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# Electron Storage Ring Distributions Near Linear Resonance

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## ABSTRACT

The standard formula for the shape of a beam in a collisionless electron storage ring can require substantial correction when the (linear) equations of motion have solutions that are nearly periodic or antiperiodic in time. We explain how to calculate approximations to beam shapes in such cases in a simple way. A similar analysis is also applied to the storage ring dispersion (or off-energy function). Our technique is similar in its logic to the "two-time" and "smoothing" methods of Keller.

# I. INTRODUCTION

The steady-state of a beam in a collisionless electron storage ring is widely assumed to be adequately described by a simple formula [1] that can in fact require significant correction under certain special circumstances. In this paper, we explain how to calculate approximations to such corrections in a simple way.

We were originaly led to consider this problem in connection with a study of statistical-mechanical issues in the physics of electron-position colliding-beam storage rings. The motivation for the present work in that context is spelled out in the last section of [2], and need not be repeated here.

Before we present our ideas in a more concrete way, we first provide some background for the general reader.

Electrons in storage rings travel in bunches. When there is no counter-rotating positron beam (and when the current is not too high), the oscillation of an electron about the center of its bunch is governed primarily by magnets and by the process of synchrotron radiation. The effects of synchrotron radiation tend to accumulate slowly. For times short enough that such effects can be ignored, the vertical or horizontal projection of the displacement of an electron from the center of its bunch, in a plane perpendicular to the bunch center's velocity, is determined by an equation of the form [F1]

$$\ddot{y} + K(t)y = 0. \tag{1.1}$$

A dot signifies differentiation with respect to time, and the coefficient K (which, strictly speaking, should carry a subscript that distinguishes horizontal from vertical) depends periodicaly on time. The period, T, is equal to the time needed for the bunch center to circle the storage ring once.

As long as motion governed by (1.1) is stable,[F2] and as long as (1.1) has no periodic or antiperiodic solution (with period or antiperiod T), the most general solution to (1.1) is [3]

$$y(t) = [I\beta(t)]^{1/2}\cos[\delta+\int_0^t \frac{ds}{\beta(s)}], \tag{1.2}$$

were I and $\delta$ are constants of integration, and where $\beta$ is a positive, period-T function, characteristic of K. The effective frequency of the quasiharmonic motion (1.2) is evidently $(2\pi T)^{-1} \int_0^T ds/\beta(s)$. In the storage ring literature, the dimensionless index

$$\frac{1}{2\pi} \oint \frac{ds}{\beta(s)} \equiv \nu \tag{1.3}$$

is referred to as the "tune." (We reserve the notation "$\oint$" for the integral of any periodic function (with period T) over any time interval of length T.) When K approaches a configuration that supports a periodic or antiperiodic solution--so that the general form (1.2) need not apply--then the tune approaches an integer or a half-integer. The general condition

$$\nu = n/2 \quad , \tag{1.4}$$

for even or odd integer n, is referred to as "linear resonance." Nearness to linear resonance is the special circumstance, motioned in the first paragraph, under which the conventional calculation of electron beam profiles requires substantial correction.

For times long enough that radiation effects cannot be ignored, the displacement of an electron from its bunch center must be described by a linear equation that includes dissipation and fluctuation terms,

$$\ddot{y} + \gamma(t)\dot{y} + K(t)y = \lambda(t)\zeta(t) \quad , \tag{1.5}$$

where $\zeta$ represents centered Gaussian noise with unit delta-function variance [F3]

$$\langle\zeta(t)\zeta(t')\rangle = \delta(t-t'), \; \langle\zeta(t)\rangle \equiv 0 \quad , \tag{1.6}$$

and where the coefficient functions $\gamma$ (non-negative) and $\lambda$ are both periodic in time, with period T, just like K. The phase space probability density $P(y,v{=}\dot{y},t)$ for a system governed by (1.5) evolves in time according to the Fokker - Planck equation

$$\frac{\partial P}{\partial t} + v\,\frac{\partial P}{\partial y} - K(t)y\,\frac{\partial P}{\partial v} = \frac{\partial}{\partial v}\left[\gamma(t)vP + \frac{1}{2}\,\lambda^2(t)\,\frac{\partial P}{\partial v}\right] \quad . \tag{1.7}$$

According to the conventional, intuitive account [1], the steady-state distribution determined by (1.5) or (1.7) is given approximately, up to normalization, by

$$\exp - \left[\oint\gamma(s)ds\right]\left[\oint\lambda^2(s)\beta(s)ds\right]^{-1}\left\{\frac{y^2}{\beta(t)} + \beta(t)\left[v - \frac{1}{2}\,\frac{\dot{\beta}(t)}{\beta(t)}y\right]^2\right\} \quad . \tag{1.8}$$

Note that if expression (1.2) and its time derivative are substituted for y and v, respectively, in (1.8), then the expression in curved brackets reduces to the amplitude I that appears in (1.2).

How accurate is (1.8)? An exact calculation, described in Appendix A, reveals that (1.8) corresponds to the formal limit

$$\gamma \to 0 \quad , \ \lambda \to 0 \quad , \ \gamma/\lambda^2 \text{ fixed} \quad , \tag{1.9}$$

and that (from now on taking $\lambda$ to be formally $O(\gamma^{1/2})$) the remainder is formally a series in powers of $\gamma$, beginning with $O(\gamma^1)$. However, the coefficients in this series diverge when the tune approaches a resonant value given by (1.4)

What is the leading behavior near resonance when these singular terms are summed? It is tempting to try reading this from the explicit tune dependence in equation (A.10). This is unrealistic, however, because the function $\beta(t)$, which enters the remainder in (A.10) not only through the definition of $\nu$, does not approach a simple limit as the system approaches linear resonance. Indeed, as we show in Appendix B, when $K(t)$ approaches a configuration for which equation (1.1) has precisely one periodic or antiperiodic solution ("noncoexistent" [5] resonance), then $\beta(t)$ becomes infinite for almost all values of t; while when $K(t)$ approaches a configuration for which equation (1.1) has two periodic or antiperiodic solutions [F4] (coexistent resonance) then $\beta(t)$ can approach a finite limit, but the limit depends nontrivially on the function-space direction along which $K(t)$ approaches its limit.

In this paper, we explain a simple technique for deriving the leading behavior of the steady-state distribution near resonance directly from the Fokker-Planck equation (1.7). Our procedure, in a nutshell, is as follows:

We make the formal identification

$$\delta K(t) \equiv K(t) - K_0(t) \equiv 0\ (\gamma) \quad , \tag{1.10}$$

where $K_0(t)$ is a coefficient function for which equation (1.1) has at least one periodic or antiperiodic solution; and we also retain the identification

$$\lambda^2(t) \equiv 0(\gamma) \quad ; \tag{1.11}$$

and then we calculate leading terms in the expression of the steady-state distribution as a formal series in powers of $\gamma$,

$$P(y,v,t) = (P_0+P_1+P_2+\ldots)N \quad , \tag{1.12}$$

where N is a normalization constant. In order to do this, we decompose equation (1.7) into a perturbative hierarchy. The first equation in the hierarchy involves only $P_0$ ; the second equation involves $P_0$ and $P_1$ ; the (n+1)'st equation involves $P_{n-1}$ and $P_n$. In view of the exact expression (A.10), we impose the boundary condition that each term in the expansion (1.12) must be periodic in time, with period T.

One might naively expect that the first equation in this hierarchy determines $P_0$, up to normalization; and that the second equation then determines $P_1$, as a linear functional of $P_0$; and so on. Unfortunately, the first equation has in fact many a priori admissible solutions. When almost any such solution is substituted into the second perturbative equation, however, there is no periodic solution for $P_1$. The requirement that a periodic solution for $P_1$ exist--without regard for its detailed construction--turns out to be sufficient to fix $P_0$ uniquely, up to normalization. In a similar fashion, $P_1$ is determined by the second equation together with the requirement that the third equation be self-consistent as an equation for $P_2$, and so on. This kind of logic is very similar to that employed in the "two-time" and "smoothing" methods that are discussed in the applied mathematics literature [6].

