simeq 1$ the behaviour changes to $a(x) \propto x^{1/2}$. For the $\beta = 1/10$ curve the functional dependence starts to change back to $a(x) \propto x^{2/3}$ as the fractional matter content increases. For reference the predicted behaviour of ρ_X/ρ_{NR} and ρ_R/ρ_{NR} in the SDA are also shown in Fig. 2. By $x = 1$, ρ_X/ρ_{NR} ($\simeq 0.37\beta^{-1}$) and ρ_R/ρ_{NR} ($\simeq 0.34\beta^{-1}$) are about equal. More interesting, $(\rho_X + \rho_R)/\rho_{NR} \simeq 0.71\ \beta^{-1}$, implying that about 30% of the initial rest energy per X (or decay product of the X) has been redshifted away. This, of course, is due to the redshifting of the energy of the decay products of Xs which decayed early on (note, a fraction $1 - e^{-1} \simeq 63\%$ of the Xs decay by $x = 1$).

As briefly mentioned in the previous section, for a chosen value of β, the solutions to the DPC equations are made into a cosmological model by using the ratio ρ_R/ρ_{NR} to identify the present epoch. The ratio ρ_R/ρ_{NR} is set equal to Ω_R/Ω_{NR}. The values of $a_o = a(x_o)$ and $x_o = \Gamma t_o$ (t_o = present age of the Universe) are compiled in Table 1 for various values of β and $\Omega_R/\Omega_{NR} = 5, 4, 3, 2, 1, 0.5$.

B. Evolution of Density Perturbations

Let's turn now to the growth of density perturbations in the linear regime. Recall in the SDA: for $x \leq 1$, $\delta\rho/\rho \propto a$; for $1 \leq x \leq x_{MD}$, $\delta\rho/\rho \propto$ cons't; for $x \geq x_{MD}$, $\delta\rho/\rho \propto a$; $x = x_{MD}$ is the epoch when the Universe again becomes matter-dominated (i.e., $\rho_R/\rho_{NR} \simeq 1$).

In Fig. 3 a^{-1} $\delta\rho/\rho$ is shown as a function of a for β = 1/3, 1/10, 1/30, 1/80, 1/300. The epoch x = 1 corresponds to a $\simeq$ 27; for reference the behaviour predicted in the SDA is also shown. The most striking feature of Fig. 3 is that the growth of linear density perturbations does not suddenly stop when the Universe becomes radiation-dominated (x = 1, a $\simeq$ 27) as expected, but instead linear density perturbations continue to grow slowly during the radiation-dominated epoch ($1 \leq x \leq x_{MD}$; $27 \leq a \leq 27\ \beta^{-1}$). There are two reasons for this. First (and most important), is that the perturbations still have velocity (i.e. $\dot{\delta} \neq 0$) at the onset of radiation domination, and so continue 'to coast' and undergo further growth. Second, since ρ_R/ρ_{NR} is finite (and not infinite), slow power law growth is still predicted (with exponent depending upon ρ_R/ρ_{NR}). Equally obvious is the fact that the behaviour $\delta\rho/\rho \propto$ a does not immediately begin again when $\rho_{NR} \simeq \rho_R$ ($a \simeq 27\beta^{-1}$), but takes several expansion times.

In the simplified picture described in Sec. IIA no growth is predicted for $\delta\rho/\rho$ for $27 \leq a \leq 27\beta^{-1}$; that is, if $\delta\rho/\rho = c \times a(x)$ for x << 1, then $\delta\rho/\rho = (c\beta) \times a(x)$ for the $x \gg x_{MD}$ ($a \gg 27\beta^{-1}$). This approximation predicts a 'deficit in growth' (compared to $\delta\rho/\rho \propto a$) of about β^{-1}. From Fig. 3 it can be seen that the deficit is considerably less. Likewise, in the SDA no growth in $\delta\rho/\rho$ is predicted from x = 1 until the present epoch; in Fig. 3 it is apparent that there is some growth. The 'total deficit' and the growth of $\delta\rho/\rho$ from x = 1 until the present epoch are quantified in Table 2.

C. The Age of the Universe

The present age of a cosmological model is just given by

$$t_o = \int_o^{a_o} da/\dot{a} \; ; \tag{26a}$$

it is most useful to express t_o in 'Hubble times' ($\equiv H_o^{-1}$):

$$H_o t_o = \int_o^{a_o} da/[a(H/H_o)] \; . \tag{26b}$$

For a matter-dominated Universe $H/H_o = (a/a_o)^{-3/2}$, and we obtain the familiar result that $H_o t_o = 2/3$. For a purely radiation-dominated model $H/H_o = (a/a_o)^{-2}$, and it follows that $H_o t_o = 1/2$. This fact is a very formidable difficulty which the DPC must face -- the prediction of a youthful Universe. [For reference, H_o is believed to be in the range 50-100 $\mathrm{kms^{-1}Mpc^{-1}}$ = $(20\ \mathrm{Byr})^{-1}$ - $(10\ \mathrm{Byr})^{-1}$, while various techniques (dating of globular clusters, nucleocosmochronology) suggest that $t_o \simeq$ 15 '± 3' Byr. On the face of it this implies $H_o t_o \simeq 0.6 - 1.8$. At present, however, systematic uncertainties in both H_o and t_o preclude a definitive determination of $H_o t_o$.[17]]

In the SDA, $H/H_o = [\Omega_R(a/a_o)^{-4} + \Omega_{NR}(a/a_o)^{-3}]^{1/2}$ for $(1+z_D)^{-1} \leq a/a_o \leq 1$, and $H/H_o = (1+z_D)^{1/2}\Omega_R^{1/2}(1+\beta)^{1/2}a^{-3/2}$ for $0 \leq a/a_o \leq (1+z_D)^{-1}$. Using this expression for H/H_o in Eqn. (26b) it follows that

$$H_o t_o = (2\Omega_{NR}^{-2}/3)[1-3\Omega_R+\Omega_R^{3/2}(1+\beta)^{-1/2}(2+\beta)] \tag{27}$$

(this is identical to the expression derived by Turner <u>etal.</u> in ref.

9). There are two effects which cause H_ot_o to deviate from 1/2 in the DPC: (1) the present contribution of matter to the energy density, quantified by Ω_{NR}; (2) the early $(a/a_o \leq (1+z_D)^{-1})$ matter-dominated epoch. For fixed Ω_{NR}, Ω_R the second effect becomes negligible as $\beta \to 0$ (i.e., as $1 + z_D \to \infty$), and H_ot_o becomes only a function of Ω_{NR}, Ω_R:

$$H_ot_o \underset{\beta\to 0}{\to} (2\Omega_{NR}^{-2}/3)[1-3\Omega_R+2\Omega_R^{3/2}]. \tag{28}$$

Fig. 4 shows $Ht[\equiv(x/x_H) \times (f_X + f_{NR} + f_R)^{1/2}]$ as a function of x for β = 1/3, 1/10, 1/30, 1/80 with the present epoch being identified by Ω_R/Ω_{NR}. The convergence of Ht (for fixed Ω_R/Ω_{NR}) to a value which is independent of β as $\beta \to 0$ is manifest. In Table 3 H_ot_o is tabulated for various values of β and Ω_R/Ω_{NR}. For comparison the values computed in the SDA, cf. Eqn. (27), are shown in parentheses. In this regard the SDA is rather good (agreement to better than 5%).

D. Age-Redshift Relationship

A closely-related relationship which is also of some cosmological interest is the relationship between redshift and the age of the Universe. This relationship is given by:

$$H_ot = \int_o^{(1+z)^{-1}a_o} da/[(H/H_o)a] \ . \tag{29}$$

For a matter-dominated Universe, $H/H_o = (a/a_o)^{-3/2}$,

$$H_o t = (2/3)\ (1+z)^{-3/2}\ ;$$

while for a radiation-dominated Universe, $H/H_o = (a/a_o)^{-2}$, and

$$H_o t = (1/2)\ (1+z)^{-2}\ .$$

At a given redshift, a radiation-dominated Universe is younger. This fact is of some importance when considering the evolution of galaxies, clusters, _etc._ The DPC cosmology, of course, lies somewhere in between these two models. This is illustrated in Fig. 5 for $\beta = 1/10$, $1/30$, $1/80$ and $\Omega_R/\Omega_{NR} = 3$. As can be seen in the figure for low redshift, $1 + z \lesssim 1 + z_D \simeq 1/3\beta$, when the Universe is radiation-dominated (_i.e._, $\rho_R \geq \rho_{NR}$) $H_o t$ evolves $\propto (1+z)^{-2}$ (as in the pure radiation case); while for higher redshifts, $1 + z \gtrsim 1 + z_D$, when the Universe is matter-dominated (_i.e._, $\rho_{NR} \geq \rho_R$) $H_o t$ evolves $\propto (1+z)^{-3/2}$ (as in the pure matter case). [For comparison a cosmological model with $\rho_R = 0$, $\Lambda \neq 0$, $k = 0$, $\Omega_\Lambda (\equiv \Lambda/3H_o^2) = 3\ \Omega_{NR}$, and $H/H_o = [\Omega_\Lambda + \Omega_{NR}\ (a/a_o)^{-3}]^{1/2}$ is also shown. At a given redshift, a $\Lambda \neq 0$ model is even older than a matter-dominated model. For further discussion of flat models with $\Lambda \neq 0$ see ref. 10.]

E. Angular Size and Comoving Volume vs. Redshift

Two kinematical quantities of significance are the observed angular size of an object at a given redshift, and the comoving volume element $dV_o \propto d\Omega dz\ f(z)$ at a given redshift. Both are related in a simple way to present proper distance to an object with redshift z. Physically that distance, $d_o(z)$, is just the present value of the scale factor times the coordinate distance covered by a photon from the epoch $a = (1+z)^{-1}a_o$ to the present epoch:

$$d_o(z) = a(t_o)\int_{t(1+z)}^{t_o} dt'/a(t') , \tag{30}$$

$$= H_o^{-1}\int_{(1+z)^{-1}a_o}^{1} a_o da/[(H/H_o)a^2] .$$

For a matter-dominated Universe $(H/H_o) = (a/a_o)^{-3/2}$, and

$$H_o d_o(z) = 2[1-(1+z)^{-1/2}];$$

while for a radiation-dominated model $(H/H_o) = (a/a_o)^{-2}$, and

$$H_o d_o(z) = z/(1+z).$$

In Fig. 6 $H_o d_o(z)$, the present proper distance to an object at redshift z in Hubble units, is shown as a function of z for pure matter, pure radiation, $\Omega_R/\Omega_{NR} = 3$ and $\beta = 1/10$, and $\Omega_\Lambda/\Omega_{NR} = 3$ models. In Hubble units, the distance to an object at redshift z increases as one goes from a pure radiation model to a pure matter model onto a model with $\Lambda \neq 0$. The distance to the (particle) horizon is $H_o\ d_o(z=\infty)$ (in Hubble units). For pure matter $H_o d_o(\infty) = 2 (= 3H_o t_o$ -- the familiar result that the horizon distance in a matter-dominated Universe is $3t_o$); for pure radiation $H_o\ d_o(\infty) = 1\ (= 2\ H_o t_o)$; and in the SDA:

$$H_o\ d_o(\infty) = 2\Omega_{NR}^{-1}\ [1 - \Omega_R^{1/2}/(1+\beta)^{1/2}] .$$

Table 4 is a compilation of $H_o\, d_o(\infty)$ for various values of β and Ω_R/Ω_{NR}; the values calculated in the SDA are also given. In general the value calculated in the SDA is higher, and the agreement is quite good ($\lesssim$ 8% difference).

