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COUPLED BUNCH INSTABILITIES IN A $p\bar{p}$ COLLIDER †

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I. INTRODUCTION

These notes summarize a small amount of work done during preparation of the Fermilab Dedicated Collider proposal [1] last year. The basic problem is as follows:

Consider a storage ring with k proton bunches and k antiproton bunches, where electrostatic deflection devices are used to separate the beams except at the collision points in the interaction regions. Then the normal betatron motions of the bunches become coupled not only by the usual beam-beam force at collision points, but also by the forces exerted in the close encounters as one bunch passes nearby another. The problem we pose is simply to determine necessary and sufficient conditions for stability, given a linear approximation to the forces and motions as well as an assumption of rigid (coherent) bunch motion. This problem-essentially one of coupled oscillators-has been studied before, and the main result here may be folklore. However, this author has had some trouble, as usual, in identifying it all from the literature.[2-4] We hope that the formalism and results here may be of use in exploring this phenomenon in more generality.

The main result of our short investigation is that sufficient conditions for stable motion are:

1. All the bunch-antibunch forces be "small" and comparable in strength (acceptable for close encounters but perhaps not acceptable for the direct encounters at the collision points), and

2. The differences in tune of all p and $\bar{p}$ bunches be of the same sign; i.e. the tunes of p and $\bar{p}$ be split apart.

3. Integer tune stopbands are avoided.

Note that nothing is assumed about the uniformity of the bunch spacing or bunch intensities. However, the first condition as noted may not be met in practice. Our use of perturbation theory for the close-encounter beam-beam force should be acceptable. But it might not extend to the usual direct beam-beam force. It will not be hard to relax this restriction. But the necessary work has not yet been done.

II. SETUP OF THE GENERAL CALCULATION

To set up the problem we essentially follow Chao & Keil [3], and let

$$\xi_m(t) \equiv \begin{pmatrix} x_m(t) \\ x_m'(t) \end{pmatrix} \qquad \bar{\xi}_n \equiv \begin{pmatrix} \bar{x}_n(t) \\ \bar{x}_n'(t) \end{pmatrix} \tag{2.1}$$

be the betatron coordinates of the $m^{\underline{th}}$ proton bunch and $n^{\underline{th}}$ antiproton bunch. We now consider the mapping of these coordinates from t=0 to T/2, where T is the period of revolution around the machine. During this time each p bunch encounters each $\bar{p}$ bunch once and only once. Provided the ring has two-fold symmetry (which we assume to the case), we may then iterate this motion. We need only set up the 4k×4k matrix which implements the mapping.

Let $M(\Theta_f,\Theta_i)$ be the usual 2×2 transport matrix for the protons and $\bar{M}(\Theta_f,\Theta_i)$ for the antiprotons*. Let Θ_{mn} be the coordinate of the collision point (for the $m^{\underline{th}}$ p bunch against the $n^{\underline{th}}$ $\bar{p}$ bunch). The change $\delta x_m'$ caused by the close encounter may be written (in linear approximation) as

$$\delta x_m' = \bar{\epsilon}_{mn}\left(x_m + \bar{x}_n\right) \tag{2.2}$$

$$\delta x_n' = \epsilon_{nm}\left(x_m + \bar{x}_n\right) \tag{2.3}$$

Notice the $\bar{\epsilon}_{mn}$ & ϵ_{nm} need not be the same because the bunch intensities are not assumed to be equal. However

$$\frac{\bar{\epsilon}_{mn}}{\epsilon_{nm}} = \frac{\bar{N}_n}{N_m} \tag{2.4}$$

where N_m is the number of protons in the $m^{\underline{th}}$ bunch and $\bar{N}_n$ is the number of antiprotons in the $n^{\underline{th}}$ bunch.