We shall perform such a calculation three times in this paper. In Section II, as a warm-up, we shall use the method just described to recover (1.8) from $P_0$ when $K(t)$ is far from resonance. For this calculation, we shall ignore the prescription (1.10) concerning the nearness of $K(t)$ to some $K_0(t)$. In Sections III and IV, we shall calculate the leading behavior of the steady-state distribution near coexistent and noncoexistent resonance, respectively. In the noncoexistent case, "leading behavior" will have to mean $P_0 + P_1$, as we shall explain in Section IV. In the coexistent case, there will be no pressing reason to calculate beyond $P_0$. In each of Sections III and IV, we shall check our work by verifying that when $\delta K$ becomes much larger than $\gamma$ and $\lambda^2$, then our approximate distribution returns to the nonresonant form (1.8).

In each of these calculations, we shall from the outset take for granted, in accordance with the exact expression (A.4), that the desired distribution--either in zeroth order in $\gamma$, or when all perturbative contributions are summed--is a Gaussian in y and v, centered at y=v=0.

This work will also be conducted under the assumption that there is no a priori connection (other than rough scale) between the coefficients $\gamma$,K, and $\lambda$. In reality, however, $\lambda$ and K are intimately related [1], and $\lambda$ can in fact become infinite when K approaches some $K_0$. (In Appendix C, we shall show how to derive the leading behavior of $\lambda$ near resonance in a fashion very similar in spirit to our derivations of probability distributions in sections II-IV.) We shall discuss in the concluding section, V, how this is to be reconciled with our scheme of perturbative calculation of probabilities.

## II. Nonresonant Steady State

In this case, the perturbative hierarchy is

$$\mathcal{L}_0 P_0 = 0 \quad , \tag{2.1a}$$

$$\mathcal{L}_0 P_1 = \mathcal{L}_1 P_0 \quad , \tag{2.1b}$$

$$\mathcal{L}_0 P_2 = \mathcal{L}_1 P_1 \quad , \tag{2.1c}$$

etc., where the linear operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are defined by

$$\mathcal{L}_0 \equiv \frac{\partial}{\partial t} + v \frac{\partial}{\partial y} - K(t) y \frac{\partial}{\partial v} \quad , \tag{2.2a}$$

$$\mathcal{L}_1 \equiv \frac{\partial}{\partial v} \left[ \gamma(t) v + \frac{1}{2} \lambda^2(t) \frac{\partial}{\partial v} \right] \quad . \tag{2.2b}$$

In view of (2.2a), the operator $\mathcal{L}_0$ satisfies the identity

$$[\mathcal{L}_0 F]\Big|_{y=y(t),\, v=\dot{y}(t)} = \frac{d}{dt} \left[ F(y=y(t),\ v=\dot{y}(t),\ t) \right] \quad , \tag{2.3}$$

where $F(y,v,t)$ is arbitrary, and where $y(t)$ is any solution of (1.1). In view of (2.3), equation (2.1a) says that when any solution to (1.1), and its time derivative, respectively, are substituted for y and v in $P_0(y,v,t)$, then the result is constant in time.

It follows that log $P_0$ must be a quadratic form in y and v with the same property, since we are assuming a priori that $P_0$ must be a centered Gaussian. Since $P_0$ is periodic in time, with period T, the quadratic form log $P_0$ must have coefficients that are also periodic functions of time, with period T.

It turns out that as long as the tune $\nu$ does not satisfy (1.4), then, up to overall scale, there is only one quadratic form - the expression in curved brackets in (1.8) - that meets these specifications. To see this, proceed as follows:

First, recognize that the following recipe yields every quadratic form in y and v that is time - independent when y satisfies (1.1) and $v=\dot{y}$: Choose two arbitrary linearly independent solutions, $y_1$ and $y_2$, to (1.1), and then form all constant-coefficient quadratic forms in the combinations $a_1$ and $a_2$, defined by

$$a_i \equiv y\dot{y}_i(t) - vy_i(t) \quad . \tag{2.4}$$

The completeness of this construction follows from the fact that the Wronskian of any two solutions of (1.1) must be constant in time.

The desired result concerning the quadratic form log $P_0$ then follows directly from this construction, once one writes $y_1$ and $y_2$ in the form (1.2).

Thus, we write, up to normalization

$$P_0 = \exp - \alpha\{\frac{y^2}{\beta(t)} + \beta(t)[v - \frac{1}{2} y \frac{\dot{\beta}(t)}{\beta(t)}]^2\} \quad . \tag{2.5}$$

This is as far as we can go with (2.1a). In order to determine $\alpha$, we now consider the requirement that (2.1b) be consistent with periodic $P_1$.

As before, we begin by replacing y and v, in both sides of (2.1b), by a solution to (1.1) and its time derivative. In view of (1.2), (2.2b),(2.3), and (2.5), this yields

$$\frac{d}{dt}\left[P_1(\dot{y}(t),y(t),t)\right] = e^{-\alpha I}\Big\{\left[\gamma(t) - \alpha\lambda^2(t)\beta(t)\right]$$

$$\cdot\left[1 - 2\alpha I\sin^2\left[\delta + \int_0^t \frac{ds}{\beta(s)}\right]\right] \tag{2.6}$$

$$+ \frac{1}{2}\alpha\gamma(t)\dot{\beta}(t) I\sin 2\left[\delta+\int_0^t\frac{ds}{\beta(s)}\right]\Big\} \quad .$$

In order to exploit the periodicity of $P_1$ let us now integrate both sides of (2.6) over a time interval equal in length to an integral multiple of T. We then have

$$P_1\left[y(t+mT),\ \dot{y}(t+mT),\ t\right] - P_1\left[y(t),\dot{y}(t),t\right]$$

$$= P_1\left[y(t+mT),\dot{y}(t+mT),t+mT\right] - P_1\left[y(t),\dot{y}(t),t\right]$$

$$= me^{-\alpha I}\left[\oint\gamma(s)ds - \alpha\oint\lambda^2(s)\beta(s)ds\right]\left[1-\alpha I\right]$$

$$+ \frac{1}{2}\alpha I e^{-aI}\Big\{\left(\frac{1-e^{4\pi i m\nu}}{1-e^{4\pi i\nu}}\right)\int_t^{t+T}\left[\gamma(t')(1-\frac{i}{2}\dot{\beta}(t')) - \alpha\beta(t')\lambda^2(t')\right]$$

$$\cdot \exp 2i\left[\delta+\int_0^{t'}\frac{ds}{\beta(s)}\right]dt' + \text{complex conjugate}\Big\} \quad . \tag{2.7}$$

Notice now that for all m, the first difference in (2.7) is the difference, at a single time (t), between the values of $P_1$ at two points in some bounded phase space domain, since, by assumption, equation (1.1) defines a stable oscillator. Therefore this difference must itself be bounded as a function of m, for all values of I. It follows that the term proportional to m in the last expression in (2.7) must, for nonresonant $\nu$, vanish for all I. This means

$$\alpha = \left[\oint \gamma(s)ds\right]\left[\oint \lambda^2(s)\beta(s)ds\right]^{-1} . \tag{2.8}$$

Upon insertion of (2.8) into (2.5), we obtain (1.8), which is what we wanted to show.