The observed angular size of an object at redshift z (in a flat cosmological model) is just its proper size then ($\equiv \ell(z)$) divided by its proper distance then ($\equiv d(z) = d_o(z)/(1+z)$),

$$\theta = \ell(z)/d(z) \ , \qquad (31)$$
$$= \ell(z)(1+z)/d_o(z).$$

For an object whose proper (i.e., physical size) does not vary (i.e., is independent of z), e.g., a galaxy, we have

$$\theta = H_o\ell_o(1+z)/H_o d_o(z);$$

the angular size is inversely proportional to $H_o d_o(z)$. On the other hand for an object whose proper size $\propto a(t)$, say, e.g., a certain comoving region of space, $\ell = \ell_o/(1+z)$, we have

$$\theta = H_o\ell_o/H_o d_o(z) \ ,$$
$$= 1.1' \ h(\ell_o/\mathrm{Mpc})/H_o d_o(z).$$

[For example the microwave temperature fluctuation on a given angular scale $\theta \ll 1°$ is related to the density constraint on the comoving length scale corresponding to that angle at the surface of last scattering, $1 + z \simeq 1500$; for more details see refs. 18, 19.]

The comoving volume element dV_o (which for simplicity we take to be the proper volume element at the present epoch) defined by the solid angle $d\Omega$ and redshift interval dz is just

$$\begin{aligned} dV_o &= d/dz\ (d_o^3(z)/3)\ d\Omega\ dz, \\ &= d/dz(d_o(z))\ d_o^2(z)dz\ d\Omega\ , \\ &= H^{-1}(z)d_o^2(z)\ dz\ d\Omega\ , \end{aligned} \tag{32}$$

where $H(z)$ is the Hubble parameter at the epoch $a = (1+z)^{-1}a_o$. It is convenient to write

$$dV_o = f(z)\ z^2\ dz\ d\Omega\ H_o^{-3}\ , \tag{33a}$$

where

$$f(z) = (H_o d_o(z))^2 (H_o/H(z)) z^{-2}\ . \tag{33b}$$

So defined, $f(z) \rightarrow 1$ as $z \rightarrow 0$. For a pure matter model,

$$f(z) = 4z^{-2}[(1+z)^{-3/4} - (1+z)^{-5/4}]^2;$$

while for a pure radiation model,

$$f(z) = (1+z)^{-4}\ .$$

The function f(z) is shown in Fig. 7 for pure radiation, pure matter, $\Omega_R/\Omega_{NR} = 3(\beta = 1/10, 1/80)$, and $\Omega_\Lambda/\Omega_{NR} = 3$ models. At fixed redshift, f(z) decreases (meaning the proper volume defined by $d\Omega$ and dz is smaller), going from the $\Lambda \neq 0$ model to the pure matter model to the DPC to the pure radiation. The comoving number density of a given set of objects in the volume defined by $d\Omega$ and dz is of course $\propto f(z)^{-1}$, implying that for a given set of objects their comoving number density is highest in a pure radiation model, and lowest in a $\Lambda \neq 0$ model. In principle, this fact could be used to differentiate between cosmological models.

IV. SUMMARY AND CONCLUDING REMARKS

The idea of the DPC is straightforward -- a massive relic species with energy density greater than that of the stable, NR particles present (baryons, etc.) decays in the recent past (since decoupling) into particles which are still relativistic today. The current motivation for the model is twofold; first, there exist particle physics theories which predict massive particle species whose relic abundances would be significant (e.g., neutrinos) and whose lifetimes are comparable to the age of the Universe (say $\gtrsim 10^8$ years).[20] Second, and more important, the DPC offers the possibility of solving the 'Ω-problem' by producing a hot, unclustered component which dominates the present mass density.

The solutions to the cosmological equations describing the DPC form a one-parameter (two, if one of the decay products becomes NR by the present epoch) family. The quantity β^{-1} is just the ratio of the energy density in the massive, unstable particle species (before it decays) to the mass density in stable, NR particles. The parameter β is also related to redshift of the decay epoch and the ratio of energy density in R particles to that in NR particles: $\beta^{-1} \simeq (\Omega_R/\Omega_{NR})(1+z_D)$. To solve the Ω-problem Ω_R/Ω_{NR} should be 3-5. A number of unpleasant things occur if $(1+z_D)$ is too big, say $\gtrsim 20$ (excessively-large fluctuations in the microwave background on both small[21] and large[9] angular scales; disruption of bound structures which form before the decay epoch[9]; kinks in the galaxy-galaxy correlation function[9] -- for further discussion see Turner etal., ref. 9). Thus the cosmologically interesting values of β are in the range of, say 1/3 to 1/100.

The cosmological kinematics (e.g., age-redshift, angular size-redshift, comoving volume-redshift relationships) of the DPC are significantly different than that of a pure matter model, pure radiation model, or a model with a significant cosmological constant. In principle, cosmological observations could differentiate between models, and perhaps even rule out such models.

The simultaneous decay approximation made by Turner etal.[9] to treat the DPC is found to provide a reasonable description of many aspects of the DPC, e.g. 10-20% or better accuracy in determining Ω_R, $H_o t_o$, and $H_o d_o(\infty)$. However, it does not provide a reliable description of the evolution of linear density perturbations. In this approximation it was assumed that growth would shut off at $t \simeq \Gamma^{-1}$ and not resume until the Universe again became matter-dominated. In fact $\delta\rho/\rho$ continues to grow after the decay epoch, however, at a slackening pace. Since the growth of density perturbations from the initial epoch of matter domination until the decay epoch is identical to that in the usual cosmology, the fact that growth does not cease for $t \geq \Gamma^{-1}$ implies that there is additional growth of linear density perturbations in the DPC (over that in the usual $k = 0$ model).

As might be expected things happen more smoothly when the decays are not assumed to be simultaneous: growth of density perturbations gradually slows to a standstill, the enormous mass density in NR Xs is slowly converted into R particles, _etc._ This gradual change from a matter-dominated to a radiation-dominated may improve the viability of the scenario by easing some of the potential difficulties with the DPC (including the 'puffing-up' of structures due to the liberation of Xs into R particles, the sudden halting of the growth of density perturbations).

The motivation for the DPC is clearly the Ω-problem. As was discussed in the Introduction 'the Ω-problem' is basically a timescale problem (although there are also difficulties with forming structure in a lo-Ω model); the DPC does not resolve this problem in a totally satisfactory way, since it solves it by introducing a new timescale -- the lifetime of the X particle. However, if nothing else, it is an interesting new cosmological model.

This research was carried out and written up at the Aspen Center for Physics during the 1984 Summer Program. I wish to acknowledge useful discussions with and comments from R. Kron, D. Seckel, and G. Steigman. This research was supported in part by the DOE (at Chicago and Fermilab), the NSF (at Aspen), the NASA (at Fermilab and Aspen), and an Alfred P. Sloan Fellowship.

References

1. R. H. Dicke and P. J. E. Peebles, in General Relativity: An Einstein Centenary Survey, eds. S. Hawking and W. Israel (Cambridge Univ. Press: Cambridge, 1979).
2. A. H. Guth, Phys. Rev. D23, 347 (1981).
3. A. Linde, Phys. Lett. 108B, 389 (1982).
4. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
5. For a recent review of new inflation see, e.g., M. S. Turner, in Proceedings of the 4th Workshop on Grand Unification, eds. H. Weldon, P. Langacker, and P. Steinhardt (Birkhäuser: Boston, 1983); P. J. Steinhardt and M. S. Turner, Phys. Rev. D29, 2162 (1984).
6. See, e.g., S. Faber and J. Gallagher, Annu. Rev. Astron. Astrophys. 17, 135 (1979); W. Press and M. Davis, Astrophys. J. 259, 449 (1982); M. Davis and P. J. E. Peebles, Annu. Rev. Astron. Astrophys. 21, 109 (1983); P. J. E. Peebles, in Physical Cosmology, eds. R. Balian, J. Audouze, and D. N. Schramm (North-Holland: Amsterdam, 1980); G. Tammann, A. Sandage, and A. Yahil, ibid.; Formation and Evolution of Galaxies, eds. J. Audouze and J. Tran Thanh Van (Reidel: Dordrecht, 1983).
7. J. M. Bardeen, in preparation (1984).
8. N. Kaiser, ITP-UCSB preprint (1984).
9. M. S. Turner, G. Steigman, and L. M. Krauss, Phys. Rev. Lett. 52, 2090 (1984); this possibility has also been discussed by G. Gelmini, D. N. Schramm, and J. Valle, in preparation (1984);

M. Fukugita and T. Yanagida, RIFP preprint 561 (1984); and K. Olive, D. Seckel, and E. T. Vishniac, in preparation (1984). The possibility of a Universe which is presently radiation-dominated seems to have first been suggested by D. Dicus, E. W. Kolb, and V. L. Teplitz, Astrophys. J. 223, 327 (1978); also see, E. W. Kolb, in Proc. ν-Mass Miniconference, eds. V. Barger and D. Cline (Telemark, WI, 1980). The role of unstable particles in galaxy formation was first discussed by M. Davis, M. Lecar, C. Pyror, and E. Witten, Astrophys. J. 250, 423 (1981); in this paper they also briefly mentioned the possibility that high velocity, NR particles produced by the decays of a more massive particle species could today provide a smooth compound of mass density. The role of unstable neutrinos has been explored in somewhat more detail by P. Hut, and S. White, unpublished (1983). The difficulties associated with having a hot, unclustered component of mass density have been discussed by S. Bludman and Y. Hoffman, Phys. Rev. Lett. 52, 2087 (1984).