We may now write down the mapping (assuming p moves clockwise, $\bar{p}$ anticlockwise).

$$\begin{aligned}\xi_m\left(\tfrac{T}{2}\right) = {} & M(\pi+\theta_m,\theta_m)\,\xi_m(0) \\ & + \bar{\epsilon}_{mn} M(\pi+\theta_m,\theta_{mn})\,\tau^-\,\bar{M}(\theta_{mn},\theta_n)\,\bar{\xi}_n(0) \\ & + \bar{\epsilon}_{mn} M(\pi+\theta_m,\theta_{mn})\,\tau^-\,M(\theta_{mn},\theta_m)\,\xi_m(0)\end{aligned} \tag{2.5}$$

$$\begin{aligned}\bar{\xi}_n\left(\tfrac{T}{2}\right) = {} & \bar{M}(\theta_n-\pi,\theta_n)\,\bar{\xi}_n(0) \\ & + \epsilon_{nm}\bar{M}(\theta_n-\pi,\theta_{mn})\,\tau^-\,M(\theta_{mn},\theta_m)\,\xi_m(0) \\ & + \epsilon_{nm}\bar{M}(\theta_n-\pi,\theta_{mn})\,\tau^-\,\bar{M}(\theta_{mn},\theta_n)\,\bar{\xi}_n(0)\end{aligned} \tag{2.6}$$

**We assume a planar machine, with electrostatic deflection in either horizontal or vertical plane, but not both.

Our notation is

$$\tau^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.7}$$

This can be written in matrix form. Let

$$\psi(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_k(t) \\ \bar{\xi}_1(t) \\ \vdots \\ \bar{\xi}_k(t) \end{pmatrix} \tag{2.8}$$

with

$$\psi\left(\tfrac{T}{2}\right) = \hat{M}\, \psi(0) \tag{2.9}$$

The matrix $\hat{M}$ has the form

$$\left(\begin{array}{c|c} \begin{matrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_k \end{matrix} & A \\ \hline B & \begin{matrix} \bar{X}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{X}_k \end{matrix} \end{array}\right) \tag{2.10}$$

where A & B are of order ε and where X_i and $\bar{X}_j$ are 2×2 matrices. It can in fact be checked that M is also symplectic.

Because M is symplectic it follows that if λ is an eigenvalue, so also is λ^{-1}. Therefore a necessary and sufficient condition for stability is that all eigenvalues of M have modulus unity. (Otherwise there will be a growing mode when the motion is iterated, i.e. when we raise M to a large power).

This then sets the problem. We need only examine the eigenvalues of $\hat{M}$, where the 2×2 elements of the matrix are

$$X_m = M(\pi+\theta_m, \theta_m) + \sum_{n=1}^{k} \varepsilon_{mn} M(\pi+\theta_m, \theta_{mn})\, \tau^{-} M(\theta_{mn}, \theta_m) \tag{2.11}$$

$$\bar{X}_n = \bar{M}(\theta_n - \pi, \theta_n) + \sum_{m=1}^{k} \epsilon_{nm} \bar{M}(\theta_n - \pi, \theta_{mn}) \tau \bar{M}(\theta_{mn}, \theta_n) \tag{2.12}$$

The off-diagonal 2×2 blocks of A & B (in hopefully obvious notation) are

$$A_{mn} = \bar{\epsilon}_{mn} M(\pi + \theta_m, \theta_{mn}) \tau \bar{M}(\theta_{mn}, \theta_n) \tag{2.13}$$

$$B_{nm} = \epsilon_{nm} \bar{M}(\theta_n - \pi, \theta_{mn}) \tau M(\theta_{mn}, \theta_m) \tag{2.14}$$

This can be done by examining the secular equation for the eigenvalues λ:

$$R(\lambda) \equiv \det(\hat{M} - \lambda) = 0 \tag{2.15}$$

III. RESULTS FOR THE SIMPLIFIED CASE

Up to this point, everything has still been general. Hereafter we adopt perturbation theory and calculate the eigenvalues of M as a perturbation series in ε; i.e. we expand the resolvent R to second order in ε.

After some calculation, a simple result emerges. The details of the calculation are relegated to an appendix. But in brief, half of the effect of the close-encounters-force goes into a coherent tune shift of each bunch [cf Eqns. (2.5) and (2.6)]

$$\Delta\nu_m = -\sum_n \frac{\epsilon_{mn}\beta_{mn}}{2\pi} \equiv \sum_n \Delta\nu_{mn} \tag{3.1}$$

$$\Delta\bar{\nu}_n = -\sum_m \frac{\bar{\epsilon}_{nm}\beta_{mn}}{2\pi} \equiv \sum_m \Delta\bar{\nu}_{nm} \tag{3.2}$$

with

$$\frac{\Delta\nu_{mn}}{\Delta\bar{\nu}_{nm}} = \frac{\bar{N}_n}{N_m} \tag{3.3}$$