## III. Steady State Near Coexistent Resonance

In this case, and also in the next section, the perturbative hierarchy is

$$L_0 P_0 = 0 \quad , \tag{3.1a}$$

$$L_0 P_1 = L_1 P_0 \quad , \tag{3.1b}$$

$$L_0 P_2 = L_1 P_1 \quad , \tag{3.1c}$$

etc. , where the differential operators $L_0$ and $L_1$ are now defined by

$$L_0 \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial y} - K_0(t)y\frac{\partial}{\partial v} \quad , \tag{3.2a}$$

$$L_1 \equiv \frac{\partial}{\partial v}\left[y\delta K(t) + v\gamma(t) + \frac{1}{2}\lambda^2(t)\frac{\partial}{\partial v}\right] . \tag{3.2b}$$

Reasoning exactly as in the preceding section, we conclude from (3.1a) that $\log P_0$ is (up to an additive constant corresponding to normalization) a quadratic form that is constant in time whenever y and v are replaced by a solution to

$$\ddot{y} + K_0(t)y = 0 \quad , \tag{3.3}$$

and its time derivative. As in the preceding section, this means that

$$\log P_o = \text{constant} - \frac{1}{2} a_i A_{ij} a_j \quad , \tag{3.4}$$

where the symmetric matrix A is time-independent, and where the $a_i$ are defined by (2.4), now in terms of solutions $y_i$ to (3.3). (Summation over repeated indices is understood.) Since coexistent resonance means that the $y_i$ are either both periodic or both antiperiodic in time T, it follows that A is not further constrained by the periodicity of $P_0$. Thus, we proceed immediately to consider the consistency of (3.1b).

As in the preceding section, we shall reason from the time-integral of (3.1b). In this case, however, we shall have to integrate only over one period of length T, not over many, because now every solution y(t) to (3.3) satisfies $y(t+mT) = (\pm)^m y(t)$, and therefore the analogue here of the left-hand side of (2.7) is in fact strictly zero for all m, since our Gaussian assumption implies $P(y,v,t) = P(\pm y, \pm v,t)$. After an easy calculation, we are led in this way to

$$0 = a_i[BA+ACA]_{ij}a_j + \oint \Upsilon(s)ds - \mathrm{Tr}CA \quad , \tag{3.5}$$

for all $a_1$ and $a_2$, where the matrices B and C are defined by

$$B \equiv W^{-1}\oint ds\left\{\delta k(s)\begin{pmatrix}-\,y_1(s)y_2(s) & -\,y_2^2(s)\\ y_1^2(s) & +\,y_1(s)y_2(s)\end{pmatrix}\right.$$

$$\left.+\ \gamma(s)\begin{pmatrix}-y_1(s)\dot{y}_2(s) & -y_2(s)\dot{y}_2(s)\\ y_1(s)\dot{y}_1(s) & y_2(s)\dot{y}_1(s)\end{pmatrix}\right\} \quad , \tag{3.6}$$

and

$$C_{ij} \equiv \frac{1}{2}\oint ds\lambda^2(s)y_i(s)y_j(s) \quad . \tag{3.7}$$

The quantity W in (3.6) is the Wronskian of $y_1$ and $y_2$, i. e.

$$W \equiv y_1\dot{y}_2 - y_2\dot{y}_1 \quad . \tag{3.8}$$

In deriving (3.5)-(3.7) we have used the inversion of (2.4),

$$\begin{aligned} y &= W^{-1}[-a_1y_2 + a_2y_1] \quad , \\ v &= W^{-1}[-a_1\dot{y}_2 + a_2\dot{y}_1] \quad . \end{aligned} \tag{3.9}$$

Equation (3.5) implies that

$$\mathrm{Tr}\ CA = \oint\gamma(s)ds \quad , \tag{3.10}$$

and also that the symmetric part of the matrix BA + ACA vanishes, i. e.

$$BA + AB^T + 2ACA = 0 \quad , \tag{3.11}$$

Where the superscript "T" signifies matrix transposition. In writing (3.11) we have used the symmetry of A and of C.

In order to solve (3.11) for A, we first rewrite it as [F5]

$$A^{-1}B + B^{T}A^{-1} = -2C \quad . \tag{3.12}$$

It follows from (3.2) that $A^{-1}B + C$ is an antisymmetric matrix. Up to overall scale, however, there is only one antisymmetric $2 \times 2$ matrix, namely

$$\sigma \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{3.13}$$

Thus, we may write

$$A = B(a\sigma - C)^{-1} = B(\sigma C - a)\sigma/[a^2 + \det C] \quad , \tag{3.14}$$

where a is some number. The second equality in (3.14) follows from the identity

$$(M_s + M_a)^{-1} = -\sigma[M_s - M_a]\sigma/[\det M_s + \det M_a] \quad , \tag{3.15}$$

satisfied by all symmetric ($M_s$) and antisymmetric ($M_a$) $2 \times 2$ matrices.

In order to determine a, we exploit the symmetry of A by writing

$$0 = \mathrm{Tr}A\sigma = \mathrm{Tr}(aB - B\sigma C)/[a^2 + \det C] \quad . \tag{3.16}$$

It follows that

$$a = \mathrm{Tr}B\sigma C/\mathrm{Tr}B = \mathrm{Tr}\sigma CB/\mathrm{Tr}B \quad . \tag{3.17}$$

It is not hard to verify that (3.10) is automatically satisfied when A is given by (3.14) and (3.17): Simply write

$$\mathrm{Tr}CA = \mathrm{Tr}AC = \mathrm{Tr}B(a\sigma-C)^{-1}((-a\sigma-C)+a\sigma) = -\mathrm{Tr}B + a\mathrm{Tr}A\sigma = -\mathrm{Tr}B \quad , \tag{3.18}$$

and then observe that the definition (3.6) implies that

$$\mathrm{Tr}\ B = -\oint\gamma(s)ds \quad . \tag{3.19}$$

This completes our steady-state calculation near coexistent resonance. Let us now verify that when $\delta K$ is much larger than the other small scales $\gamma$ and $\lambda^2$, but still itself small, then the right-hand-side of (3.4) approaches the nonresonant exponent in (1.8), with $\beta$ given by the small-$\delta K$ form derived in Appendix B. In this limit, we have, for most purposes,

$$B \sim -D\sigma \quad , \tag{3.20}$$

where the symmetric matrix D corresponds to the $\delta K$-part of B, times $\sigma$. Approximation (3.20) may not be applied to the trace of B, to which $\delta K$ does not contribute, according to (3.19). From (3.20) and (3.19), and the definitions (3.14) and (3.17), it follows that in this limit, A becomes

$$A \sim -D\sigma[-\sigma(\mathrm{Tr}CD/\oint\gamma(s)ds) - C]^{-1} \sim [\oint\gamma(s)ds][\mathrm{Tr}CD]^{-1}D \quad . \tag{3.21}$$

Therefore, up to an additive constant,

$$\log P_0 = -\frac{1}{2}a_iA_{ij}a_j \sim -[\oint\gamma(s)ds][\oint ds\lambda^2(s)y_i(s)D_{ij}y_j(s)]^{-1} \cdot a_pD_{pq}a_q \quad . \tag{3.22}$$

Upon comparison with (B.14)-(B.16), we see that this is the desired result.

## IV. Steady State Near Noncoexistent Resonance

In this section, we shall determine both $P_0$ and $P_1$ because, as we shall see, $P_0$ alone gives only a very poor characterization of the approach to noncoexistent resonance. Indeed, we shall see that $P_0$ is completely independent of $\delta K$ in this case. Moreover,we shall also see that the integral of $P_0$ over the entire y-v plane is infinite, so that one cannot compute normalization even in principle without explicit reference to higher-order corrections.

It will be convenient in what follows to let $y_1$, in the definition of $a_1$, be the one periodic or antiperiodic solution of (3.3). It then follows from the constancy of the Wronskian of $y_1$ and $y_2$ that

$$y_2(t+T) = \pm[y_2(t) + \Delta y_1(t)] \quad , \tag{4.1}$$

for all t, where $\Delta$ is some nonzero constant, and where the signs plus and minus, respectively, correspond to periodic and antiperiodic $y_1$. The combinations $a_1$ and $a_2$ then satisfy

$$\begin{aligned} a_1(y,v,t+T) &= \pm a_1(y,v,t) \\ a_2(y,v,t+T) &= \pm[a_2(y,v,t) + \Delta a_1(y,v,t)] \quad . \end{aligned} \tag{4.2}$$

As in the preceding section, we conclude From (3. 1a) That (up to an additive constant that we may ignore without consequence) log $P_0$ takes the Form (3. 3), again with constant $A_{ij}$. Because of (4. 2), however, periodicity of $P_0$ now leads to the constraints

$$0 = A_{22} = \Delta A_{22} + A_{12} \quad . \tag{4.3}$$

We are thus left with

$$\log P_0 = -\frac{1}{2} A_{11} a_1^2 \quad . \tag{4.4}$$

The integral of $P_0$ over the entire y - v plane is then infinite because, according to (4.4), $P_0$ at any one time depends on only one linear combination of y and v.