10. P. J. E. Peebles, Astrophys. J., in press (1984); P. H. Frampton and G. Lipton, unpublished report (1983); M. S. Turner, G. Steigman, and L. M. Krauss, Phys. Rev. Lett. 52, 2090 (1984). The possibility of a non-zero cosmological constant has been discussed previously in other contexts; see, for example, J. Gunn and B. Tinsley, Nature (London) 257, 454 (1975) and references therein.

11. J. N. Fry, Univ. of Chicago preprint (1984).

12. J. Yang, M. S. Turner, G. Steigman, D. N. Schramm, and K. Olive, Astrophys. J. 281, 493 (1984).

13. During a radiation-dominated epoch the growth of linear density perturbations is suppressed; from the time $\rho_R >> \rho_{NR}$ until $\rho_R \simeq \rho_{NR}$ the growth in $\delta\rho_{NR}/\rho_{NR}$ is about a factor of 2; see, P. Meszaros, Astron. Astrophys. 37, 225 (1974).

14. P. J. E. Peebles, The Large Scale Structure of the Universe (Princeton Univ. Press: Princeton, 1980).

15. M. Davis etal., see ref. 9; J. Bond, G. Efstathiou, and J. Silk, Phys. Rev. Lett. 45, 1980 (1980); J. Bond and A. Szalay, Astrophys. J. 274, 443 (1983).

16. R. Scherrer and M. S. Turner, Fermilab/Univ. of Chicago preprint (1984).

17. Given that the range of values for H_o reported in the literature (40-110 kms^{-1} Mpc^{-1}) is some 10 times the typical uncertainty quoted in a given determination, it is clear that at present systematic uncertainties preclude a definitive determination of H_o. Some recent determinations include: R. Buta and G. DeVaucouleurs, Astrophys. J. 266, 1 (1983); M. Aaronson and J. Mould, ibid. 265, 1 (1983); A. Sandage and G. Tammann, ibid. 256, 339 (1982); D. Branch etal., ibid. 270, 123 (1983); A. Sandage and G. Tammann, Nature, in press (1984). At present the determinations of the age suggest a very firm lower bound of 10 Byr, with a 'best value' for the age of 15 '± 3' Byr (the '± 3' does not account for systematic uncertainties, e.g., due to the distance scale, or theoretical modeling which must be done). Recent discussions of the determination of the age of the Universe include, I. Iben, Annu. Rev. Astron. Astrophys. 12, 215 (1974); A. Sandage, Astrophys. J. 252, 553 (1982); D. N. Schramm, in Highlights of

Astronomy as Presented to the 18th General Assembly of the IAU, vol. 6, ed. R. West (Reidel, Dordrecht, 1983); Astrophysical Parameters for Globular Clusters, IAU Colloquium 68, eds. A. G. Davis Philip and D. S. Hayes (L. Davis Press, Schenectady, NY, 1981).

18. J. Bond and G. Efstathiou, Astrophys. J. in press (1984).

19. N. Vittorio and J. Silk, Univ. of California at Berkeley preprint (1984).

20. F. Wilczek, Phys. Rev. Lett. 49, 1549 (1982); D. Reiss, Phys. Lett. 115B, 217 (1982); G. Gelmini and J. Valle, Phys. Lett. B, in press (1984); Y. Chikashige, R. Mohapatra, and R. Peccei, Phys. Rev. Lett. 45, 1926 (1980); G. Gelmini, S. Nussinov, and T. Yanagida, Nucl. Phys. B120, 31 (1983).

21. N. Vittorio and J. Silk, in preparation (1984).

Appendix - One of the X's Decay Products is not Always Relativistic

Here we consider one example of the general case where the decay products of the X are not always relativistic. The scenario we have in mind is one where the X decays to a + b, where the mass of the a(m_a) is large enough so that by the present epoch the decay produced a's are nonrelativistic, but the mass of the b is sufficiently small so that the decay produced b's are ultrarelativistic. Before we go on to discuss this case, let's briefly motivate it.

Solving the 'Ω-problem' only requires a 'hot' component, not necessarily a relativistic component ($v >> 10^{-3}c$ will do just fine), and so this case is also a logical possibility. In addition, allowing the smooth component to be hot but not relativistic can help to alleviate the age problem associated with the DPC. The idea then is to have (at the present epoch): $10^{-3}c << \langle v_a \rangle << c$, $(\Omega_a + \Omega_R)/\Omega_{NR}$ large (say 3 or 4), and Ω_R small (say less than 0.1). [Note, in this case the R component is the relativistic b's.] The attraction of this scenario is the possibility of circumventing the age problem (i.e., pushing $H_o t_o$ closer to 2/3), while still solving the Ω problem. There are some difficulties with this scenario however.

In addition to the fact that such a scenario requires a coincidence of 2 masses and a lifetime, there is another difficulty. Not only are there decay-produced a's present, but there will, in general, also be a cold primordial component of a's. If we denote the initial relic abundances of the X and a (relative to photons) by r_X and r_a respectively, then after the decay-produced a's become NR we will have:

$$\Omega_{ha}/\Omega_{ca} = r_X/r_a,$$
$$\Omega_{NR} = \Omega_B + \Omega_{ca} ,$$
$$\Omega_{ha}/\Omega_{NR} \le r_X/r_a ,$$

where 'ca' and ha' refer to the cold and hot components of a's respectively, and it has been assumed that $\Omega_R << 1$. Therein lies the rub -- unless $r_X/r_a > 3$ or 4, the Ω - problem will not have been solved. If both X and a are neutrinos, say, then $r_X/r_a = 1$, implying $\Omega_{ha}/\Omega_{NR} < 1$. Of course it is possible to have $r_X/r_a >> 1$ (e.g., if the a's abundance is suppressed because it decoupled very early, T >> 1 MeV, or if the X has many more degrees of freedom).

So much for motivation. Introduce another parameter α,

$$\alpha \equiv m_a/m_X . \tag{A1}$$

In terms of α the initial momentum of a decay produced a or b is:

$$p_o = m_X(1-\alpha^2)/2 . \tag{A2}$$

Let ρ_R denote the energy density of the decay produced b's and ρ_a the energy density of the decay produced a's. The equation governing ρ_R is very similar to the one we derived earlier in Sec. III,

$$\dot{\rho}_R = -4H\rho_R + (1-\alpha^2)\Gamma\rho_X/2 , \tag{A3}$$

the only difference being the factor of $(1-\alpha^2)/2$ which accounts for the fact that only that fraction of the decay energy goes into relativistic b particles. Using Eqn. (16) for ρ_X, ρ_R can be expressed as before:

$$\rho_R = (a/a_i)^{-4}\rho_{Xi}[(1-\alpha^2)/2]\int_o^t a(t')/a_i \, e^{-\Gamma t'} d\Gamma t' \, . \tag{A4}$$

The evolution of ρ_a is quite a bit more complicated. First consider the evolution of the number density of a's,

$$\dot{n}_a = -3Hn_a + \Gamma n_X \, , \tag{A5}$$

where n_X is the number density of Xs ($\equiv \rho_X/m_X$) and

$$n_X = (a/a_i)^{-3} n_{Xi} \, e^{-\Gamma t}.$$

Using this expression for n_X, Eqn. (A5) is straightforward to integrate:

$$n_a = (a/a_i)^{-3} \, n_{Xi} \int_o^t e^{-\Gamma t'} \, d\Gamma t' \, , \tag{A6}$$

where the contribution to $n_a(t)$ from X's decaying in the time interval $t' \to t' + dt'$ is clearly:

$$(a/a_i)^{-3} n_{Xi} \, e^{-\Gamma t'} \, d\Gamma t'.$$

The energy density in a's is just

$$\rho_a(t) = (a/a_i)^{-3} n_{Xi} \int_o^t E(t,t') e^{-\Gamma t'} d\Gamma t' \, , \tag{A7}$$

where $E(t,t')$ is the energy of an a which was produced by an X decay at time t', at time t. This energy is simple to compute:

$$E(t,t')^2 = m_a^2 + p_o^2[a(t')/a(t)]^2. \tag{A8}$$

Bringing everything together we have for ρ_a

$$\rho_a = \alpha(a/a_i)^{-3}\rho_{Xi}\int_o^t[p(t,t')^2/m_a^2+1]^{1/2}e^{-\Gamma t'}d\Gamma t' \ , \tag{A9}$$

where $p(t,t') = p_o\ a(t')/a(t)$.

Since we have arranged things so that the a's are always hot and cannot clump on interesting scales, $\delta_a = 0$ and the equation for the evolution of δ_{NR} is unchanged, cf. Eqn. (20). Employing the dimensionless variables from Sec. II and introducing the additional dimensionless variable

$$f_a = \rho_a/\rho_{Xi} \ ,$$

the equations governing this DPC can be written as

$$a'/a = (f_X + f_R + f_a + f_{NR})^{1/2}x_H^{-1} \ , \tag{A10a}$$

$$f_R = a^{-4}[(1-\alpha^2)/2]\int_o^x a(x')e^{-x'}\ dx' \ , \tag{A10b}$$

$$f_a = \alpha a^{-3}\int_o^x[p(x,x')^2/m_a^2 + 1]^{1/2}e^{-x'}dx' \ , \tag{A10c}$$

$$\delta''_{NR} + 2(a'/a)\delta'_{NR} - 3/2(a'/a)^2\delta = 0 \ , \tag{A10d}$$

$$f_X = a^{-3}e^{-x} \ , \tag{A10e}$$

$$f_{NR} = \beta a^{-3} \ , \tag{A10f}$$

where $\delta \equiv \delta_{NR}\ (f_X + f_{NR})/(f_X + f_R + f_a + f_{NR})$ and $p(x,x') = p_o\ a(x')/a(x)$. It should be clear that the solutions to this set of equations (supplemented by $\delta_{NR}(x_i)$ and $\delta'_{NR}(x_i)$) form a two parameter (α and β) family of solutions.

In these models $\rho_a/\rho_{NR} \to \alpha/\beta$ as the a's become NR; as before ρ_R/ρ_{NR} decreases as a^{-1} (for $x \gg 1$). Since Ω_a/Ω_{NR} is fixed by α/β, the present epoch is specified by choosing the value of Ω_R today. Assuming that at the present epoch the a's are NR so that $\Omega_a/\Omega_{NR} = \alpha/\beta$, the ratio of 'hot material' to clustered material is:

$$\begin{aligned}\Omega_{HOT}/\Omega_{NR} &= (\Omega_R + \Omega_a)/\Omega_{NR},\\ &= \alpha/\beta + \Omega_R/\Omega_{NR}\ ,\\ &= \alpha/\beta + \Omega_R(1+\alpha/\beta)/(1-\Omega_R)\ .\end{aligned}$$

Ω_R can be calculated in terms of $\bar{m}_{100}$, $\tau(\equiv\Gamma^{-1})$, and α as in Sec. II, with the result that

$$\Omega_R h^2/\theta^4 = 0.249[(1-\alpha^2)/2]\bar{m}_{100}^{\ 4/3}\tau_9^{2/3}\ .$$

[The only difference between this and the analogous expression, Eqn. (25a), in Sec. II is the factor $(1-\alpha^2)/2$, which is just the fraction of the X's rest mass which goes into relativistic b particles.]