The other half provides the coupling between p and $\bar{p}$ motions. And to this order the secular equation becomes, essentially (provided one is not near integer tunes)

$$R(\nu) \cong \sum_{m,n=1}^{k} \frac{\Delta\nu_{mn}\,\Delta\bar{\nu}_{nm}}{(\nu-\nu_m)(\nu-\bar{\nu}_n)} - 1 = 0 \tag{3.4}$$

where

$$\nu_m = \nu_0 + \Delta\nu_m \tag{3.5}$$

$$\bar{\nu}_m = \bar{\nu}_0 + \overline{\Delta\nu}_m \tag{3.6}$$

are the tunes of the relevant bunches, and the parameter ν is related to eigenvalues λ of M by the expression

$$\lambda = e^{\pm i\pi\nu} \tag{3.7}$$

Thus if one can find 2k real eigenvalues of Eqn.(3.4), stability is assured. This is the case provided the p-$\bar{p}$ tune differences $\nu_m - \bar{\nu}_n$ are all of the same sign, independent of m and n. In other words if the unperturbed tunes of p and $\bar{p}$ bunches are split by amounts large compared to the tune shifts (and, of course, one avoids integer-tune stop bands), then stability is assured. To see this one simply examines the function $R(\nu)$ and observe that the residues of all the "proton" poles are of one sign and those of the the "antiproton" poles of the opposite sign. Thus the function is as shown in Fig. 1 and the 2k zeroes can be explicitly counted. As the two groups of tunes overlap, however, this argument breaks down, accidental degeneracies (to this order) occur, and a more accurate analysis is needed.

IV. COMMENTS

1. We may remark that the main result seems to be contained in ancient work of Pellegrini and Sessler.[2] More recent work by Chao and Keil[3] and by Piwinski[4] has concentrated on motion of a small number of bunches. Stability seems to be the general condition away from integer stopbands, provided beam-beam tuneshifts are small compared to 0.1. This includes some study of nonlinear effects as well. Thus there is no reason (yet) to be especially apprehensive about the results of the more extensive calculations not yet done.

2. The close-encounter beam-beam force directly affects the sinuous equilibrium orbits of p and $\bar{p}$, in particular changes the wavelength. The compensation can be made by carefully locating the electrostatic deflection plates and varying the voltage on them.[4]

3. The tune shift and coupling of p and $\bar{p}$ bunches come from the first moment of the close-encounter force. The tune shift is of opposite sign to the the linear tune shift from the direct beam-beam interaction.[4]

4. The second moment of the close-encounter beam-beam force causes a tune-spread. This is small compared to the tune shift (of order of the ratio of beam size to beam separation), unlike the case of the usual direct beam-beam interaction, and may indicate that the nonlinear dynamics of the close-encounter force may be less important than for the direct beam-beam interaction.[4]

5. We have assumed rigid bunch motion. This again seems to be safe because of the relative weakness of the higher moments of the close-encounter force.

6. The calculations here are also applicable to the close-encounter beam-beam interactions occuring near the collision points in pp colliders (provided the machine has two-fold symmetry).

7. The aforementioned assumption of twofold symmetry is probably not crucial. The generalization should be no problem.

8. It seems to this author that reasonable next steps are as follows:

a. Calculate exactly the effects of direct beam-beam encounters at collision points, and then again use the second order perturbation-theory approach for the effects of the close encounters. This program should be no more involved than what has been done by Piwinski, Chao, and Keil.
b. There is probably a systematic, "diagrammatic" expansion of the exact secular equation. This would be interesting to set up.
c. Brute-force computer diagonalization of the full matrix $\hat{M}$ is probably feasible and may be the best way to go.

We thank Jonathan Schonfeld and Tom Collins for helpful discussions.

REFERENCES

1. C. Pellegrini and A. Sessler, CERN ISR report 67-19; also 6th International Conference on High Energy Accelerators, Cambridge 1967, 135 (1967).
2. A. Piwinski, 8th International Conference on High Energy Accelerators, CERN 1971, 357 (1971).
3. A. Chao and E. Keil, SLAC PEP NOTE 300; CERN-ISR-TH/79-31.
4. Fermilab Dedicated Collider proposal (available from this author).