In order to determine $A_{11}$, we begin, as in the preceding sections, by integrating (3.1b) with respect to time, with y and v replaced by a solution to (3.3) and its time derivative. Since we want to calculate $P_1$ as well, however, we shall leave the range of integration arbitrary, instead of restricting it to an interval of length an integral multiple of T, as we have done so far. The result is

$$\begin{aligned} & P_1[y(t'), \dot{y}(t'), t'] - P_1[y(t), \dot{y}(t), t] \\ & = P_0 \int_t^{t'} ds \{ \gamma(s) - \frac{1}{2} A_{11} \lambda^2(s) y_1^2(s) + a_1^2 A_{11} [ \frac{1}{2} A_{11} \lambda^2(s) y_1^2(s) \\ & - W^{-1} [\gamma(s) \dot{y}_2(s) + \delta K(s) y_2(s)] y_1(s) ] + a_1 a_2 A_{11} W^{-1} [\gamma(s) \dot{y}_1(s) \\ & \qquad + \delta K(s) y_1(s) ] y_1(s) \} \quad , \end{aligned} \tag{4.5}$$

where $a_1$ and $a_2$ are defined by substituting $y(t)$ and $\dot{y}(t)$ for y and v in (2.4), and W is as in (3.8). In order to proceed further, we require more detailed information concerning the structure of $P_1$. We reason as follows:

Since the Full P(y,v,t) is Gaussian, we expect that log P can be written as [F6]

$$\log P = \log N - \frac{1}{2} A_{11}a_1^2 - \frac{1}{2} a_i E_{ij}(t) a_j - F(t) + O(\gamma^2) \quad , \qquad (4.6)$$

where the (not necessarily constant) coefficients $E_{ij}(= E_{ji})$ and F are formally $O(\gamma)$. Thus we can write

$$P_1 = -\left[\frac{1}{2} a_i E_{ij} a_j + F\right] P_0 \quad . \qquad (4.7)$$

In view of (4.2), periodicity of $P_1$ means that

$$E_{11}(t) = E_{11}(t+T) + 2\Delta E_{12}(t+T) + \Delta^2 E_{22}(t+T) \quad , \qquad (4.8a)$$

$$E_{12}(t) = E_{12}(t+T) + \Delta E_{12}(t+T) \quad , \qquad (4.8b)$$

$$E_{22}(t) = E_{22}(t+T) \quad , \qquad (4.8c)$$

$$F(t) = F(t+T) \quad . \qquad (4.8d)$$

When (4.7) is substituted into (4.5), and then coefficients of identical monomials in $a_1$ and $a_2$ on the left and right sides are equated, the result is

$$-\frac{1}{2}[E_{11}(t') - E_{12}(t)] = A_{11}\int_t^{t'} ds[\frac{1}{2}A_{11}\lambda^2(s)y_1^2(s) - W^{-1}[\gamma(s)\dot{y}_2(s) + \delta K(s)y_2(s)]y_1(s)] \quad , \tag{4.9a}$$

$$-[E_{12}(t') - E_{12}(t)] = A_{11}W^{-1}\int_t^{t'} ds[\gamma(s)\dot{y}_1(s) + \delta K(s)y_1(s)]y_1(s) \ , \tag{4.9b}$$

$$E_{22}(t') - E_{22}(t) = 0 \quad , \tag{4.9c}$$

$$-[F(t') - F(t)] = \int_t^{t'} ds[\gamma(s) - \frac{1}{2}A_{11}\lambda^2(s)y_1^2(s)] \quad . \tag{4.9d}$$

In view of the periodicity constraint (4.8d), it follows from (4.9d), by setting t'=t+T, that

$$A_{11} = 2[\oint\gamma(s)ds][\oint\lambda^2(s)y_1^2(s)ds]^{-1} \quad . \tag{4.10}$$

As promised, this contains no reference to $\delta K$.

In view of (4.10) and the periodicity constraint (4.8b), it follows from (4.9b), upon setting $t'=t+T$, that the constant $E_{22}$ is given by

$$E_{22} = 2\Delta^{-1}[\oint\gamma(s)ds][\oint\lambda^2(s)y_1^2(s)ds]^{-1}W^{-1}\oint ds[\gamma(s)\dot{y}_1(s) + \delta K(s)y_1(s)]y_1(s) \quad . \tag{4.10}$$

Similarly, it follows from (4.8a),(4.9a),(4.10), and (4.11), that $E_{12}$ is given by

$$E_{12}(t) = \frac{-1}{2}\Delta^2 E_{22} + A_{11}\{\oint\gamma(s)ds - W^{-1}\int_{t-T}^{t} ds[\gamma(s)\dot{y}_2(s) + \delta K(s)y_2(s)]y_1(s)\} \tag{4.11}$$

$$= \Delta^{-1}[\oint\gamma(s)ds][\oint\lambda^2(s)y_1^2(s)ds]^{-1}\{\oint ds[2\gamma(s)+\Delta W^{-1}[\gamma(s)\dot{y}_1(s)+\delta K(s)y_1(s)]$$

$$\cdot\, y_1(s) - 2W^{-1}\int_t^{t+T} ds[\gamma(s)\dot{y}_2(s)+\delta K(s)y_2(s)]y_1(s)\} \quad . \tag{4.12}$$

The second equality in (4.12) follows from (4.1).

With (4.9a), (4.9d), and (4.10)-(4.12), we have a complete determination of $P_0$ and $P_1$, up to an inessential additive constant in F (it can be absorbed by normalization) and a significant additive constant in $E_{11}$.

In order to determine $E_{11}$ completely, we exploit the requirement that periodic $P_2$, be consistent with (3.1c). An economical way of doing this is as follows:

Begin, as before, by writing

$$P_2[y(t'),\dot{y}(t'),t']-P_2[y(t),\ \dot{y}(t),t] = \int_t^{t'} ds(L_1P_1)[y(s),\ \dot{y}(s),s] \quad . \tag{4.13}$$

Next observe that $P_2$, just like $P_1$, must be equal to $P_0$ times a polynomial in $a_1$ and $a_2$. Thus, when both sides of (4.13) are multiplied by $P_0^{-1}$, we have an equality between two polynomials. The zeroth-order part of this equality is

$$\chi(t')-\chi(t) = -\int_t^{t'} ds\{\gamma(s)F(s) + \tfrac{1}{2}\lambda^2(s)[y_i(s)E_{ij}(s)y_j(s) - A_{11}y_1^2(s)F(s)]\}, \tag{4.14}$$

where $\chi$ is the zeroth order monomial in $P_0^{-1}P_2$. Since periodicity of $P_2$ implies periodicity of $\chi$, it follows from (4.14) that

$$0 = \int_t^{t+T} ds\{\gamma(s)F(s) + \frac{1}{2}\lambda^2(s)[y_i(s)E_{ij}(s)y_j(s) - A_{11}y_1^2(s)F(s)]\} \quad . \tag{4.15}$$

Upon inserting (4.9a),(4.9d),(4.10)-(4.12) into (4.15), one obtains a complete determination of $E_{11}$.

