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Comparison of the canonical hamiltonian and the hamiltonian of Callan and Rubakov for the monopole fermion system

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ABSTRACT

We compare the canonical hamiltonian of the monopole-fermion system, derived by Goldstein and Yamagishi, with the hamiltonian derived by Callan in the bosonized formulation.

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The dynamics of the monopole fermion system has been considered by many authors[1-5]. In particular, Callan[2] has derived the effective hamiltonian for an SU(2) monopole interacting with fermions belonging to the doublet representation of the SU(2) group. The hamiltonian takes a simple form in the bosonized version of the model. For massless fermions, the hamiltonian is the sum of a free hamiltonian for scalar fields, and an interacting part, given by,

$$\int_{r_0}^{\infty} \frac{e^2}{32\pi^2 r^2} \left(\Phi(r)\right)^2 dr$$

$$= \int_{r_0}^{\infty} \frac{e^2}{32\pi^2 r^2} \left[\Phi(r_0)^2 + 2\,\Phi(r_0)\,\left(\Phi(r) - \Phi(r_0)\right) + \left(\Phi(r) - \Phi(r_0)\right)^2 \right] \tag{1}$$

where $e\Phi(r)/2\sqrt{\pi}$ measures the total electric charge of the system inside a sphere of radius r. Here r_0 is the radius of the monopole core, hence $e\Phi(r_0)/2\sqrt{\pi}$ measures the total charge inside the core.

An alternative formulation of the problem, based on the canonical quantization of the dyon fermion system, has been given by Yamagishi[4] and Goldstein[5]. In Yamagishi's notation, the effective interaction hamiltonian is given by,

$$\frac{I\,\dot{\phi}^2}{2} + \frac{e^2}{8\pi} \int_0^{\infty} dr \int_0^{\infty} dr'\; \rho(r)\, G(r,r')\, \rho(r') \tag{2}$$

where the variable $\dot{\phi}$ is associated with the dyonic excitation, and $\rho(r)$ denotes the contribution to the electric charge density (multiplied by $4\pi r^2$) from the fermion fields. Here,

$$G(r, r') = \theta(r - r')\,\bar{J}(r)\,J(r') + \theta(r' - r)\,\bar{J}(r')\,J(r) \tag{3}$$

where J, $\bar{J}$ are solutions of the equation,

$$\frac{d}{dr}\left(r^2 \frac{d}{dr} J(r)\right) - 2\left(K(r)\right)^2 J(r) = 0 \tag{4}$$

satisfying the boundary conditions,

$$J(r) \sim r \quad \text{as } r \to 0, \qquad J(r) = 1 - \frac{Ie^2}{4\pi r} + O(e^{-m_w r}) \quad \text{as } r \to \infty$$

$$\bar{J}(r) \sim r^{-2} \quad \text{as } r \to 0, \quad \bar{J}(r) = 1/r + O(e^{-m_w r}) \quad \text{as } r \to \infty \tag{5}$$

m_W being the scale of breaking of the SU(2) group. K(r) is a function which appears in the expression for the gauge field of a classical 't Hooft-Polyakov monopole. K(r) satisfies,

$$K(r) \to 1 \quad \text{as } r \to 0; \qquad K(r) \sim e^{-m_w r} \quad \text{as } r \to \infty \tag{6}$$

The quantity I in Eq.(2) and (5) is of order m_W^{-1}, and may be expressed as a function of the parameters of the theory. The total charge of the system is given by,

$$Q_{tot} = I\dot{\phi} + \int_0^\infty \rho(r)\,J(r)\,dr \tag{7}$$

The purpose of this report is to show that the hamiltonian given in (2) reduces to (1) in the proper limit. To do this we define the monopole radius to be $r_0 = m_W^{-1}/\varepsilon$, where ε is a small but fixed number. We shall ignore all terms of order $\exp(-1/\varepsilon)$. Using (3) and (5), we may write,

$$G(r, r') = \frac{1}{r} J(r') \qquad \text{for } r \geq r_0, \quad r' \leq r_0 \tag{8}$$

