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Weak-Coupling Analysis of the Supersymmetric Liouville Theory*

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ABSTRACT

A weak-coupling expansion is developed for the supersymmetric Liouville quantum field theory defined on a circle. The zero-mode system (supersymmetric quantum mechanics with an exponential potential) is first solved exactly, and then nonzero-mode effects are incorporated as perturbations. The theory is translationally invariant, conformally covariant, and has a zero-energy ground state with $\langle\bar{\psi}\psi\rangle \neq 0$.

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The supersymmetric Liouville theory is a two-dimensional model of real scalar (ϕ) and Majorana spinor (ψ) fields with exponential interactions. In superfield form, the classical action is given by

$$A = \int d^2\theta d^2x\left(\frac{1}{2}\, d_R\Phi d_L\Phi - \frac{2Mi}{g^2}\, e^{g\Phi}\right) \tag{1}$$

with

$$\Phi = \phi + \overline{\theta}\psi + \frac{1}{2}\,\overline{\theta}\theta F \tag{2}$$

and

$$d_{R,L} = \frac{\partial}{\partial\theta_{R,L}} - i\theta_{R,L}\,(\partial_\tau \pm \partial_\sigma) \tag{3}$$

Here $\theta \equiv \begin{pmatrix}\theta_R \\ \theta_L\end{pmatrix}$ is the usual Majorana Grassman variable with $\theta_R^2 = \theta_L^2 = \{\theta_R, \theta_L\} = 0$. Our other conventions are: $\gamma^0 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$; $\gamma^1 = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}$; $\overline{\theta} = i(\theta_L, -\theta_R)$; $d^2\theta = d\theta_R d\theta_L$; $x^\mu = (\tau, \sigma)$; $g^{\mu\nu} = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$. Note that the auxiliary field, F, may be eliminated from (1) since it only appears quadratically. The result is $F = -\frac{2M}{g}\, e^{g\phi}$.

The model defined by (1) was proposed in Ref. [1] as a supersymmetric extension of the pure Liouville theory (obtained by setting ψ=0) which would maintain the latter's complete classical integrability. More recently the model has been extensively analyzed [2-7] as a result of the Polyakov approach [8] to the covariant quantization of the spinning string. A goal of this analysis (as yet unachieved) is to compute quantum correlation functions involving arbitrary exponentials of superfields, for all values of the coupling constant, g.

In this paper we shall consider the weak-coupling limit, $g \to 0$, of the supersymmetric Liouville theory defined on a circle. We shall extend to the supersymmetric case some results which were previously obtained [9] for the pure Liouville theory.

We show that a weak-coupling expansion can be defined for the supersymmetric theory by first solving the zero-mode problem exactly and then perturbing in nonzero modes. Within this framework, we compute the first nontrivial corrections to matrix elements of $\exp(\alpha g \Phi)$ between states whose energies are $O(g^2)$. We also check the conformal covariance of the model using perturbation theory, and confirm some results obtained from a formal operator analysis.

Consider the model in component field form. In that form, a close analogue of the Hamiltonian analysis of the pure Liouville theory, as given in [9], is obtained through the use of the supercurrent. Classically, the supercurrent is

$$S_\mu = (i\partial\!\!\!/\phi - \frac{2M}{g} e^{g\phi})\gamma_\mu \psi + iC[\gamma_\mu, \gamma_\nu]\partial^\nu \psi \tag{4}$$

where the last term is a conformal improvement. The classical supercurrent is traceless on-shell, i.e. $\gamma . S = 0$, if the improvement coefficient is $C = 1/g$. For the quantized theory, we anticipate that C will require corrections, as is true of the conformal improvement coefficient for the pure Liouville theory [10]. Indeed, we shall argue below that

$$C = \frac{1}{g}\left(1 + \frac{g^2}{4\pi}\right) \tag{5}$$

is necessary for the conformal covariance of the model.

We now quantize the supersymmetric Liouville theory on a periodic spatial interval (i.e. a circle), $0 \leq \sigma \leq 2\pi$, using operator methods. At time $\tau = 0$, we define canonical ϕ and ψ fields using the mode expansions

$$\phi(\sigma) = q + \frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{1}{n} \left(a_n e^{-in\sigma} + b_n e^{in\sigma}\right) \tag{6a}$$

$$\pi(\sigma) = \frac{p}{2\pi} + \frac{1}{\sqrt{4\pi}} \sum_{n\neq 0} \left(a_n e^{-in\sigma} + b_n e^{in\sigma}\right) \tag{6b}$$

$$\psi(\sigma) = \begin{pmatrix}\psi_R(\sigma)\\ \psi_L(\sigma)\end{pmatrix}, \quad \overline{\psi}(\sigma) = i\left(\psi_L(\sigma), - \psi_R(\sigma)\right) \tag{7a}$$

$$\psi_R(\sigma) = \frac{1}{\sqrt{4\pi}}\, r + \frac{1}{\sqrt{2\pi}} \sum_{n\neq 0} e^{in\sigma} u_n \tag{7b}$$

$$\psi_L(\sigma) = \frac{1}{\sqrt{4\pi}}\, \ell + \frac{1}{\sqrt{2\pi}} \sum_{n\neq 0} e^{-in\sigma} d_n . \tag{7c}$$