We need not write out the solution in full. Let us record here only the part of (4.6) (ignoring normalization) that dominates when $\delta K$ is much larger than $\gamma$ and $\lambda^2$, but still small, in order to verify that we can recover the small-$\delta K$ form of the nonresonant exponent in (1.8). In this limit, we have, up to an additive constant,

$$-\frac{1}{2}A_{11}a_1^2-\frac{1}{2}a_iE_{ij}a_j-F--\left[\oint\gamma(s)ds\right]\left[\oint\lambda^2(s)y_1^2(s)ds\right]^{-1}W^{-1}$$

$$\cdot\left\{a_2^2\Delta^{-1}\oint\delta K(s)y_1^2(s)ds+a_1a_2\left[\oint\delta K(s)y_1^2(s)ds-2\Delta^{-1}\int_t^{t+T}\delta K(s)y_1(s)y_2(s)ds\right]\right.$$

$$+a_1^2\left[W+\left[\oint\lambda_1^2(s)y_1^2(s)ds\right]^{-1}\right.$$

$$\cdot\left[2\Delta^{-1}\int_t^{t+T}ds\lambda^2(s)y_1(s)y_2(s)\int_s^{s+T}ds'\delta K(s')y_1(s')y_2(s')\right.$$

$$-\oint\delta K(s)y_1^2(s)ds\int_t^{t+T}\lambda^2(s')y_2(s')\left[\Delta^{-1}y_2(s')+y_1(s')\right]ds'-$$

$$\left.\left.\left.-2\int_t^{t+T}ds\lambda^2(s)y_1^2(s)\int_t^s ds'\delta K(s')y_1(s')y_2(s')\right]\right]\right\} \tag{4.16}$$

Using the (anti)periodicity of $y_1$, as well as the identities

$$\int_s^{s+T}\delta K(s')y_2^2(s')ds'=\int_t^{t+T}\delta K(s')y_2^2(s')ds'$$

$$+2\Delta\int_t^s\delta K(s')y_1(s')y_2(s')ds'+\Delta^2\int_t^s\delta K(s')y_1^2(s')ds'\ , \tag{4.17}$$

$$\int_s^{s+T}\delta K(s')y_1(s')y_2(s')ds'=\int_t^{t+T}\delta K(s')y_1(s')y_2(s')ds'$$

$$+\Delta\int_t^s\delta K(s')y_1^2(s')ds'\ , \tag{4.18}$$

it is straightforward (if tedious) to establish that (4.16) is equivalent--to within a remainder of $O((\delta K)^2)$--to the result of substituting (B.24) and (B.25) into the exponent in (1.8), which is what we wanted to show.

## V. STRUCTURE OF $\lambda$

In real storage ring applications [1], the noise envelope $\lambda(t)$ is more properly written as a product $H(t)\eta(t)$, where $\eta$ ("dispersion," or "off-energy function") is the periodic solution of the differential equation

$$\ddot{\eta} + \gamma(t)\dot{\eta} + K(t)\eta = G(t) \quad , \tag{5.1}$$

and where neither H nor G (both periodic) is singular at resonance. An exact expression for the periodic solution to (5.1) is given in equation (A.12). One sees that this can be infinite whenever the tune $\nu$ approaches an integer.

In Appendix C we show that $\eta$ is generally $O(\gamma^{-1})$ when $\gamma$ is small, and when the identification (1.10) holds, and when the limitting tune at K is an integer, and when the scale of G has no formal connection with the scale of $\gamma$.

We are thus forced to conclude that in the case of near-integral tune, the calculations in the last two sections correspond to the limit $\gamma \to 0$, together with the formal identification (1.10), and with the formal identification

$$H(t)G(t') = O\,(\gamma^{3/2}) \quad , \tag{5.21}$$

for all t and t', which guarantees (1.11).

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## Appendix A: Exact Probability; Exact $\eta$

### 1. Steady State Probability

The general solution of (1.1) is

$$y(t) = y(0)Q(t,0) + \dot{y}(0)R(t,0) + \int_0^t ds R(t,s)\lambda(s)\zeta(s) \quad , \tag{A.1}$$

where Q and R satisfy

$$\left[\frac{d^2}{dt^2} + \gamma(t)\frac{d}{dt} + K(t)\right]\begin{Bmatrix} Q(t,s) \\ R(t,s) \end{Bmatrix} = 0 \quad , \tag{A.2}$$

with

$$Q(s,s) = 1 \qquad \frac{d}{dt}Q(t{=}s,s) = 0$$

$$R(s,s) = 0 \qquad \frac{d}{dt}R(t{=}s,s) = 1 \quad . \tag{A.3}$$

Since $\zeta$ is a Gaussian random variable, it follows [7] that when $y(0)$ and $\dot{y}(0)$ are fixed, then $y$ and $v \equiv \dot{y}$ are also Gaussian random variables, and they are distributed according to the probability density

$$(2\pi)^{-1}\ [\det Z]^{1/2} \exp -\frac{1}{2}\ \{Z_{11}[y-y(0)Q(t,0)\ -\ \dot{y}(0)R(t,0)]^2$$

$$+\ 2Z_{12}[y-y(0)Q(t,0)-\dot{y}(0)R(t,0)][v-y(0)\dot{Q}(t,0)-\dot{y}(0)\dot{R}(t,0)]$$

$$+\ Z_{22}[v-y(0)\dot{Q}(t,0)-\dot{y}(0)\dot{R}(t,0)]^2\}\quad , \tag{A.4}$$

where the symmetric matrix Z is defined by the quadratic form

$$x_i(Z^{-1})_{ij}x_i = \langle[x_1\int_0^t ds R(t,s)\lambda(s)\zeta(s) + x_2 \frac{d}{dt}\int_0^t ds R(t,s)\lambda(s)\zeta(s)]^2\rangle$$

$$= \int_0^t ds\lambda^2(s)[x_1 R(t,s) + x_2 \frac{\partial}{\partial t} R(t,s)]^2 \ . \tag{A.5}$$

The variables $x_1$ and $x_2$ are introduced here for formal purposes only. In deriving the second equality in (A.5), we have used (1.6).

We are especially concerned here with the limiting form to which Z, $\langle y\rangle$, and $\langle v\rangle$ relax as t becomes very large. For this purpose, it is desirable explicitly to decompose Q and R into damped and oscillating factors. A lengthy but straightforward calculation establishes that

$$R(t,s) = [\beta(t)\beta(s)]^{1/2}\sin\left(\int_s^t \frac{ds'}{\beta(s')}\right)\exp\left(-\frac{1}{2}\int_s^t \gamma(s')ds'\right) \quad , \tag{A.6}$$

and

$$Q(t,s) = \frac{1}{2}\gamma(s)\, R(t,s)$$

$$+ [\beta(t)/\beta(s)]^{1/2}\left\{\cos\left(\int_s^t \frac{ds'}{\beta(s')}\right) - \frac{1}{2}\dot{\beta}(s)\sin\left(\int_s^t \frac{ds'}{\beta(s')}\right)\right\}\exp\left(-\frac{1}{2}\int_s^t \gamma(s)ds\right) \quad , \tag{A.7}$$

where $\beta$ now corresponds as in (1.2) to the homogeneous undamped linear equation

$$\ddot{z} + z\left[K - \frac{1}{4}\gamma^2 - \frac{1}{2}\dot{\gamma}\right] = 0 \quad . \tag{A.8}$$

It follows from (A.6) and (A.7), using the periodicity of $\beta$ and $\gamma$, that as t becomes large, we have

$$\langle y\rangle = y(0)Q(t,0) + \dot{y}(0)R(t,0) \to 0 \quad ,$$

$$\langle v\rangle = y(0)\dot{Q}(t,0) + \dot{y}(0)\dot{R}(t,0) \to 0 \quad , \tag{A.9}$$

and also

$$<(x_1 y + x_2 v)^2> \to x_i (Z^{-1})_{ij} x_j \to$$

$$\frac{1}{2}\beta(t)\{[x_1 + \frac{1}{2} x_2(\frac{\dot\beta(t)}{\beta(t)} - \gamma(t))]^2 + [x_2/\beta(t)]^2\}[1-e^{-\oint\gamma(s)ds}]^{-1}$$

$$\cdot \int_{t-T}^{t} ds\lambda^2(s)\beta(s)e^{-\int_s^t \gamma(s')ds'}$$

$$+ \frac{1}{4}\beta(t)\{[(x_2/\beta(t))] - i\ (x_1 + \frac{1}{2} x_2(\frac{\dot\beta(t)}{\beta(t)} - \gamma(t))]^2[1-e^{2i\oint ds/\beta(s)-\oint\gamma(s)ds}]^{-1}$$

$$\cdot\int_{t-T}^{t} ds\lambda^2(s)\beta(s)\exp[2i\int_s^t \frac{ds'}{\beta(s')} - \int_s^t\gamma(s)ds']$$

$$+ \text{complex conjugate}\} \quad , \tag{A.10}$$

which is what we wanted to show.