APPENDIX: DETAILS OF THE CALCULATION

We start with the secular equation, and temporarily choose a basis where we have diagonalized the matrices X_m and $\bar{X}_n$. Then all off-diagonal elements are of order ε. To second order in ε we have

$$0 = \det(\hat{M}-\lambda) = \prod_{m,n=1}^{2k} (X_{mm}-\lambda)(\bar{X}_{nn}-\lambda)\left[1-\sum_{i,j=1}^{2k} \frac{A_{ij}B_{ji}}{(X_{ii}-\lambda)(\bar{X}_{jj}-\lambda)}+\dots\right] \tag{A.1}$$

Provided there is no degeneracy (something we demonstrate later), this implies

$$1 - \mathrm{Tr}\ \frac{1}{X-\lambda}\, A\, \frac{1}{\bar{X}-\lambda}\, B = 0 \tag{A.2}$$

where the trace is over a 2k×2k matrix. Alternatively we may write

$$\mathrm{Tr} \sum_{m,n=1}^{k} \frac{1}{(X_m-\lambda)}\, A_{mn}\, \frac{1}{(\bar{X}_n-\lambda)}\, B_{nm} = 1 \tag{A.3}$$

where X_m, $\bar{X}_n$, A_{mn}, B_{nm} are known explicitly; they are the 2×2 matrices we have already written down. The form of Eqn (A.3) liberates us from any specific set of basis functions. For any 2×2 nonsingular matrix

$$\frac{1}{X-\lambda} = \frac{(\det X)X^{-1}-\lambda}{\lambda^2-\lambda\,\mathrm{Tr}X+\det X} \tag{A.4}$$

Because X is symplectic, det X=1. And because X is a transport matrix, TrX=2cos μ, where μ is the "unperturbed" phase advance around half the ring; i.e. the machine tune ν is μ/π. Thus

$$\frac{1}{X-\lambda} = \frac{X^{-1}-\lambda}{(\lambda-e^{i\mu})(\lambda-e^{-i\mu})} \tag{A.5}$$

The calculation of the trace in Eqn.(A.3) now is reasonably straight-forward.

In the numerator, it is sufficient to use the unperturbed matrices X_m and $\bar{X}_n$; hence their inverses are immediately obtainable using the group property of transport matrices. In the numerator, only the term proportional to λ^2 survives.

In the denominator, we take care to keep the perturbations to X and $\bar{X}$. The result for the secular equation is then

$$1 = \lambda^2 \sum_{m,n=1}^{k} \frac{\epsilon_{nm}\bar{\epsilon}_{mn}\beta_{mn}^2 \sin\mu_m \sin\bar{\mu}_n}{\left[1+\lambda^2-\lambda\left(2\cos\mu_m+\sum_{n'}\bar{\epsilon}_{mn'}\beta_{mn'}\sin\mu_m\right)\right]\left[1+\lambda^2-\lambda\left(2\cos\mu_n+\sum_{m'}\epsilon_{nm'}\beta_{m'n}\sin\mu_n\right)\right]} \tag{A.6}$$

where β_{mn} is the value of the β-function at θ_{mn}, the location the collision of the $m^{\underline{th}}$ proton bunch with the $n^{\underline{th}}$ antiproton bunch.

We recognize in the denominator the familiar form for a first-order tune shift. The "tune shift" $\Delta\nu_{mn}$ of the $m^{\underline{th}}$ proton bunch due to close encounters with the $n^{\underline{th}}$ antiproton bunch is evidently given by

$$\Delta\nu_{mn} = -\frac{\epsilon_{mn}\beta_{mn}}{2\pi} \tag{A.7}$$

$$\Delta\bar{\nu}_{nm} = -\frac{\bar{\epsilon}_{nm}\beta_{mn}}{2\pi} \tag{A.8}$$

and this is summed over all encounters (cf Eqns. (2.11) and (2.12)) to give the total coherent tune shift.

$$\Delta\nu_m = \sum_n \Delta\nu_{mn} \tag{A.9}$$

$$\Delta\bar{\nu}_n = \sum_m \Delta\bar{\nu}_{nm} \tag{A.10}$$

With this notation, and with the convention (cf Eqns. (2.11) and 2.12))

$$\lambda = e^{i\mu} \equiv e^{i\pi\nu} \tag{A.11}$$

we may expand in small quantities and obtain the secular equation

$$\sum_{m,n=1}^{k} \frac{(\Delta\nu_{mn})(\Delta\bar{\nu}_{nm})}{(\nu-\nu_0-\Delta\nu_m)(\nu-\bar{\nu}_0-\Delta\bar{\nu}_n)} = 1 \tag{A.12}$$

which is what was quoted in Section II.

