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Perturbative Corrections to Universality and Renormalization Group Behaviour

R. K. Ellis

Fermi National Accelerator Laboratory, Batavia, IL. 60510

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ABSTRACT

The influence of weak coupling corrections on tests of the universality of lattice actions is discussed. The two loop corrections described in this paper were performed using the background field method. An estimate is also provided of the perturbative corrections to asymptotic scaling for the Wilson action.

Monte Carlo simulation of Lattice QCD is carried out at finite, large values of the bare coupling g, but the continuum limit of the theory corresponds to vanishing g. When extracting physical results from lattice calculations it is therefore important to correct for the non-zero value of the coupling constant. In this paper I report on the result of some perturbative calculations, mostly perormed in collaboration with G. Martinelli[1,2], which address this question.

On the lattice any physical quantity (of mass dimension d) is proportional to Λ_L^d. Under a change of the lattice action Λ_L changes but, according to universality, physical quantities must remain the same. So in the continuum limit, the value of a mass m calculated using two different lattice actions is,

$$m = k\Lambda_L = k'\Lambda_L' \tag{1}$$

The measurement of the ratio k'/k provides an estimate of Λ_L/Λ_L'. The ratio of Λ parameters is calculable in weak coupling perturbation theory so that eq.(1) can be used to check universality in the continuum limit. In the limit as the lattice spacing a tends to zero the bare coupling of SU(N) gauge theory varies according to the renormalization group equation,

$$a\,\frac{dg(a)}{da} = -\beta(g) = b_0 g^3(a) + b_1 g^5(a) + b_2 g^7(a) + O\left(g^9(a)\right) \tag{2}$$

The scale parameter Λ_L is fixed by the solution to this equation,

$$\Lambda_L^2 a^2 = \left(1 + \frac{1}{b_0^3}\,(b_1^2 - b_2 b_0)g^2\right) \exp\left(-\frac{1}{b_0 g^2} - \frac{b_1}{b_0^2}\ln(b_0 g^2)\right) \tag{3}$$

The first two coefficients of the beta function $\beta(g)$ are universal and given in pure SU(N) gauge theory by,

$$b_0 = \frac{11}{3}\frac{N}{16\pi^2} \qquad b_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2 \tag{4}$$

The coefficient b_2 is dependent on the regularization scheme and is known only for the dimensionally regularized continuum theory. Since it is unknown for the lattice regulated theory, Λ_L is determined "experimentally" from eq. (3) retaining only the exponential factor. We refer to this simplified form of eq. (3) as asymptotic scaling. The relationship between the b_2 of different lattice actions, necessary for a check of universality, can be obtained from a two loop calculation. Consider two lattice actions whose coupling constants are related in the weak coupling region by,

$$\frac{1}{g'^2(a')} = \frac{1}{g^2(a)}\left[1+g^2(a)(b_0L + c_0) + g^4(a)(b_1L + c_1) + \ldots\right] \tag{5}$$

and the coefficients of the logarithm $L = \ln\left(a^2/a'^2\right)$ are governed by the renormalization group equation. If the coupling constant $g(a)$ satisfies eq (2) with coefficients b_0, b_1 and b_2 then $g'(a)$ satisifes the same equation with the same coefficients b_0, b_1, but with b_2' given by,

$$b_2' = b_2 + (b_1c_0 - b_0c_1) \tag{6}$$

The theoretical ratio of the Λ parameters is fixed by a one loop calculation,

$$\frac{\Lambda_L}{\Lambda_L'} = \exp\left(\frac{c_0}{2b_0}\right) \tag{7}$$

Assuming the g^2 and g'^2 are approximately equal and given by $\bar{g}^2$ in the scaling window of Monte Carlo data, the experimental ratio of Λ parameters can be corrected[3],

$$\frac{\Lambda_L}{\Lambda_{L'}} \simeq \left(1 - \bar{g}^2\delta_{L,L'}\right)\left(\frac{\Lambda_L}{\Lambda_{L'}}\right)_{\exp} \tag{8}$$

The correction factor δ is given by,

$$\delta_{L,L'} = \frac{1}{2b_0^2}(b_0c_1 - b_1c_0) \tag{9}$$

The constants c_0 and c_1 will be obtained from the effective action of the two lattice theories calculated using the background field method.