2. Periodic Solution To (5.1)

As in the preceding subsection, the general solution to (5.1) is

$$\eta(t) = \eta(0)Q(t,0) + \dot\eta(0)R(t,0) + \int_0^t dsR(t,s)G(s) \quad . \tag{A.11}$$

Using the definitions of Q and R in (A.6) and (A.7), and the periodicity of $\beta, \gamma$, and G, it is a straightforward matter to show as t becomes large, the right hand side of (A.11) relaxes to the periodic limit

$$\eta(t)\to\left[(1-\xi)^2 + 4\xi\sin^2\pi\nu\right]^{-1}\beta^{1/2}(t)\int_{t-T}^{t}ds\ \beta^{1/2}(s)G(s)$$

$$\cdot\ \left\{(1-\xi) + 2\xi(\sin\pi\nu)\cos\left(\pi\nu - \int_s^t \frac{ds'}{\beta(s')}\right)\right\}\exp\left(-\frac{1}{2}\int_s^t\gamma(s'')ds''\right) \qquad (A.12)$$

$\left(\xi \equiv \exp - \oint\gamma(s)ds\right)$ , which is then the desired result.

Appendix B: I And $\beta$ Near Linear Resonance

In this Appendix we derive the leading terms in the expansion of $\beta(t)$, and of the quadratic form defined by the curved brackets in (1.8), in powers of $\delta K = K(t)-K_0(t)$, where $K_0$ is as in Sections III and IV.

For this purpose, we have found it convenient to calculate $\beta$ and I as follows: Let $y_1'(t)$ and $y_2'(t)$ be arbitrary linearly independent solutions F7 of (1.1); let W' be their Wronskian, as in (3.8); and let $a_1'$ and $a_2'$ be defined in terms of $y_1'$ and $y_2'$ as in (2.4). Then we can write

$$I = \left|\sin 2\pi\nu\right|^{-1}\left|W'\right|^{-1}\left|a_2'(y,v,t+T)a_1'(y,v,t)\right.$$

$$\left. - a_1'(y,v,t+T)a_2'(y,v,t)\right| \quad , \qquad (B.1)$$

and

$$\beta = \left|\sin 2\pi\nu\right|^{-1}\left|W'\right|^{-1}\left|y_2'(t+T)y_1'(t) - y_1'(t+T)y_2'(t)\right| \quad , \qquad (B.2)$$

where the tune $\nu$ is obtained from $y_1'$ and $y_2'$ according to

$$2\cos 2\pi\nu = (W')^{-1}\left[y_1'(t+T)\dot{y}_2'(t) - y_2'(t+T)\dot{y}_1'(t) - \dot{y}_1'(t+T)y_2'(t)\right.$$

$$\left. + \dot{y}_2'(t+T)y_1'(t)\right] \quad . \tag{B.3}$$

We need not derive these expressions here. One can easily verify them by using the explicit form (1.2); one can also derive them directly from the more abstract formalism explained in [3]. In either case, note that (B.2) is simply the coefficient of $v^2$ in (B.1), as is required by the explicit expression in (1.8).

1. Near Coexistent Resonance

In this case, our plan is pertubatively to construct solutions, $y_1'$ and $y_2'$, to (1.1) out of arbitrary solutions $y_1$ and $y_2$ to (3.3), and then to insert the results of this construction into the right-hand sides of (B.1)-(B.3).

Thus, we write

$$y_i'(t) = y_i(t) + y_{i1}(t) + y_{i2}(t) + \ldots \quad , \tag{B.4}$$

where each $y_{im}$ is $O\left((\delta K)^m\right)$. We shall be concerned here primarily with the first order corrections $y_{i1}$. Before we characterize these corrections in greater detail, however, let us first explain some simplifications of a general nature that result when an expansion of the form (B.4) is substituted into (B.1)-(B.3).

We begin with (B.3). When (B.4) is substituted into (B.3), we have

$$2\cos 2\pi\nu = \pm W^{-1}\{2W-2W_1 + (1-\frac{W_1}{W})[y_1(t)\dot{y}_{21}(t)$$

$$- y_2(t)\dot{y}_{11}(t) - \dot{y}_1(t)y_{21}(t) + \dot{y}_2(t)\, y_{11}(t) + (t\to t+T)]$$

$$- 2W_2 + 2(W_1^2/W) + [y_1(t)\dot{y}_{22}(t) - y_2(t)\dot{y}_{12}(t) - \dot{y}_1(t)y_{22}(t)$$

$$+ \dot{y}_2(t)y_{12}(t) + (t\to t+T)]\} + W^{-1}\{y_{11}(t+T)\dot{y}_{21}(t) - y_{21}(t+T)\dot{y}_{11}(t)$$

$$- \dot{y}_{11}(t+T)y_{21}(t) + \dot{y}_{21}(t+T)y_{11}(t)\} + O((\delta K)^3) \quad , \tag{B.5}$$

where $W$, $W_1$, and $W_2$ are the first three terms in the pertubative expansion of the Wronskian $W'$, and the plus and minus signs correspond, respectively, to periodic and antiperiodic $y_1$ and $y_2$. It follows from (B.5), together with the identities

$$W_1 = y_1\dot{y}_{21} + y_{11}\dot{y}_2 - y_2\dot{y}_{11} - y_{21}\dot{y}_1 \quad ,$$

$$W_2 = y_1\dot{y}_{22} + y_{12}\dot{y}_2 - y_2\dot{y}_{12} - y_{22}\dot{y}_1 + y_{11}\dot{y}_{21} - y_{21}\dot{y}_{11} \quad , \tag{B.6}$$

and with the fact that all terms in W must be constant in time, that the tune $\nu$ must satisfy

$$|\sin 2\pi\nu| = [W^{-1}\{[y_{11}(t+T) \mp y_{11}(t)][\dot{y}_{21}(t+T) \mp \dot{y}_{21}(t)]$$

$$- [y_{21}(t+T) \mp y_{21}(t)][\dot{y}_{11}(t+T) \mp \dot{y}_{11}(t)]\}]^{1/2} + O((\delta K)^2) \quad . \quad (B.7)$$

In addition, we also have

$$I = |W|^{-1}|\sin\pi\nu|^{-1}|a_2[y(\dot{y}_{11}(t+T) \mp \dot{y}_{11}(t)) - v(y_{11}(t+T) \mp y_{11}(t))]$$

$$-a_1[y(\dot{y}_{21}(t+T) \mp \dot{y}_{21}(t)) - v(y_{21}(t+T) \mp y_{21}(t))]| + O(\delta K) \quad , \quad (B.8)$$

and

$$\beta = |W|^{-1}|\sin 2\pi\nu|^{-1}|y_2(t)[y_{11}(t+T) \mp y_{11}(t)]$$

$$- y_1(t)[y_{21}(t+T) \mp y_{21}(t)]| + O((\delta K)) \quad , \quad (B.9)$$

where $a_1$ and $a_2$ are defined in terms of $y_1(t)$ and $y_2(t)$ as in (2.4).