What are the interesting values of α and β. Since $\rho_a/\rho_{NR} \to \alpha/\beta$ at late times we will want to have $\alpha/\beta \simeq 3$-4 to solve the 'Ω-problem'. Next, we want to arrange to have the contribution of the R particles small (say $\rho_R/\rho_{NR} \lesssim 1$), so that the model is close to being a matter-dominated cosmology. At the decay epoch $\rho_R/\rho_{NR} \simeq \beta^{-1}/2$; in order that this ratio be ≤ 1 today we need to have $(1+z_D) \gtrsim \beta^{-1}/2$. The condition that $(1+z_D) \gtrsim \beta^{-1}/2$ also insures that the a's are NR today ($v_a/c \simeq \alpha^{-1}(1+z_D)^{-1}/2 \lesssim \beta/\alpha \simeq 1/3 - 1/4$). To summarize, solving the 'Ω-problem' requires $\alpha/\beta \simeq 3$-4. Specifying Ω_R(or Ω_R/Ω_{NR}) defines the present epoch for a given model (i.e., α and β). The decay epoch for that model is

$$(1+z_D) \simeq (\beta^{-1}/2)(\Omega_{NR}/\Omega_R).$$

Until the decay-produced a's become non-relativistic, $a/a(x=1) \simeq 1/2\alpha \simeq (1/6 - 1/8)\beta^{-1}$, this scenario is essentially identical to the DPC discussed in Sec. II (which in this parameterization corresponds to $\alpha=0$). Thus the evolution of a, ρ_R, ρ_{NR}, and ρ_X/ρ_{NR} shown in Fig. 2 is applicable here (with ρ_R there to be identified with $\rho_a + \rho_R$). The same is true for the evolution of δ_{NR}. However as the a's become NR there is a new twist. Instead of the growth of linear perturbations asymptotically returning to $\delta_{NR} \propto a$ as Ω_R becomes small, δ_{NR} asymptotically approaches $\delta_{NR} \propto a^m$, $m < 1$. The reason for this behaviour is easy to understand. As the a's become NR and $\Omega_R \to 0$, the scale factor $a \propto x^{2/3}$; however, by assumption the a's remain 'too hot' to cluster and thus remain smooth. Therefore asymptotically $\delta \to (\beta/\alpha)\delta_{NR}$. With $\delta = (\beta/\alpha)\delta_{NR}$ and $a \propto x^{2/3}$ it is straightforward to show that the growing mode perturbation

$$\delta_{NR} \propto a^m \text{ or } x^{2m/3},$$

$$m = 1/4(1+24\beta/\alpha)^{1/2} - 1/4 \quad .$$

The reason for the impeded growth is simple; the NR particles must try to cluster in the presence of a smooth background component. [This suppressed growth is known as the Meszaros effect.[13]]. The evolution of δ_{NR} is shown in Fig. A1 for $\alpha/\beta = 4$, and $\beta^{-1} = 20$. At late times $\delta_{NR} \propto a^{0.41}$ as predicted.

The age-redshift relation ($H_o t$ vs. $1+z$) for this scenario is shown in Fig. A2. Table A1 is a compilation of the values of $H_o t_o$ for various models. The age problem has been improved considerably, with most of the models having $H_o t_o$ greater than 0.6, and one as high as 0.632.

The distance-redshift relation ($H_o d_o(z)$ vs. $1+z$) and comoving volume factor $f(z)$ vs. $(1+z)$ are shown in Figs. 6 and A3 respectively for a model with $\alpha/\beta = 3/20$, $\beta^{-1} = 20$, and $\Omega_R = 0.1$. Note, for this canonical model:

$$\Omega_a = .675 \quad ,$$

$$\Omega_{NR} = .225 \quad ,$$

$$\Omega_R = .100 \quad ,$$

$$\Omega_{HOT}/\Omega_{NR} \simeq 3.44 \quad ,$$

$$(1+z_D) \simeq 23 \quad .$$

Figure Captions

Figure 1 - Lines of constant $\Omega_R h^2/\theta^4$ and constant $1+z_D$ in the $\tilde{m}[\equiv m_X(r/(3/11))] - \tau$ plane. For reference β is also indicated on the right hand scale. Note that $\Omega_R h^2/\theta^4 = 0.249\tilde{m}_{100}^{4/3}\tau_9^{2/3}$ and we have used $\beta^{-1} \simeq (\Omega_R/\Omega_{NR})(1+z_D)$ to calculate $1+z_D$.

Figure 2 - The evolution of ρ_X/ρ_{NR} (curves marked 'X'), ρ_R/ρ_{NR} (curves marked 'R'), and a(x) for β = 1/10 and 1/100. The dotted (broken) curves show the evolution of ρ_R/ρ_{NR} (ρ_X/ρ_{NR}) in the SDA. All three curves are universal so long as $\rho_X + \rho_R >> \rho_{NR}$; in this limit a(x) is independent of β, while ρ_X/ρ_{NR} and ρ_R/ρ_{NR} scale as β^{-1}.

Figure 3 - The evolution of $\delta_{NR}/a \equiv (\delta\rho_{NR}/\rho_{NR})/a$ as a function of a for β^{-1} = 3, 10, 30, 80, 300. The broken curves show the evolution in the approximation that linear perturbations grow like a^n (n = 1, matter-dominated; n = 0, radiation-dominated). The different symbols denote the epochs when Ω_R/Ω_{NR} takes on the values indicated. The decay epoch (x = 1, a $\simeq$ 27) is indicated by the arrow.

Figure 4 - The evolution of Ht as a function of x = Γt for β^{-1} = 3, 10, 30, 80, 300. The different symbols denote the epochs when Ω_R/Ω_{NR} takes on the values indicated. The limiting values of Ht (for fixed Ω_R/Ω_{NR}) as $\beta \to 0$ are indicated.

Figure 5 - The age (in Hubble units) vs. redshift relationship for a pure matter model, β = 1/10, 1/30, 1/80 (all with Ω_R/Ω_{NR} = 3 at the present epoch), and a pure radiation model. The curve labeled $\Lambda \neq 0$

corresponds to a $k = 0$ cosmology with $\Lambda/8\pi G = 3\rho_{NR}$ (at the present epoch), that is: $(H/H_o) = [0.75 + 0.25\,(a/a_o)^{-3}]^{1/2}$.

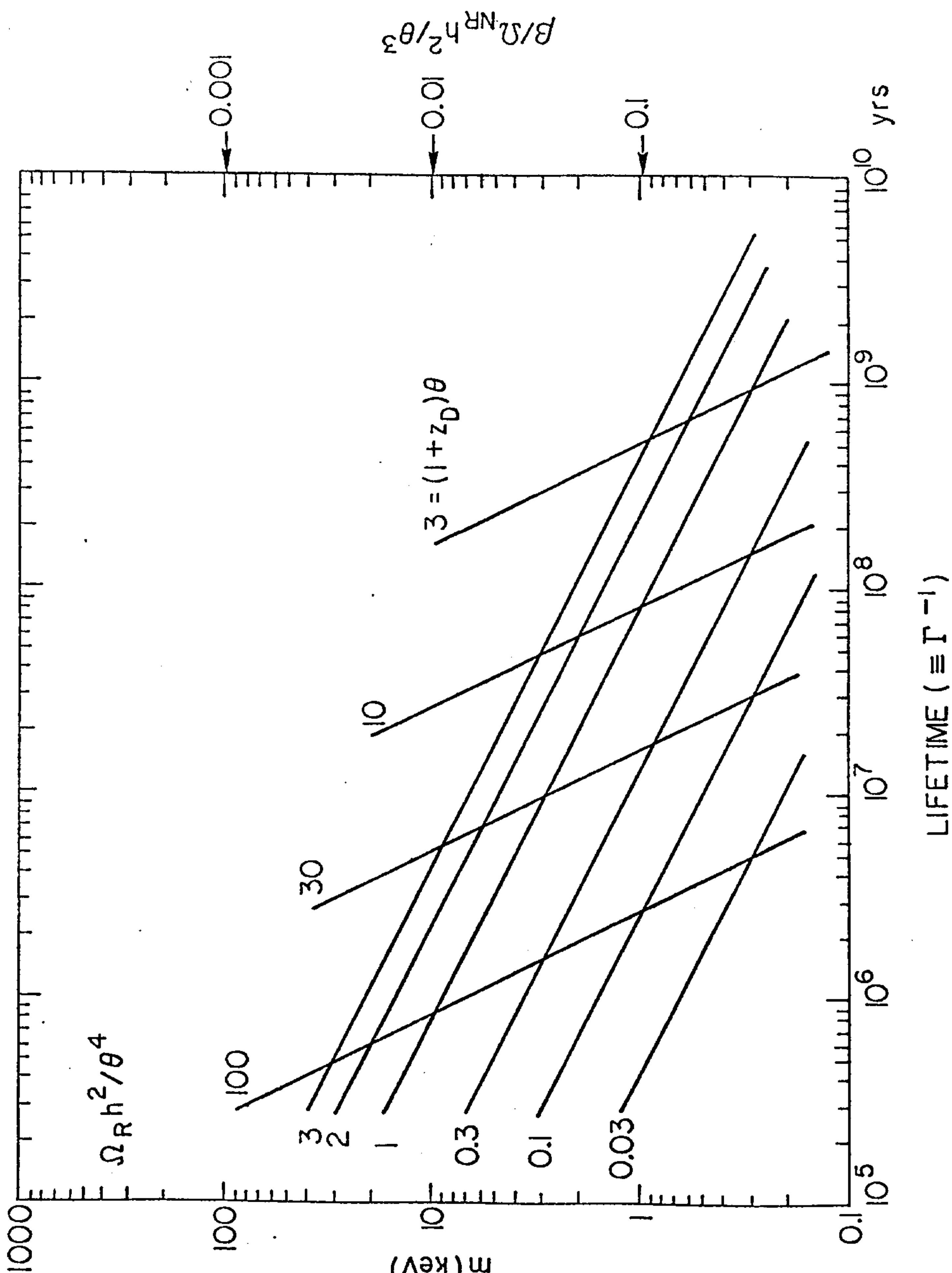
Figure 6 - The present proper distance to an object at redshift z (in Hubble units) vs. redshift for the $\Lambda \neq 0$ model described in Fig. 4, a pure matter model, $\beta = 1/10$ ($\Omega_R/\Omega_{NR} = 3$ at the present epoch), and a pure radiation model. The arrows on the left axis indicate $H_o d_o(z=\infty)$. The curve labeled '3,20' corresponds to a model with $\alpha = 3/20$, $\beta = 1/20$, and $\Omega_R = 0.1$ (at the present epoch) -- see the appendix for details.