We also should mention the instabilities which occur when the tunes are near an integer. We have implicitly assumed the situation shown in Fig. 2a, where the relevant eigenvalues on the unit circle are well-separated from their complex conjugate images. Instability occurs when the eigenvalues and their images approach each other at integer tunes, e.g. Fig. 2b. If p and $\bar{p}$ tunes are widely split, with neither near an integer, but straddling an integer value, then there can again be instability. This occurs[3] when proton eigenvalues become degenerate with the antiproton images, as shown in Fig. 2c. It is unlikely that this kind of configuration would be chosen as an operating point in a real machine, and we shall not give it further consideration here.

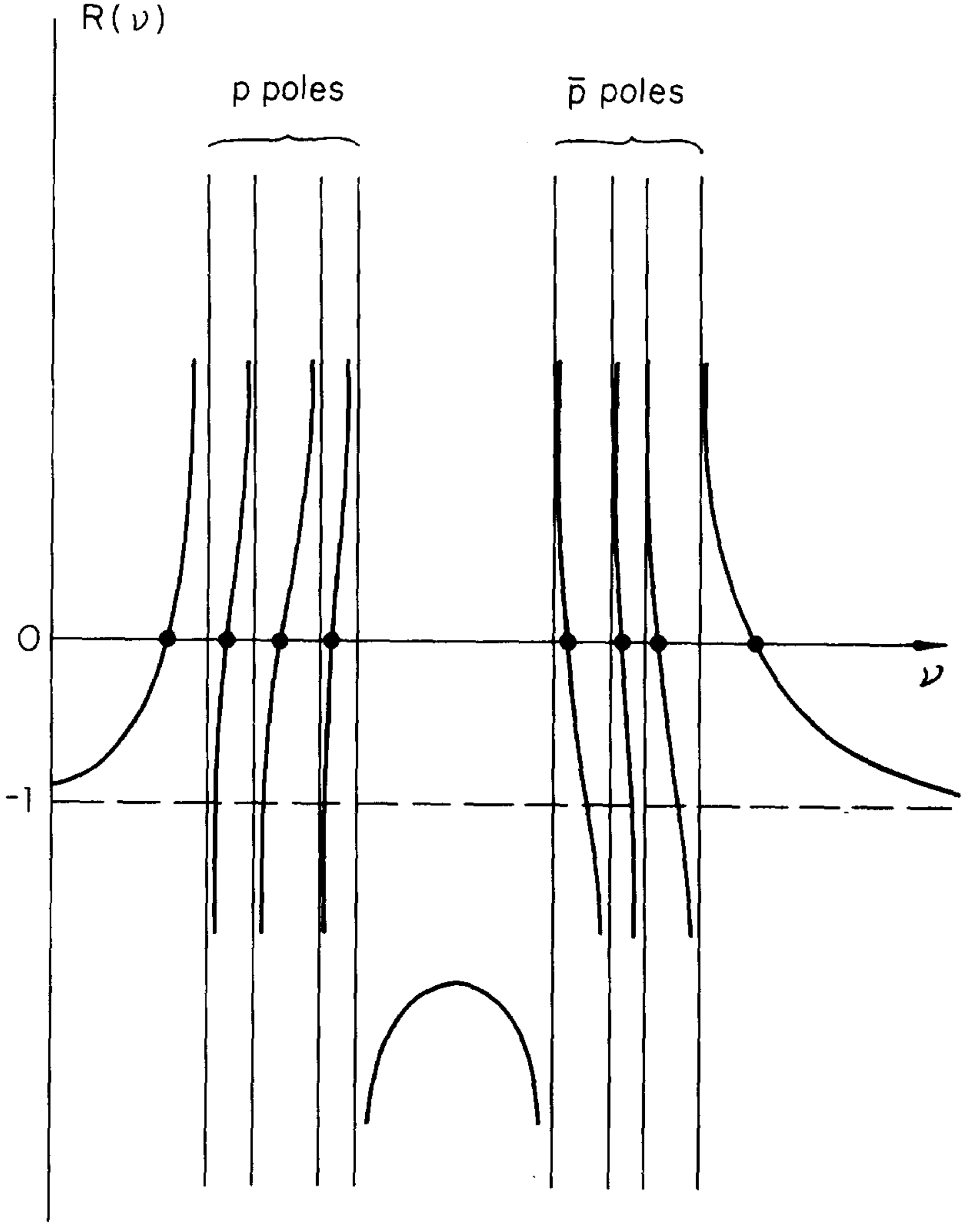

Fig. 1 Behavior of the function R(ν) for k = 4.

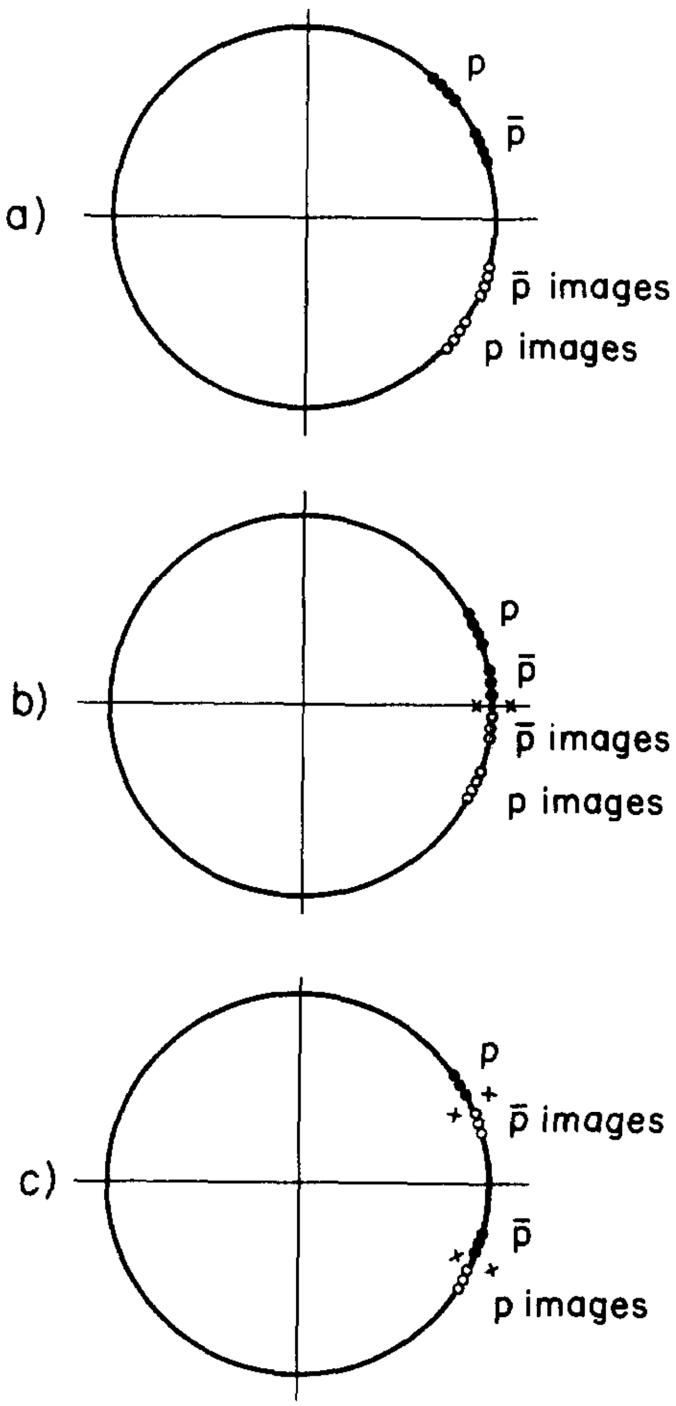

Fig. 2 Zeroes of the determinant $\hat{M}$ for (a) Stable behavior away from integer tunes, (b) unstable behavior when the $\bar{p}$ tunes are near integer, (c) unstable behavior when the sum of p and $\bar{p}$ tunes is a even integer.