In order to proceed further, we now need to characterize the $y_{i1}$ in greater detail. We begin by observing that each $y_{i1}$ must satisfy

$$\ddot{y}_{i1}(t) + K_0(t)y_{i1}(t) = - y_i(t)\delta K(t) \quad . \quad (B.10)$$

The general solution of (B.10) is

$$y_{i1}(t) = \mu_{ij}y_j(t) - W^{-1}\int_{t_0}^{t} ds\ y_i(s)\delta K(s)\left[y_1(s)y_2(t)-y_2(s)y_1(t)\right] \quad , \tag{B.11}$$

where the reference time $t_0$ is arbitrary, and the $\mu_{ij}$ are constants of integration. A complete determination of the $\mu_{ij}$ requires that we choose boundary conditions for the $y_{i1}$. This will not be necessary here, however, since in fact all reference to the $\mu_{ij}$ and to $t_0$ dissappears when (B.11) is substituted into (B.7)-(B.9). Indeed, it follows from (B.11) and the (anti) periodicity of the the $y_i$ that

$$y_{i1}(t+T) \mp y_{i1}(t) = \mp W^{-1}\oint ds y_i(s)\delta K(s)\left[y_1(s)y_2(t)-y_2(s)y_1(t)\right] \quad . \tag{B.12}$$

In view of (B.12), (B.7)-(B.9) are equivalent to

$$|\sin 2\pi\nu| = \left[\det D\right]^{1/2} + 0\left((\delta K)^2\right) \quad , \tag{B.13}$$

$$I = |W|^{-1}\left[\det D\right]^{-1/2}|a_i D_{ij} a_j| + 0\left((\delta K)\right) \quad , \tag{B.14}$$

$$\beta(t) = |W|^{-1}\left[\det D\right]^{-1/2}|y_i(t)D_{ij}y_j(t)| + 0\left((\delta K)\right) \quad , \tag{B.15}$$

where the symmetric matrix D is defined by

$$D = W^{-1} \oint ds\ \delta K(s) \begin{pmatrix} y_2^2(s) & -y_1(s)y_2(s) \\ -y_1(s)y_2(s) & y_1^2(s) \end{pmatrix} , \qquad (B.16)$$

Note that the second absolute value in (B.14) is actually a quadratic form in y and v if and only if $a_i D_{ij} a_j$ never changes sign as a function of $a_1$ and $a_2$, i.e. if and only if both eigenvalues of D have the same sign, i.e. detD≥0. Thus, as long as we assume that motion governed by $K_0 + \delta K$ is stable and nonresonant -- i.e. positive $\sin^2 2\pi\nu$ - det D -- then (B.14) is a self-consistent representation of a quadratic invariant. Positivity of detD also ensures that the absolute value signs in (B.14) and (B.15) can be deleted in forming the ratio $\beta/I$, which is required for the conclusion at the end of Section III.

2. Near Noncoexistent Resonance

Our plan in this case is the same as in the preceding subsection, except that now only $y_2$ is a completely arbitrary solution of (3.3). We require, as in Section IV, that $y_1$ be either periodic or antiperiodic.

We shall calculate $|\sin 2\pi\nu|$, $|\sin 2\pi\nu| I$, and $|\sin 2\pi\nu|\beta$ to the highest order in $\delta K$ determined completely by $y_1$ and $y_2$ and the first-order corrections to $y_1$ and $y_2$. This will be leading order for $|\sin 2\pi\nu|$, and next-to-leading order for $|\sin 2\pi\nu|$ I and $|\sin 2\pi\nu|\beta$. These results permit one to compute I and $\beta$ to leading order in $\delta K$, and $\beta/I$ to next-to-leading order. This will be sufficient to justify any remark, made elsewhere in this paper, that concerns noncoexistent resonance and that refers to this Appendix.

Proceeding as before, one can now show easily that

$$|\sin 2\pi\nu| = \{W^{-1}\Delta[y_1(t)[\dot{y}_{11}(t)-\dot{y}_{11}(t+T)]-\dot{y}_1(t)[y_{11}(t)-y_{11}(t+T)]]\}^{1/2}$$

$$\cdot\ \{1+O(\delta K)\} \quad , \tag{B.17}$$

and also

$$|\sin 2\pi\nu| I = |\Delta/W| [a_1{}^2(1-W^{-1}W_1)$$

$$+\ \Delta^{-1}a_2[y(\dot{y}_{11}(t) \mp \dot{y}_{11}(t+T)) - v(y_{11}(t) \mp y_{11}(t+T))]$$

$$-\ \Delta^{-1}a_1[y(\dot{y}_{21}(t) \mp \dot{y}_{21}(t+T)) - v(y_{21}(t) \mp y_{21}(t+T))]$$

$$+\ a_1[y\dot{y}_{11}(t) - vy_{11}(t)] + O[(\delta K)^2]] \quad , \tag{B.18}$$

and

$$|\sin 2\pi\nu| \beta(t) = |\Delta/W| [y_1^2(t)(1-W^{-1}W_1)$$

$$+\ \Delta^{-1}y_2(t)[y_{11}(t) \mp y_{11}(t+T)] - \Delta^{-1}y_1(t)[y_{21}(t) \mp y_{21}(t+T)]$$

$$+\ y_1(t)y_{11}(t) + O[(\delta K)^2]] \quad , \tag{B.19}$$

where $\Delta$ is as defined in (4.1), and $W_1$ is the first-order term $W'-W$.

Expression (B.11) for $y_{11}$ and $y_{21}$ implies that

$$W^{-1}W_1 = \mathrm{Tr}\mu = \mu_{11} + \mu_{22} \quad , \tag{B.20}$$

$$y_{11}(t) \mp y_{11}(t+T) = y_1(t)\Big[-\Delta\mu_{12} + \Delta W^{-1}\int_{t_0}^{t+T} ds\,\delta K(s) y_1^2(s)$$

$$- W^{-1}\int_t^{t+T} ds\,\delta K(s) y_1(s) y_2(s)\Big]$$

$$+ y_2(t) W^{-1}\oint ds\,\delta K(s) y_1^2(s) \quad , \tag{B.21}$$

$$y_{21}(t) \mp y_{21}(t+T) = y_1(t)\Big[-\Delta\mu_{22} + \Delta W^{-1}\int_{t_0}^{t+T} ds\,\delta K(s) y_1(s) y_2(s)$$

$$- W^{-1}\int_t^{t+T} ds\,\delta K(s) y_2^2(s)\Big] + y_2(t) W^{-1}\int_t^{t+T} ds\,\delta K(s) y_1(s) y_2(s) \ .$$

When (B.20)-(B.22) are substituted into (B.17)-(B.19), we obtain

$$|\sin 2\pi\nu| = \left[\Delta\oint ds\,\delta K(s) y_1^2(s)\right]^{1/2}\left[1 + O(\delta K)\right] \quad , \tag{B.23}$$

$$\begin{aligned}|\sin 2\pi\nu| I = |\Delta/W| \Big[ a_1^2 &+ \Delta^{-1} a_i \tilde{D}_{ij} a_j \\ &+ a_1 W^{-1}\left[a_2 \oint ds\,\delta K(s) y_1^2(s) - a_1 \int_t^{t+T} ds\,\delta K(s) y_1(s) y_2(s)\right] \\ &+ O\left((\delta K)^2\right)\Big] \quad ,\end{aligned} \tag{B.24}$$

$$\begin{aligned}|\sin 2\pi\nu| \beta(t) = |\Delta/W| \Big[ y_1^2(t) &+ \Delta^{-1} y_i(t) \tilde{D}_{ij} y_j(t) \\ &+ y_1(t) W^{-1}\left[y_2(t) \oint ds\,\delta K(s) y_1^2(s) - y_1(t) \int_t^{t+T} ds\,\delta K(s) y_1(s) y_2(s)\right] \\ &+ O\left((\delta K)^2\right)\Big] \quad ,\end{aligned} \tag{B.25}$$

where the matrix $\tilde{D}$ is defined as in (B.16), but with $\oint$ replaced by $\int_t^{t+T}$.

## Appendix C: $\eta$ Near Resonance

In this Appendix we determine the leading term in the formal expansion of $\eta$

$$\eta = \eta_{-1} + \eta_0 + \eta_1 + \ldots \tag{C.1}$$

in powers of $\gamma$, assuming the identification $\delta K = K - K_0 = O(\gamma)$ , as in Sections III and IV and Appendix B. Each term in this expansion is to be periodic in time, with period T.

Since the exact expression for $\eta$, (A. 12), becomes sngular only when the tune $\nu$ approaches an integer, we assume in what follows that $K_0$ generates resonant dynamics of the periodic kind.