Figure 7 - The weighting factor f(z) for the comoving volume element dV_o ($= f(z)z^2\, dz\, d\Omega$) vs. redshift z. The models indicated are the same as those described in Fig. 4.

Figure A1 - The evolution of δ_{NR}/a vs. a for a model with $\alpha/\beta = 4$, $\beta = 1/20$. Asymptotically δ_{NR}/a is predicted to vary as $a^{-0.59}$; a line with slope -0.59 is shown for comparison.

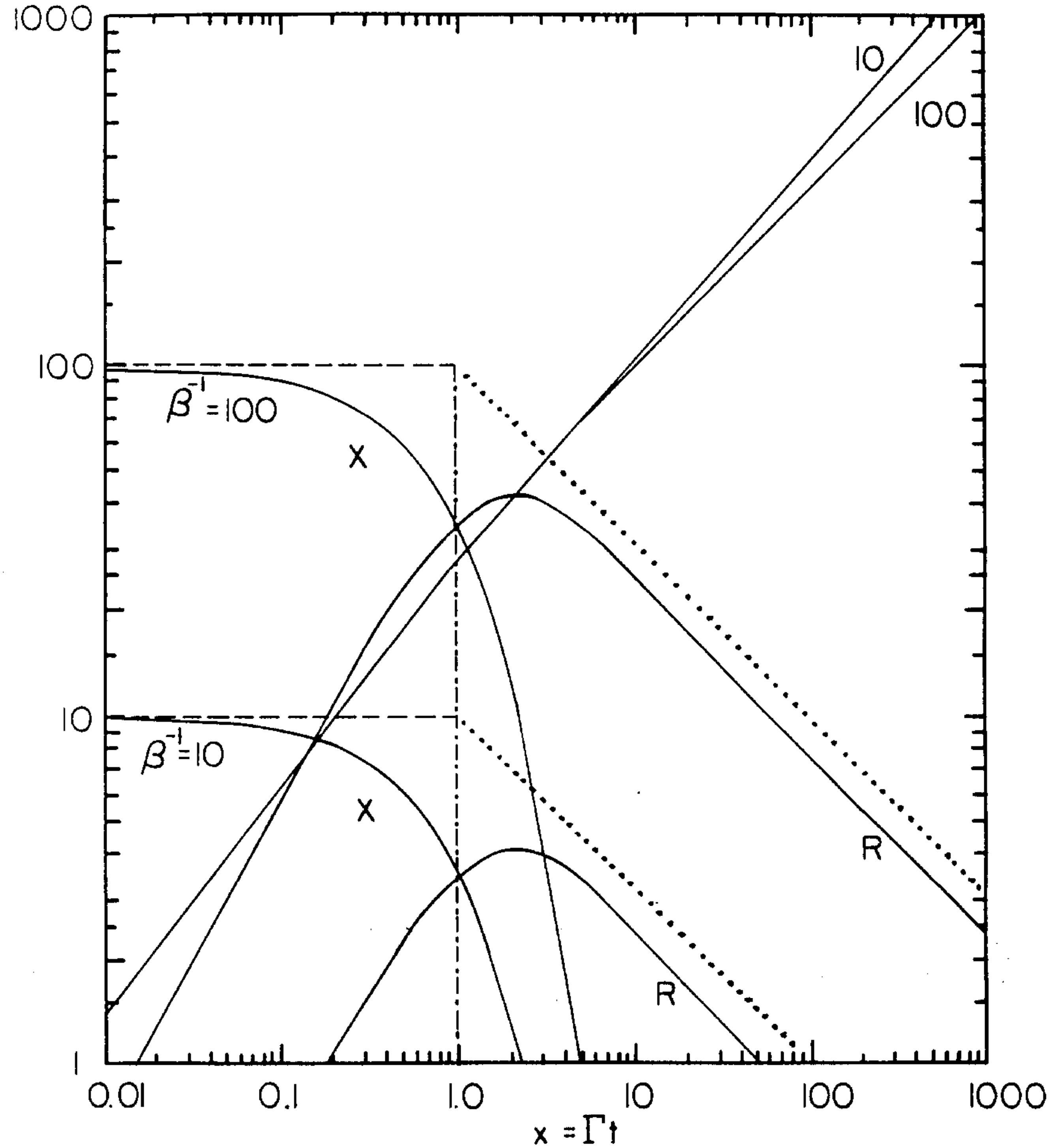
Figure A2 - The age (in Hubble units) vs. redshift relationship for a model with $\alpha/\beta = 3$, $\beta = 1/20$, and $\Omega_R = 0.1$ at the present epoch. For comparison the $\Lambda \neq 0$ described in Fig. 3, a pure matter model, and a pure radiation model are also shown.

Figure A3 - The weighting factor f(z) for the comoving volume element dV_o($= z^2\, f(z)\, dz\, d\Omega$) vs. redshift for a model with $\alpha/\beta = 3$, $\beta = 1/20$, and $\Omega_R = 0.1$ at the present epoch. For comparison the $\Lambda \neq 0$ model described in Fig. 3, a pure matter model, and a pure radiation model are also shown.



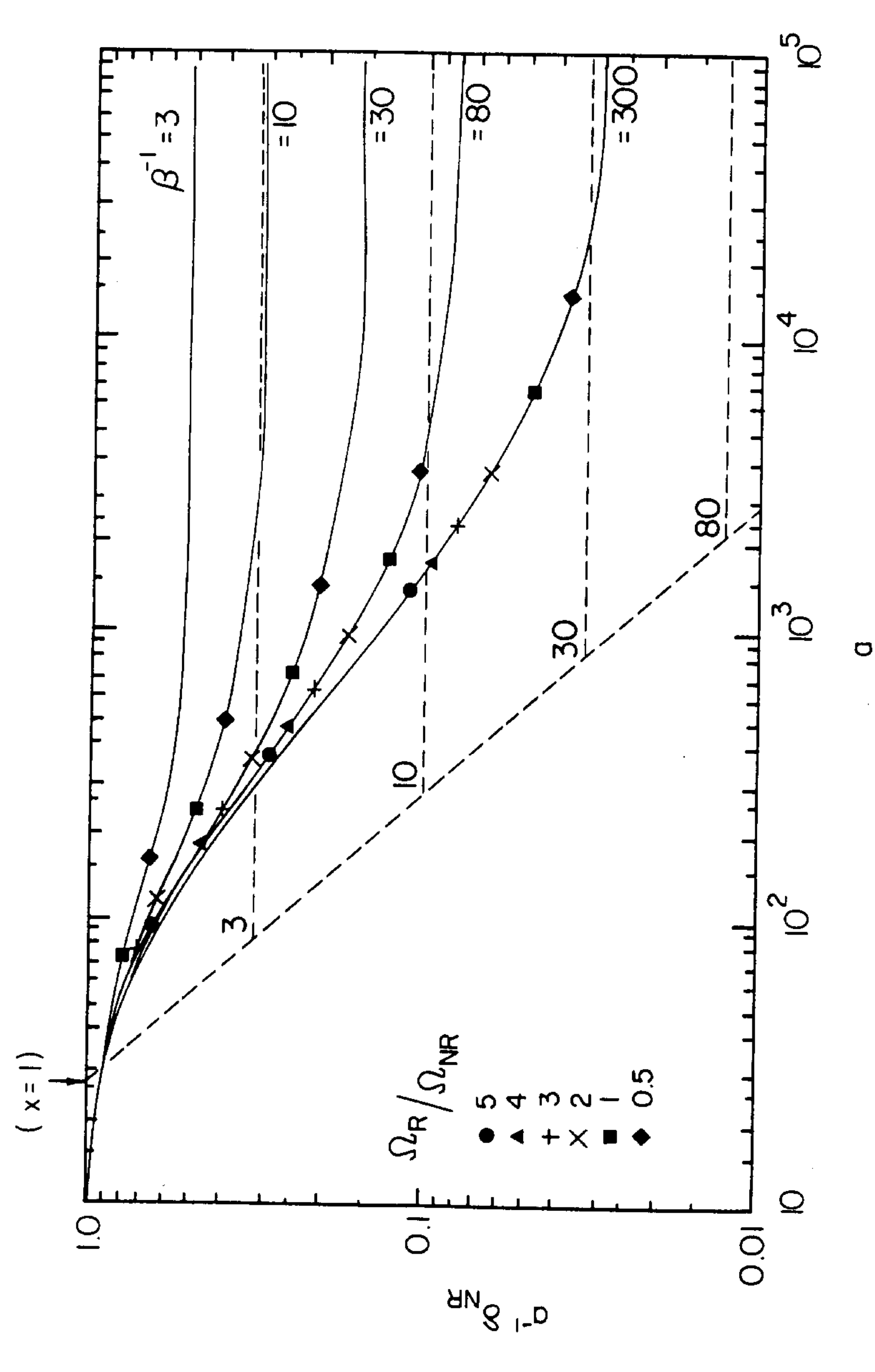
$\tilde{m}$ (keV)
1000
100
10
1
0.1
10^5
10^6
10^7
10^8
10^9
10^{10} yrs
LIFETIME ($\equiv \Gamma^{-1}$)
$\beta/\Omega_{NR} h^2/\theta^3$
0.001
0.01
0.1
$\Omega_R h^2/\theta^4$
3
2
1
0.3
0.1
0.03
100
30
10
$3 = (1+z_D)\theta$

FIG. 1



1000
100
10
1
0.01
0.1
1.0
10
100
1000
$x = \Gamma t$
$\beta^{-1} = 100$
$\beta^{-1} = 10$
X
X
R
R
10
100

FIG. 2



(x = 1)
Ω_R / Ω_{NR}
5
4
3
2
1
0.5
$\beta^{-1} = 3$
= 10
= 30
= 80
= 300
3
10
30
80
1.0
0.1
0.01
10
10^2
10^3
10^4
10^5
a