THE BACKGROUND FIELD METHOD

In this section I outline the background field method[4] and give an example of its use in the continuum. The background field is introduced by writing the normal Yang-Mills Lagrangian as the sum of the quantum field Q_μ and the background field B_μ. The gauge fixing term which breaks the gauge invariance with respect to transformations of the quantum field is chosen in such a way that the invariance of the action under gauge transformation of the background field is preserved. The generating functional is given by,

$$Z[J,B] = \int [dQ] \; \frac{\delta G^C}{\delta \omega^D} \; \exp\, i \int d^4x \left(L(Q+B_0) - \frac{1}{2\alpha_0}(G^A)^2 + J^A_\mu Q^A_\mu \right) \tag{10}$$

where the gauge fixing function is,

$$G^A = \left(\partial^\mu \delta^{AC} - g_0 f^{ABC} B^\mu_{0B} \right) Q^\mu_C \tag{11}$$

It can be shown that the effective action of the background field is equal to the normal effective action of the theory calculated with an unusual gauge fixing term. Wave function renormalization of the quantum field is not necessary since the quantum field occurs only on internal lines. However renormalization of the gauge parameter is still necessary. the result for the two point function of the theory - the effective action - in the gauge specified by eq (11) is,

$$\Gamma^{\mu\nu}(p,g,\alpha,\mu) = - (g^{\mu\nu}p^2 - p^\mu p^\nu)(1-g^2 d_0 - g^4 d_1) \tag{12}$$

where[5],

$$d_0 = \frac{N}{16\pi^2} \left\{ \frac{205}{36} + \frac{3}{2}\alpha + \frac{1}{4}\alpha^2 + \frac{11}{3}\rho \right\}$$

$$d_1 = \left(\frac{N}{16\pi^2}\right)^2 \left\{ \frac{2687}{72} + \frac{34}{3}\rho - \frac{57}{8}\zeta(3) - \alpha\left(\frac{187}{48} + \frac{13}{4}\rho - \frac{5}{4}\zeta(3)\right) \right.$$

$$\left. - \alpha^2\left(\frac{161}{144} + \frac{1}{3}\rho + \frac{1}{8}\zeta(3)\right) - \alpha^3\left(\frac{3}{16} - \frac{1}{4}\rho\right) - \frac{\alpha^4}{16} \right\} . \tag{13}$$

and $\rho = \ln 4\pi - \gamma_E - \ln(-p^2/\mu^2)$. Eq (13) is the renormalized two point function in the $\overline{\text{MS}}$ scheme. The values of the renormalization constants in this scheme are[4],

$$\alpha_0 = Z_Q \alpha \ , \qquad Z_Q = 1 + \frac{g^2 N}{16\pi^2}\left(\frac{13}{6} - \frac{\alpha}{2}\right)\frac{1}{\varepsilon} + O(g^4)$$

$$g_0 = g\,\mu^{\varepsilon} Z_g, \qquad B_0^{\mu} = \sqrt{Z_B}\,B^{\mu} \ , \qquad Z_g \sqrt{Z_B} = 1$$

$$Z_B = 1 + g^2\frac{b_0}{\epsilon} + g^4\ \frac{b_1}{2\varepsilon} + O(g^6) \tag{14}$$

and renormalization is performed as usual.

$$\Gamma_0^{\mu\nu}\ \left(p, g_0, \alpha_0, \epsilon\right) = Z_B^{-1}\ \Gamma^{\mu\nu}(p, g, \alpha, \mu) \tag{15}$$

The results given in eq (13) constitute the first step in the calculation of the relationship between continuum and lattice Λ parameters in two loops using the background field method. However the lattice part of this calculation has not yet been performed.