$$G(r, r') = \Theta(r - r') \frac{1}{r} \left(1 - \frac{Ie^2}{4\pi r'}\right) + \Theta(r' - r) \left(1 - \frac{Ie^2}{4\pi r}\right) \frac{1}{r'},$$

$$\text{for} \quad r, r' \geq r_0 \tag{9}$$

Using (7) and (5), we get,

$$I\dot{\phi} = Q_{tot} - \int_0^{r_0} \rho(r)\, J(r)\, dr - \int_{r_0}^{\infty} \left(1 - \frac{Ie^2}{4\pi r}\right) \rho(r)\, dr \tag{10}$$

We define a variable $\Phi(r)$ for $r \geq r_0$, such that,

$$\frac{\Phi(r)}{2\sqrt{\pi}} = Q_{tot} - \int_r^{\infty} \rho(r')\, dr' \tag{11}$$

is the total charge inside a sphere of radius r, since $\int \rho(r')\, dr'$ is the total charge outside a sphere of radius r. Another way to see that this is the total charge inside a sphere of radius r is to compute the radial electric field at r, which comes out to be $e\Phi(r)/2\sqrt{\pi}r^2$.

From (11) we get,

$$\rho(r) = \Phi'(r)/2\sqrt{\pi} \qquad \text{for} \quad r > r_0 \tag{12}$$

and,

$$Q_{tot} = \frac{\Phi(r_0)}{2\sqrt{\pi}} + \int_{r_0}^{\infty} \rho(r)\, dr \tag{13}$$

Using Eqs.(10) and (13) we may eliminate $Q_{tot.}$ and $\dot{\phi}$ from (2), and a straight calculation using Eqs. (8), (9) and (11) shows that (2) may be brought into the form,

$$\frac{1}{2I}\left\{ \frac{\Phi(r_0)}{2\sqrt{\pi}} - \int_0^{r_0} \rho(r)\, J(r)\, dr \right\}^2$$

$$+ \frac{e^2}{8\pi} \int_0^{r_0} dr \int_0^{r_0} dr'\, G(r,r')\, \rho(r)\, \rho(r') + \frac{e^2}{16\pi^2}\, \Phi(r_0) \int_{r_0}^{\infty} \frac{\Phi(r) - \Phi(r_0)}{r^2}\, dr$$

$$+ \frac{e^2}{32\pi^2} \int_{r_0}^{\infty} \frac{\{\Phi(r) - \Phi(r_0)\}^2}{r^2}\, dr \tag{14}$$

Let us now define,

$$f(r) = \int_0^{r_0} G(r,r')\, \rho(r')\, dr' \tag{15}$$

Then, for $r<r_0$

$$\left(r^2 f'(r)\right)' - 2\left(K(r)\right)^2 f(r) = -\rho(r) \tag{16}$$

since $G(r,r')$ satisfies the equation,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} G(r,r')\right) - 2\left(K(r)\right)^2 G(r,r') = -\delta(r-r') \tag{17}$$

Using (15), (16) and (5), and doing an integration by parts carefully treating the boundary term at r_0, we may show that,

$$\frac{e^2}{8\pi}\int_0^{r_0} dr \int_0^{r_0} dr' \; G(r,r')\;\rho(r)\;\rho(r')$$

$$= \frac{e^2}{4\pi}\int_0^{r_0}\left(K(r)\,f(r)\right)^2 dr + \frac{e^2}{8\pi}\int_0^{r_0} r^2\left(f'(r)\right)^2 dr$$

$$+ \frac{e^2}{8\pi r_0}\left(\int_0^{r_0} J(r)\;\rho(r)\,dr\right)^2 \qquad (18)$$