FIG. 3

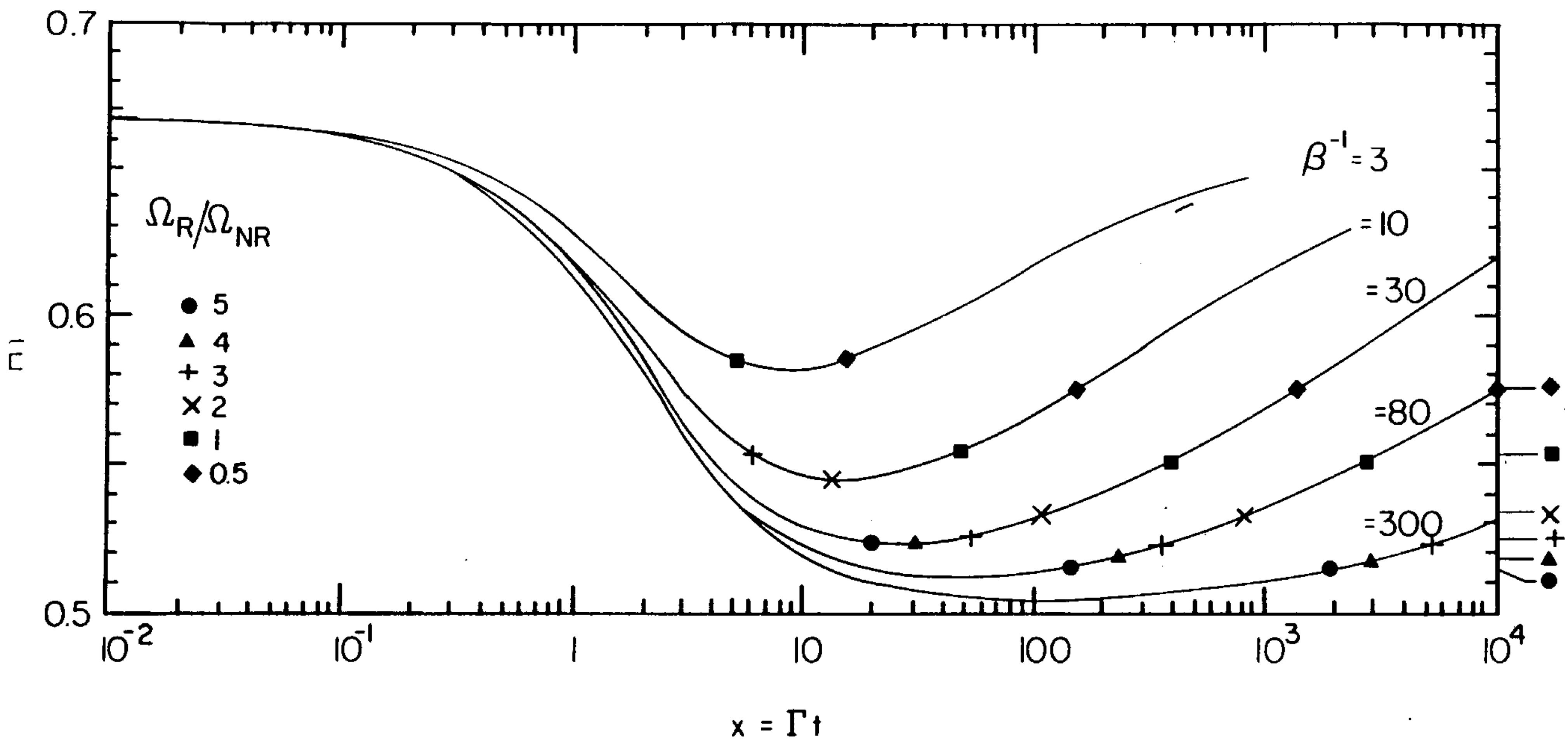
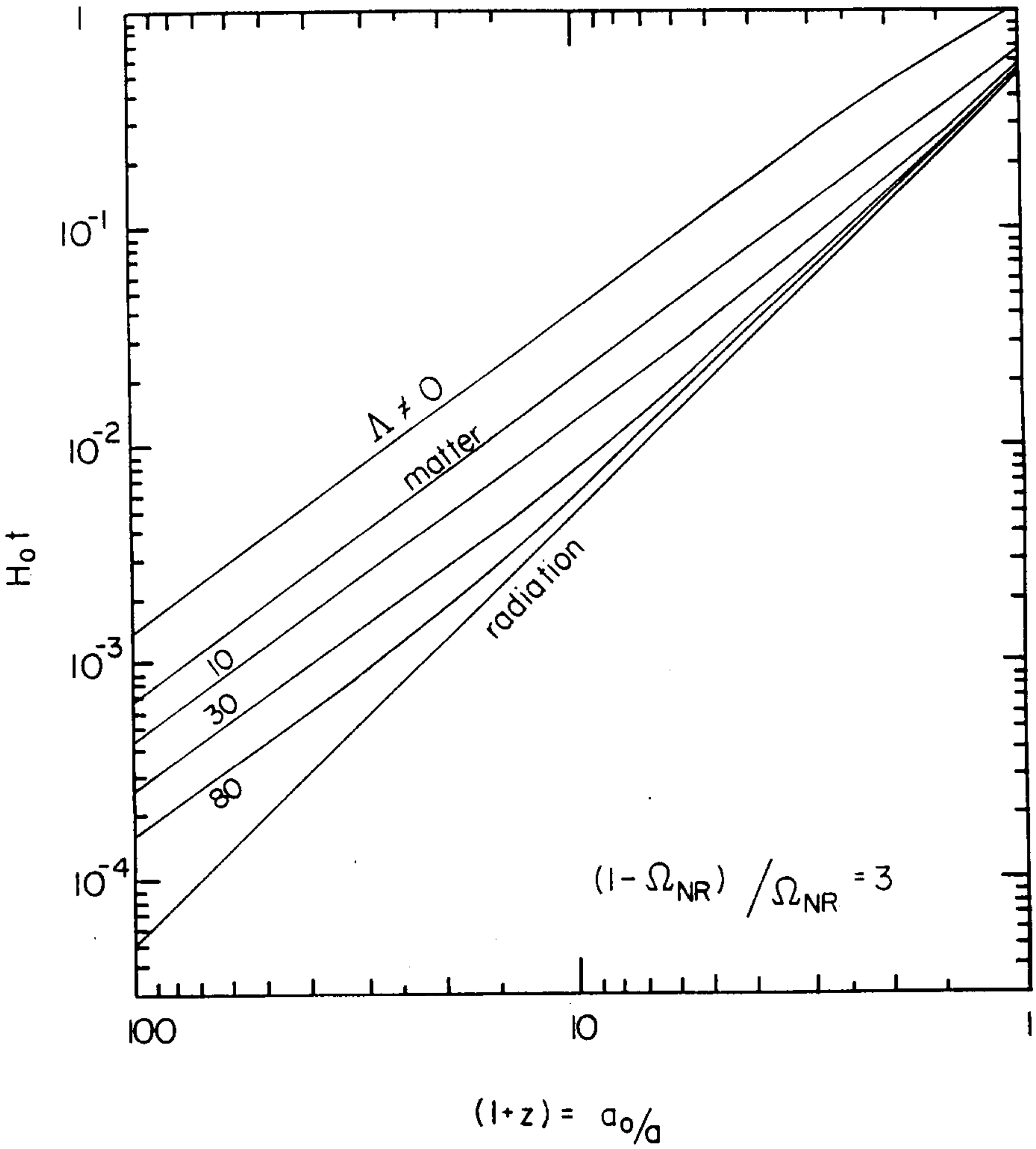

0.7
0.6
0.5
$\bar{m}$
Ω_R/Ω_{NR}
● 5
▲ 4
+ 3
× 2
■ 1
◆ 0.5
$\beta^{-1} = 3$
=10
=30
=80
=300
10^{-2}
10^{-1}
1
10
100
10^{3}
10^{4}
$x = \Gamma t$

FIG. 4



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$H_o t$
$\Lambda \neq 0$
matter
10
30
80
radiation
$(1-\Omega_{NR}) / \Omega_{NR} = 3$
100
10
1
$(1+z) = a_o/a$