PERTURBATIVE CORRECTIONS AND TESTS OF UNIVERSALITY

I now report the results of the two loop lattice calculation which relates the Λ parameters of different lattice actions. The implementation of the background field method on the lattice has been described in the literature.[6] I work with a field strength ϕ which is defined from the lattice quantum field strength $F_{\mu\nu}$ and the background field strength $f_{\mu\nu}$ as follows,

$$\exp\ i\ \varphi = \exp\ igF_{\mu\nu}\ \exp\ i\ a^2 f_{\mu\nu} \tag{16}$$

The general one plaquette lattice action can be expressed in terms of ϕ. Retaining all gauge invariant terms which can contribute in two loop order we have,

$$S(\phi) = S_2 + S_I \qquad\qquad S_2 = \frac{1}{g^2}\sum_x \sum_{\mu\nu} \mathrm{Tr}\,\frac{\phi^2}{2} \tag{17}$$

and

$$S_I = \frac{1}{g^2}\sum_x \sum_{\mu,\nu}\ s_4(\mathrm{Tr}\phi^4) + s_6(\mathrm{Tr}\phi^6) + t_4(\mathrm{Tr}\phi^2)^2 + t_6(\mathrm{Tr}\phi^2)^3$$
$$+ u_6(\mathrm{Tr}\phi^2\mathrm{Tr}\phi^4) + v_6(\mathrm{Tr}\phi^3)^2 \tag{18}$$

The term S_2 is common to all lattice actions; its normalization is fixed because it contains the only term which survives in the naive continuum limit,

$$S_2 \to \frac{1}{2} \sum_x \sum_{\mu,\nu} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} \tag{19}$$

The coefficients s_4, s_6 etc. determine the particular form of the lattice action. As an example I quote the Wilson action, which before summatión over plaquettes is,

$$S_W(P) = \beta_F\left(1 - \frac{1}{2N} \mathrm{Tr}(U(P)+U^+(P))\right) \tag{20}$$

Another action which has been used extensively contains in addition to the Wilson action an admixture of the adjoint representation,

$$S_{FA}(P) = \beta_F\left(1 - \frac{1}{2N} \mathrm{Tr}\ (U(P)+U^+(P))\right)+\beta_A\left(1 - \frac{1}{N^2}\ |\mathrm{Tr}U(P)|^2\right) \tag{21}$$

The full details of the two loop calculation are given in ref. (1). The final result is the relationship between the coupling constant g of a arbitrary action, specified by the constants s_4, s_6, t_4 etc. and the coupling constant g_M of Manton action in which all s_4, s_6, t_4 etc. are equal to zero.

$$\frac{1}{g_M^2} = \frac{1}{g^2} + s_4 \frac{2N^2-3}{N} + t_4(N^2+1) + g^2 a_R\left(s_4(2N^2-3) + t_4 N(N^2+1)\right)$$

$$+ g^2\left(s_6 \frac{15}{8} \frac{(N^4-3N^2+3)}{N^2} + v_6 \frac{9}{8} \frac{(N^2-4)}{N} + u_6 \frac{3}{8} \frac{(2N^2-3)(N^2+3)}{N}\right.$$

$$\left. + t_6 \frac{3}{8} (N^2+1)(N^2+3)\right) \tag{22}$$

$$- g^2\left(s_4^2 \frac{9N^4-30N^2+36}{2N^2} + 2s_4 t_4 \frac{(2N^2-3)(N^2+2)}{N} + t_4^2(N^2+1)(N^2+2)\right)$$

The result is completely analytic except for the quantity a_R calculated in ref. (7) and approximately given by,

$$a_R = -\ .0011 \pm .0002 \tag{23}$$

From eq. (22) we can compute the correction factors to the ratios of lattice Λ parameters using eq. (8). The results are given in Table II. The results of the full two loop calculation given in Table II are very similar to the results obtained in ref. (3) where only tadpole diagrams were calculated. The significance of the corrections can be estimated using the approximate values of the coupling constant in the scaling window ($g^2=2$ for SU(2), $\bar{g}^2=1$ for SU(3)). A comparison with the available data on SU(2) is shown in Table III. The conclusion for the particular actions shown in Table III is that the corrections are modest in size, and tend to bring to data into better agreement with the theory. The remaining discrepancy should be attributed to order a^2 corrections which may be large at presently investigated values of the coupling constant. However in view of the large errors it is possible to argue that there is no further discrepancy.