Thus the hamiltonian reduces to,

$$\Big[\frac{1}{2I}\Big\{\frac{\Phi(r_0)}{2\sqrt{\pi}} - \int_0^{r_0}\rho(r)\;J(r)\,dr\Big\}^2 + \frac{e^2}{8\pi r_0}\left(\int_0^{r_0}\rho(r)\,J(r)\,dr\right)^2$$

$$+ \frac{e^2}{4\pi}\int_0^{r_0}\left(K(r)\,f(r)\right)^2 dr + \frac{e^2}{8\pi}\int_0^{r_0} r^2\left(f'(r)\right)^2 dr\Big]$$

$$+ \frac{e^2}{16\pi^2}\,\Phi(r_0)\int_{r_0}^{\infty}\frac{\Phi(r)-\Phi(r_0)}{r^2}\,dr + \frac{e^2}{32\pi^2}\int_{r_0}^{\infty}\frac{\left(\Phi(r)-\Phi(r_0)\right)^2}{r^2}\,dr$$

$$(19)$$

The last two terms of the above hamiltonian are identical to the last two terms of (1). In Ref.(2), however, the fermionic degrees of freedom for $r<r_0$ were ignored, and hence (1) must be regarded as an effective hamiltonian involving the fermionic degrees of freedom for

$r \geq r_0$. In order to compare the two hamiltonians, we must eliminate the fermionic coordinates for $r<r_0$ by using their equations of motion, and obtain the effective hamiltonian involving the fields $\Phi(r)$ for $r \geq r_0$. Looking at the first two terms inside [...] in (19), we see that this contribution is bounded from below by

$$\left(2I + 8\pi r_0/e^2\right)^{-1} \left(\Phi(r_0)\right)^2/4\pi \tag{20}$$

and hence the contribution from the terms inside [...] in (19) may be written as $ae^2\Phi(r_0)^2/(32\pi^2 r_0)$, where a is a constant of order unity. The net effective hamiltonian is then given by,

$$\int_{r_0}^{r} \frac{e^2}{32\pi^2 r^2}\, dr \left[a\, \Phi(r_0)^2 + 2\,\Phi(r_0)\left(\Phi(r) - \Phi(r_0)\right) + \left(\Phi(r) - \Phi(r_0)\right)^2 \right] \tag{21}$$

which is identical to (1), except for the factor a multiplying the first term. The deviation of a from unity is due to the interaction energy inside the core, and does not significantly affect any physical result[6].

One of the major lessons that we learn from this treatment is that for a dyon fermion system we should not interprete the charge $I\dot{\phi}$ to be stored inside the monopole core, since parts of the second and third terms in (1) come from the $I\dot{\phi}^2$ term in (2)[8]. The fact that $I\dot{\phi}$ should not be treated as a localized charge inside the core was also noted in Ref.4.

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These authors, together with those of Ref. 4 and 5 have treated the degree of freedom corresponding to dyonic excitation as a separate dynamical variable. For a complete list of references on monopole fermion dynamics, see H.Panagopoulos and H.Yamagishi, M.I.T. report No. CTP #1159.

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[6] At the first sight the constant a does not seem to affect the dynamics at all, since the equations of motion for the Φ field for $r>r_0$, as well as the boundary condition on Φ at $r=r_0$ is independent of a. However, as was pointed out in Ref.7, the boundary condition $\Phi'(r_0)=0$ is inconsistent with the conservation of the total energy for $a\neq 1$, and we must change the boundary conditions to $\Phi'(r_0)=(a-1)(e^2/16\pi^2 r_0)\Phi(r_0)$ in order to ensure the conservation of energy. It was shown in Ref.7 that this modified boundary condition does not significantly affect the calculation of any condensate, so long as a is positive.

[7] Y. Kazama and A. Sen, FERMILAB-PUB-83/58-THY (revised version), to appear in Nucl. Phys. B.

[8] On the other hand the charge $\hat{Q}=Q_{tot}-\int_0^\infty \rho(r')\, dr' = I\hat{\phi}+\int_c^\infty J(r')\rho(r')\, dr'$ is localized inside the core, since the electric field at a point $r \gg m_W^{-1}$ is given by $e(\hat{Q}+\int_c^r \rho(r')\, dr')/4\pi r^2$. Hence $\hat{Q}$ may be interpreted as the charge associated with the dyonic degree of freedom.