The perturbative hierarchy into which (5.1) decomposes when (C.1) is introduced is

$$\ddot{\eta}_{-1} + K_0(t)\,\eta_{-1} = 0 \quad , \tag{C.2}$$

$$\ddot{\eta}_0 + K_0(t)\eta_0 = -\gamma(t)\dot{\eta}_{-1} - \delta K(t)\eta_{-1} + G(t) \quad . \tag{C.3}$$

1. Near Coexistent Resonance

In this case, the general periodic solution to (C.2) is

$$\eta_{-1}(t) = c_1 y_1(t) + c_2 y_2(t) \quad , \tag{C.4}$$

where the $c_i$ are constants, and the $y_i$ are as in Section III. The $c_i$ are determined by the requirement that (C.3) be consistent with periodic $\eta_0$, as follows:

The general solution of (C.3), for $\eta_0$, is

$$\eta_0(t) = d_1 y_1(t) + d_2 y_2(t) + W^{-1}\int_{t_0}^{t} ds\left[G(s)-\gamma(s)\dot{\eta}_{-1}(s)-\delta K(s)\eta_{-1}(s)\right] \cdot \left[y_1(s)y_2(t) - y_2(s)y_1(t)\right] \quad , \tag{C.5}$$

where the reference time $t_0$ is arbitrary, and where the $d_i$ are constants of integration. W is as defined in (3.8). This is periodic with period T if and only if $\eta_0(t_0+T) = \eta_0(t_0)$ and $\dot{\eta}_0(t_0+T) = \dot{\eta}_0(t_0)$. This is equivalent to

$$0 = \oint ds\left[G(s) - \gamma(s)\dot{\eta}_{-1}(s)-\delta K(s)\eta_{-1}(s)\right] \begin{Bmatrix} y_1(s) \\ y_2(s) \end{Bmatrix} \quad . \tag{C.6}$$

It follows from (C.6) that

$$c_i = W^{-1}(\sigma B)^{-1}_{ij}\oint dsG(s)y_j(s) \quad , \tag{C.7}$$

where B and $\sigma$ are the matrices defined in (3.6) and (3.13). We shall not comment on the conditions under which B cannot be inverted.

2. Near Noncoexistent Resonance

In this case, the general periodic solution to (C.2) is

$$\eta_{-1}(t) = cy_1(t) \quad , \tag{C.8}$$

where $y_1$ is as in Section IV. Periodicity of $\eta_0$ now requires that

$$0 = d_2\Delta y_1(t_0+T) + W^{-1}\int_{t_0}^{t_0+T} ds\left[G(s)-\gamma(s)\dot{\eta}_{-1}(s)-\delta K(s)\eta_{-1}(s)\right]$$

$$\cdot \left[y_1(s)y_2(t_0+T) - y_2(s)y_1(t_0+T)\right] \quad ,$$

$$0 = d_2\Delta\dot{y}_1(t_0+T) + W^{-1}\int_{t_0}^{t_0+T} ds\left[G(s)-\gamma(s)\dot{\eta}_{-1}(s)-\delta K(s)\eta_{-1}(s)\right]$$

$$\cdot \left[y_1(s)\dot{y}_2(t_0+T)-y_2(s)\dot{y}_1(t_0+T)\right] \quad . \tag{C.9}$$

These two equations provide a consistent determination of $d_2$ as long as

$$0 = \oint ds\left[G(s) - \gamma(s)\dot{\eta}_{-1}(s) - \delta K(s)\eta_{-1}(s)\right]y_1(s) \quad . \tag{C.10}$$

It follows from (C.10) that c is given by

$$c = \left[\oint ds\left(\gamma(s)\dot{y}_1(s) + \delta K(s)y_1(s)\right)y_1(s)\right]^{-1}\oint dsG(s)y_1(s) \quad . \tag{C.11}$$

According to the analysis in Appendix B, the inverse in (C.11) is nonsingular when system (A.8) is nonresonant in linear order in $\gamma$ and $\delta K$.

## FOOTNOTES

F1. Strictly speaking, (1.1) actually describes only one contribution--the betatron oscillation--to the displacement. It also omits terms that couple horizontal and vertical motion. For simplicity, we shall ignore such complications here. We shall also ignore oscillations along the beam direction.

F2. We shall not consider K (or, where equation (A.8) is more appropriate, $K-(1/2)\dot{\gamma}-(1/4)\gamma^2$) that can lead to explosive growth.

F3. The assumption of centered Gaussian noise is made here for convenience. Strictly speaking, the right-hand side of (1.5) should more properly be defined by a Poisson-like process

$$\sum_i \delta(t-t_i) \cdot u_i \ ,$$

where the $t_i$ and $u_i$ are both random variables, and $\langle u_i \rangle$ is not zero [1]. If this were analyzed according to the general framework developed in Section I. 4. d of [4], one would find, upon comparison with the calculation in our Appendix A, that the steady-state distribution is the same--up to a translation--as the steady-state distribution corresponding to (1.5), as long as $\lambda(t)$ is related to the (periodic) density $\rho(t)$ of the time variables $t_i$, and the (periodic) mean square value of the $u_i$ at time t, according to

$$\lambda^2(t) = \rho(t)\langle u^2 \rangle(t) \quad .$$

Thus, the theory in the body of the present paper actually refers to the distribution of the variables y and v about their (periodically time-dependent) means. These means themselves are given, respectively, by the periodic solution to

$$\ddot{y} + \gamma(t)\dot{y} + K(t)y = \rho(t)\langle u\rangle(t) \ ,$$

and its time derivative. These are small and nonsingular as long as the system is far from linear resonance, as is clear from equation (A.12). The leading behavior of these mean values near resonance can be obtained using the method described in Appendix C. For convenience, we have also idealized the precise way in which noise drives the damped oscillator in (1.5). In a real storage ring, the coupling to noise actually takes the form (within the Gaussian assumption, for simplicity).

$$\dot{p} + \gamma p + Ky = \lambda\zeta \ ,$$

$$\dot{y} - p = \lambda'\zeta \ ,$$

where $\lambda'$ is in general not zero, as would be implied by (1.5). When a nonzero $\lambda'$ is present, the right-hand side of (1.7) should be augmented by $1/2 \ (\lambda')^2\partial^2P/\partial y^2 + \lambda\lambda' \ \partial^2P/\partial y\partial v$. The analysis described in this paper is easily extended to this more general setting.

F4. Equation (1.1) cannot have one solution periodic in time T and one solution antiperiodic in time T, because the Wronskian of two such functions could not be constant and non-zero, as the Wronskian of two distinct solutions to (1.1) must be.

F5. Because we are concerned here only with stable systems, and therefore with normalizable probabilities, we proceed here as if A is generally invertible. If one wishes not to make this assumption a priori, one can supplement our analysis of (3.11) as follows: (3.11) implies that $(B+AC)A=b\sigma$, where b is a number, and $\sigma$ is as in (3.13). Either $b\neq 0$, in which case A is invertible and we can proceed as in the text, or $b=0$. In the latter instance, either both eigenvalues of A are nonzero, in which case A is again invertible, or at last one eigenvalue of A vanishes. In the latter case, in an orthogonally rotated basis in which $A=\mathrm{diag}\,(w,0)$, nonzero A and $(B+AC)A=0$ imply $B_{21}=0$, which is only possible when $(\mathrm{Tr}B)^2-4\det B\gtrless 0$. According to (3.6) and (B.13), this mean that (A.8) is unstable to linear order in $\gamma$ and $\delta K$. In this case the general solution (1.2) must be replaced by one involving hyperbolic functions. Although this line of thought can be developed further, the details are not especially illuminating, and we shall not pursue them beyond this point.

F6. Note that when P satisfies (4.6), then the condition that the integral $\int P dy dv$ be unity implies that $N\sim(2\pi)^{-1}W[A_{11}E_{22}]^{1/2}=O(\gamma^{1/2})$ in the limit $\gamma\to 0$, according to the identifications (1.10) and (1.11).

F7. In what follows, we shall need to distinguish solutions of (1.1) from solutions of (3.3). Thus, in this Appendix we denote solutions of (1.1) with primes. Solutions of (3.3) will carry no special marking.

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