By way of contrast the correction $\delta_{W,FA}$ which relates the Wilson and mixed fundamental adjoint actions is quite large. This is due to the large coefficient of r^2 as shown in Table II. Despite the size of the correction it is still not sufficient to bring data into agreement with theory for $\beta_A>0$ where the perturbative approximation should work best. The experimental results on the mixed action can be understood using another approach. In the limit of large N the mixed action can be written in terms of a equivalent Wilson action with a coupling constant g_W defined using the iterative relation,

$$B_W = B_F + B_A\ \omega\left(\frac{1}{NB_W}\right) + 0\left(\frac{1}{N}\right); \tag{24}$$

where

$$\omega(g^2) = \left\langle\frac{\mathrm{TrU(P)}}{N}\right\rangle \ ;\quad B_W = \frac{1}{Ng_W^2},\ B_F = \frac{\beta_F}{2N^2}\ ,\ B_A = \frac{\beta_A}{N^2} \tag{25}$$

Eq. (24) may be improved using perturbative results and becomes;

$$B_W = B_F + B_A\ \omega\left(\frac{1}{NB_W}\right)$$

$$- \frac{B_A}{(N^2-1)}\left(1-\omega^2\left(\frac{1}{NB_W}\right)\right) - \frac{B_A}{B_W^2}\frac{1}{48N^2}\left(1-\frac{3}{N^2}\right) - \frac{B_A^2}{B_W^3}\frac{(N^2+1)}{64N^2} \tag{26}$$

The expectation value of the plaquette variable has a perturbative expansion

$$\omega\left(\frac{1}{NB_W}\right) = 1 - \frac{N^2-1}{8N^2}\frac{1}{B_W} - \frac{N^2-1}{8N^2}\left(a_R + \frac{2N^2-3}{48N^2}\right)\frac{1}{B_W^2} \tag{27}$$

so that the perturbative corrections in eq. (26) are seen explicitly to be of order $1/N^2$. Eq.(26), taken from ref. (1), corrects the expression of Jurkiewicz, Korthals Altes and Dash,[11] who overlooked the renormalization of the gauge parameter. The difference between the expression of ref. (11) and eq.(26) is numerically small for the values of the coupling constants B_A, B_F of interest. The numerical results of ref. (11) therefore remain valid.

CORRECTIONS TO RENORMALIZATION GROUP BEHAVIOR.

A separate question which is unanswered by the calculation of ref. (1) is whether or not the quantity b_2 is large for all lattice actions. In this case even though the corrections to the ratio of Λ_L parameters is small, the use of eq. (3) to extract the experimental Λ without the inclusion of $O(g^2)$ terms would be unjustified. The continuum value of b_2 is known from the work of Tarasov et. al[12] and is given in SU(N) gauge theory by,

$$b_2^{cont} = \frac{2857}{54}\left(\frac{N}{16\pi^2}\right)^3 \tag{28}$$

The easiest way to calculate b_2^L is to perform the two loop calculation which relates the lattice coupling constant to the continuum coupling constant. The first step in this program using the background field method is given in eq. (13). A crude estimate of the size of b_2^L was obtained in ref. (2) by evaluating only the tadpole diagrams. The result for the Wilson action is as follows,

$$\Lambda_W a = \left(1+\delta_W g^2 + O(g^4)\right) \exp\left(-\frac{1}{2b_0 g^2} - \frac{b_1}{2b_0^2}\ln(b_0 g^2)\right) \tag{29}$$

where the correction term δ_W is made up of three parts,

$$\delta_W = \delta_{cont} + \delta_{cont,M} + \delta_{M.W} \tag{30}$$

δ_{cont} and $\delta_{M,W}$ are known exactly from refs. (12,1) respectively. $\delta_{cont,M}$ is estimated using the tadpole approximation. Note that even if the estimate of $\delta_{cont,M}$ were too small by an order of magnitude the change in δ_W would be less than 5% for SU(3). The conclusion to be drawn from this estimate is that if large departures from renormalization group behaviour are observed in Monte Carlo measurements of physical quantities they will not be removed by the inclusion of the first perturbative corrections. At fixed β the perturbative corrections may be as much as 10%. In Monte Carlo experiments performed in a finite range of β, such a correction would lead to an observable deviation from renormalization group behaviour which is much smaller.