FIG. 5

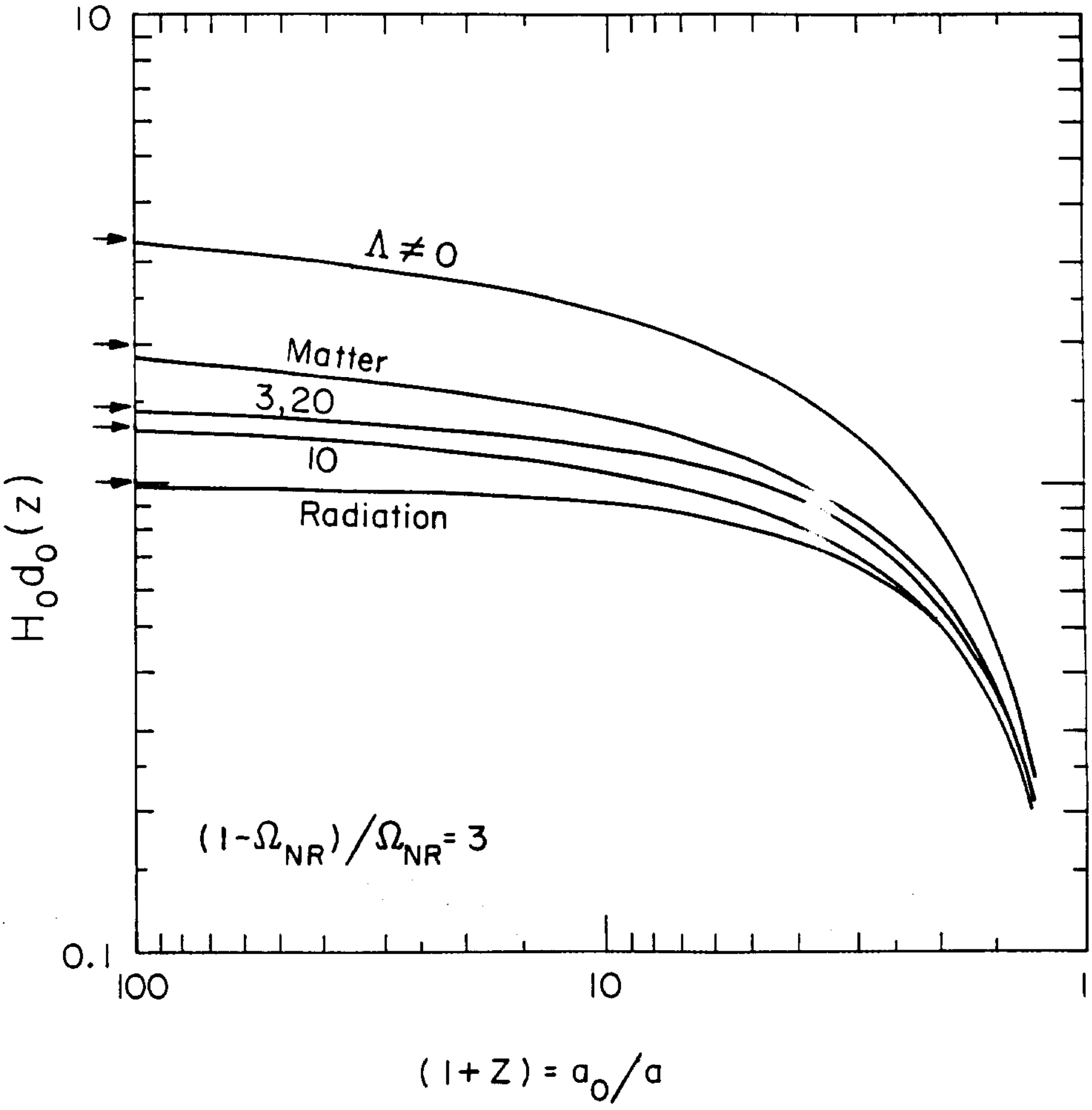
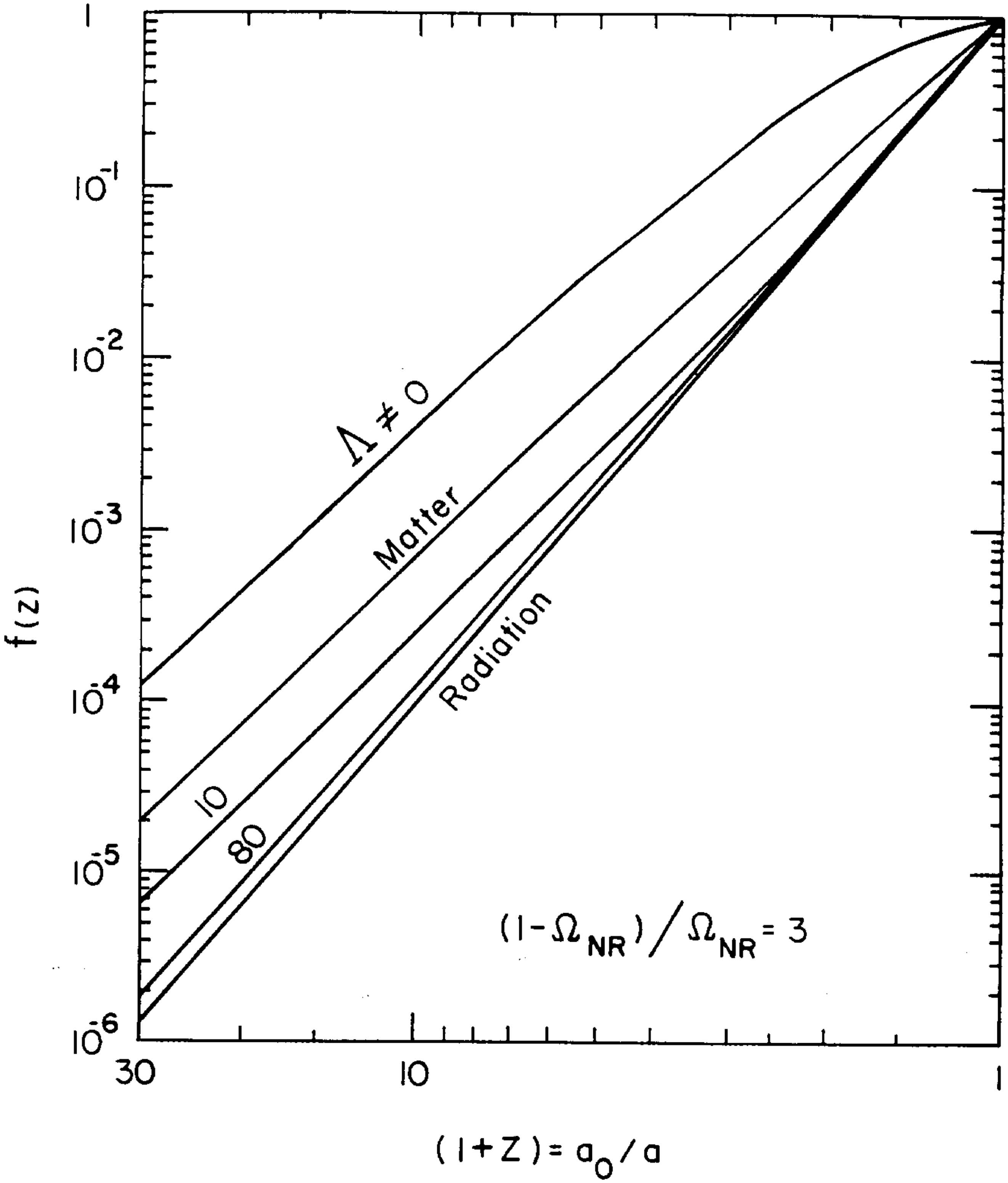
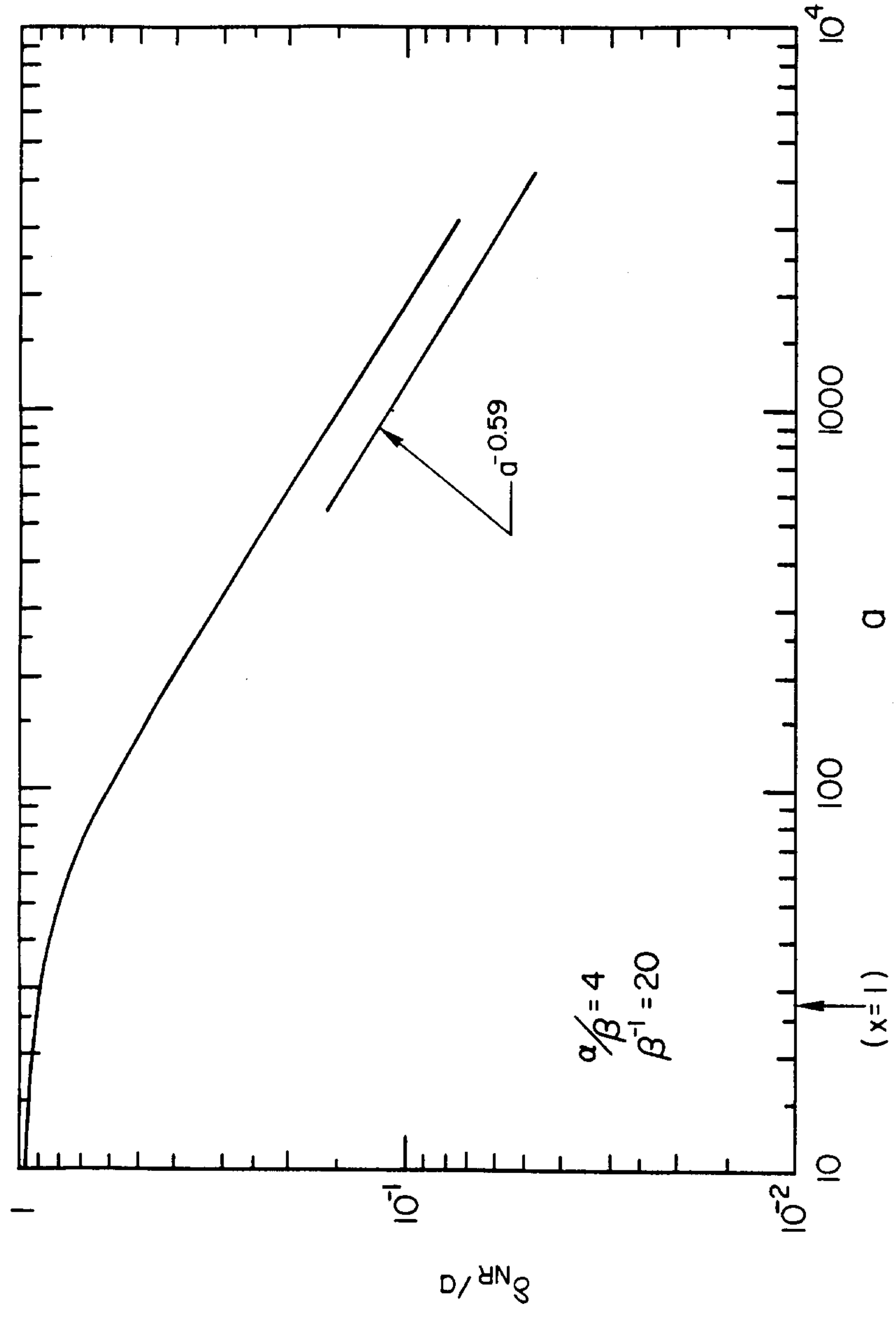

10
0.1
100
10
1
$\Lambda \neq 0$
Matter
3,20
10
Radiation
$H_0 d_0(z)$
$(1-\Omega_{NR})/\Omega_{NR} = 3$
$(1+Z) = a_0/a$

FIG. 6



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
f(z)
$\Lambda \neq 0$
Matter
10
80
Radiation
$(1-\Omega_{NR})/\Omega_{NR} = 3$
30
10
1
$(1+Z) = a_0/a$

FIG. 7



1
10^{-1}
10^{-2}
δ_{NR}/a
10
100
1000
10^4
a
$a^{-0.59}$
$\alpha/\beta = 4$
$\beta^{-1} = 20$
(x=1)