REFERENCES

1. R. K. Ellis and G. Martinelli, Nucl Phys B135 [FS11] 93 (1984)
2. R. K. Ellis and G. Martinelli, Frascati preprint LNF-84/1(P)(1984)
3. H. Sharatchandra and P. Weisz, DESY preprint DESY 81-083 (1981)
4. L. F. Abbott, Nucl. Phys. B185, 189 (1981) and references therein.
5. R. K. Ellis, unpublished.
6. R. Dashen and D. Gross, Phys. Rev.D23, 2340 (1981) A. Gonzales-Arroyo and C. P. Korthals Altes, Nucl. Phys. B205 [FS5], 46(1982) A. and P. Hasenfratz, Nucl. Phys. B193, 210 (1981)
7. A. Di Giacomo and G. C. Rossi, Phys. Lett. 100B, 481 (1981)
8. C. B. Lang et al., Phys Rev. D26, 2028, (1982)
9. G. Bhanot and C. Rebbi, Nucl. Phys. B180 [FS2], 469 (1981)
10. R. V. Gavai et al., Nucl. Phys. B220 [FS8], 223 (1983)
11. J. Jurkiewicz et al., CERN preprint TH.3621-CERN (1983)
12. O. Tarasov et al., Phys. Lett.93B, 429 (1980)

TABLE I

The coefficients in eq.(18) which determine the form of one plaquette actions. r is defined as $r = -2\beta_A/(\beta_F+2\beta_A)$.

	s_4	s_6	t_4	t_6	u_6	v_6
WILSON	$-\frac{1}{24}$	$\frac{1}{720}$	0	0	0	0
FUNDAMENTAL + ADJOINT	$-\frac{1}{24}$	$\frac{1}{720}$	$\frac{r}{8N}$	0	$-\frac{r}{48N}$	$\frac{r}{72N}$
MANTON	0	0	0	0	0	0
HEAT KERNEL	$-\frac{Ng^2}{5760}$	0	$-\frac{3g^2}{5760}$	0	0	0

TABLE II

The correction parameter $\delta_{L,L'}$ for various of actions in SU(2) and SU(3).

$\delta_{L,L'}$	SU(2)	SU(3)
$\delta_{M,W}$	$4.48\ 10^{-2}$	0.132
$\delta_{M,FA}$	$4.48\ 10^{-2} + 3.37\ 10^{-2}r + 1.26r^2$	$0.132-1.26r + 1.37r^2$
$\delta_{M,HK}$	$2.26\ 10^{-3}$	$3.39\ 10^{-3}$

TABLE III.

Comparison of the theoretical ration of Λ parameters with SU(2) data with and without our correction included. The string tension data is taken from refs. (8,9). The data on the deconfinement temperature T_c comes from ref. (10).

	Theory	String tension data	String tension data (Corrected)	T_c data	T_c data (Corrected)
$\frac{\Lambda_M}{\Lambda_W}$	3.07	5.1± 1.0	4.7	4.08	3.71
$\frac{\Lambda_M}{\Lambda_{HK}}$	2.45	3.0±0.3	3.0	2.60	2.59

TABLE IV

Contributions to δ_W

N	δ_{cont}	$\delta_{cont,M}$	$\delta_{M,W}$	δ_W
2	-8.4×10^{-3}	$-.3\times10^{-3}$	44.8×10^{-3}	$37.\times10^{-3}$
3	-12.6×10^{-3}	$-.4\times10^{-3}$	$132.\times10^{-3}$	$120.\times10^{-3}$