FIG. A1

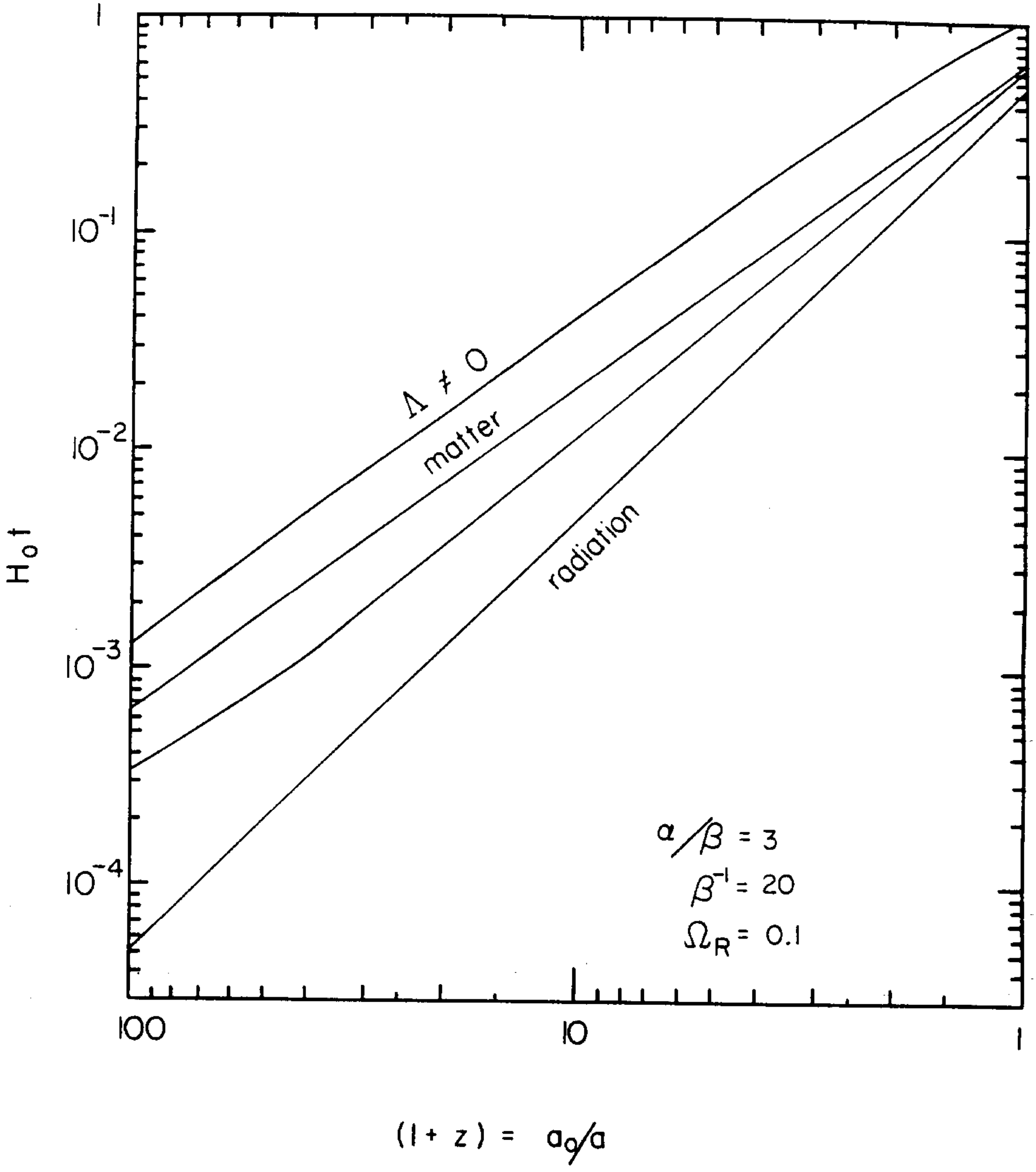
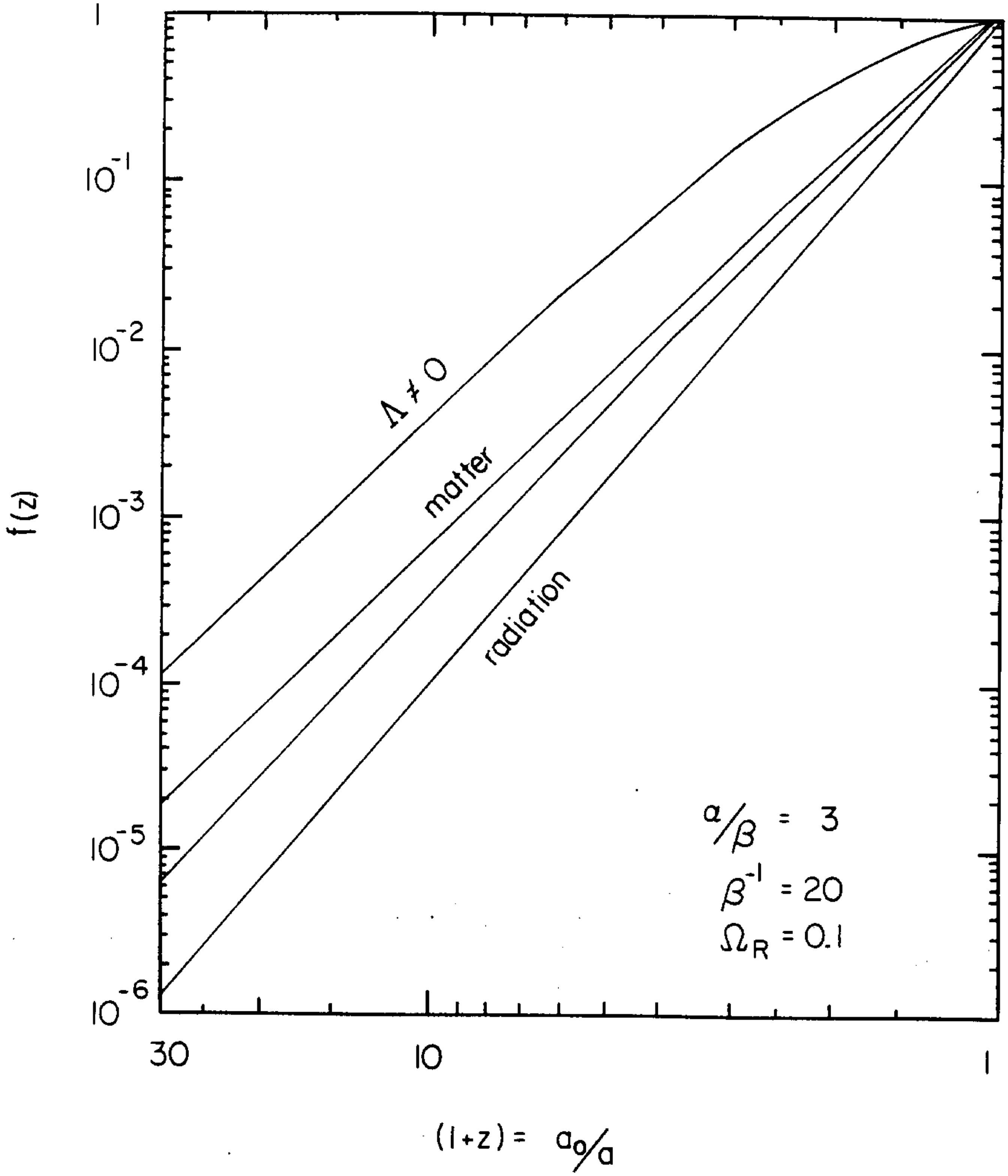

1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$H_0 t$
$\Lambda \neq 0$
matter
radiation
$\alpha/\beta = 3$
$\beta^{-1} = 20$
$\Omega_R = 0.1$
100
10
1
$(1+z) = a_0/a$

FIG. A2



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
f(z)
$\Lambda \neq 0$
matter
radiation
$\alpha/\beta = 3$
$\beta^{-1} = 20$
$\Omega_R = 0.1$
30
10
1
$(1+z) = a_0/a$

FIG. A3

TABLE 1 - Parameters for the present epoch: x_o/a_o.

Ω_R/Ω_{NR} =	5	4	3	2	1	0.5
β^{-1} = 7	-	-	-	6.5/85	25/180	80/340
10	-	-	6/80	13/120	45/245	160/490
15	-	7.5/85	14/120	30/180	95/340	350/710
20	9/95	14/120	24/160	50/240	190/490	620/950
30	20/140	30/180	53/230	110/350	400/710	1400/1450
40	33/180	54/230	90/310	190/470	720/950	2500/1900
80	150/380	220/470	370/630	850/950	2800/1850	10^4/3700
300	2000/1400	3000/1750	5500/2400	11000/3600	37000/6800	$1.4\times10^5/1.4\times10^4$

TABLE 2 - Long term 'deficit' in growth of $\delta\rho/\rho$ [prediction in SDA (= β^{-1}) show in parentheses] and growth from x = 1 until the present

β^{-1}	'Long-term Deficit'	Growth from x=1 until today: Ω_R/Ω_{NR} = 4	3
10	3.2(10)	-	2.4
20	4.7(20)	2.9	3.6
30	6.3(30)	3.6	4.0
40	7.3(40)	4.0	4.5
80	12.5(80)	5.1	5.6
300	33(300)	7.0	8.0

TABLE 3 - $H_o t_o$ [numbers in parentheses are those calculated in the sudden decay approximation, cf. Eqn. (27)].

Ω_R/Ω_{NR} =	5	4	3	2	1	0.5
β^{-1} = 7	--(.596)	--(.571)	--(.554)	.557(.547)	.560(.556)	.579(.579)
10	--(.556)	--(.545)	.552(.539)	.545(.539)	.557(.554)	.578(.578)
15	--(.534)	.541(.530)	.536(.530)	.538(.535)	.554(.553)	.578(.578)
20	.535(.526)	.531(.525)	.530(.527)	.536(.534)	.553(.553)	.577(.577)
30	.524(.520)	.524(.521)	.526(.525)	.534(.533)	.553(.533)	.577(.577)
40	.520(.518)	.521(.521)	.525(.524)	.533(.532)	.552(.552)	.577(.577)
80	.516(.516)	.519(.519)	.524(.523)	.532(.532)	.552(.552)	.577(.577)
300	.515(.515)	.518(.518)	.523(.523)	.532(.532)	.552(.552)	.577(.577)

TABLE 4 - $H_o d_o(z=\infty)$ (numbers in parentheses are those calculated in the sudden decay approximation).

Ω_R/Ω_{NR} =	5	4	3	2	1
β^{-1} = 7	--(1.75)	--(1.63)	--(1.52)	1.31(1.42)	1.28(1.35)
10	--(1.56)	--(1.47)	1.29(1.39)	1.25(1.33)	1.25(1.30)
30	1.16(1.22)	1.15(1.20)	1.14(1.18)	1.15(1.18)	1.19(1.22)
80	1.08(1.11)	1.08(1.11)	1.09(1.11)	1.11(1.13)	1.17(1.19)
300	1.05(1.06)	1.05(1.07)	1.07(1.08)	1.10(1.11)	1.16(1.18)

TABLE A1 - $H_o t_o$

	$\alpha/\beta=3$		$\alpha/\beta=4$	
Ω_R =	0.25	0.11	0.25	0.11
β^{-1} = 10	.607	.625	.625	.632
20	.587	.620	.591	.622
30	.584	.619	.586	.620