All these operators are by construction periodic in σ: $\phi(\sigma) = \phi(\sigma+2\pi)$, etc. One may also contemplate ψ's which are antiperiodic in σ (cf. [5] and references therein). However, we shall not discuss that possibility here.

The bosonic mode operators, $\{q,p,a_n,b_n\}$, satisfy the usual commutation relations (cf. [10]) such that $[\phi(\sigma),\pi(\sigma')] = i\delta(\sigma-\sigma')$. The fermionic nonzero-mode operators satisfy the anticommutation relations

$$\{u_n, u_k\} = \delta_{n+k,0} = \{d_n, d_k\} \tag{8}$$

The zero-mode fermionic operators satisfy

$$r^2 = \ell^2 = 1, \quad \{r,\ell\} = 0 \tag{9}$$

Thus we have $\{\psi_R(\sigma),\psi_R(\sigma')\} = \delta(\sigma-\sigma') = \{\psi_L(\sigma),\psi_L(\sigma')\}$. All other anti-commutators vanish. In addition the fermionic mode operators commute with the bosonic operators.

Next we define the supercurrent operator as in Eq. (4), but with $e^{g\phi}$ replaced by $:e^{g\phi}:$, where colons denote normal-ordering with respect to the a_n and b_n operators, as in [10]. It turns out that this simple normal-ordering prescription leads to a calculational scheme (at least for weak-coupling) which is ultraviolet finite.

It is also convenient to take spatial Fourier transforms of the left- and right-handed projections of the supercurrent operator. So we define

$$G_N^{\pm} = \frac{1}{2}\int_0^{2\pi} d\sigma\ e^{\pm iN\sigma}\Big((\pi(\sigma)\pm \frac{\partial}{\partial\sigma}\ \phi(\sigma))\psi_{L,R}(\sigma)\ \mp\ 2C\,\frac{\partial}{\partial\sigma}\,\psi_{L,R}(\sigma)$$

$$\mp\ \frac{2M}{g}\ :e^{g\phi(\sigma)}:\ \psi_{R,L}(\sigma)\Big). \tag{10}$$

The usual supercharges are obtained for N = 0, for which case we write

$$Z^{\pm} = Z_0^{\pm} + Z_1^{\pm} \tag{11a}$$

$$Z_0^{\pm} = z^{\pm} + R^{\pm} \tag{11b}$$

$$\binom{z^+}{z^-} = p\ \binom{\ell}{r}\ \mp\ \frac{4\pi M}{g}\ e^{gq}\ \binom{r}{\ell} \tag{11c}$$

$$\binom{R^+}{R^-} = \sqrt{8\pi}\ \sum_{n\neq 0}\ \begin{pmatrix} a_n\ d_{-n} \\ b_n\ u_{-n}\end{pmatrix} \tag{11d}$$

$$Z_1^{\pm} = \mp\ \frac{2M}{g}\ \sqrt{4\pi}\ \int_0^{2\pi} d\sigma\ \left(:e^{g\phi(\sigma)}:\ -\ e^{gq}\right)\ \psi_{R,L}(\sigma) \tag{11e}$$

Hence $Z_1^{\pm}$ represents all interactions involving nonzero-modes.

A weak-coupling expansion is now defined by perturbatively constructing exact eigenstates of Z^+ or Z^-, treating $Z_1^{\pm}$ as the perturbation. This is analogous to constructing exact eigenstates of $H \pm P$ for the pure Liouville theory using the Lippman-Schwinger formalism (cf. [9]). Consider the case of Z^-. We have

$$Z^- \,\| k\rangle = gk \,\| k\rangle \tag{12}$$

where

$$\begin{aligned} \| k\rangle &\equiv |k\rangle + \frac{1}{gk - Z^- + i\varepsilon} Z_1^- \, |k\rangle \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{gk - Z_0^- + i\varepsilon} Z_1^-\right)^n |k\rangle \end{aligned} \tag{13}$$

Here we have chosen $|k\rangle$ to be a nonzero-mode vacuum, but a nontrivial zero-mode eigenstate. Thus

$$R^-|k\rangle = 0, \; R^+|k\rangle = 0, \tag{14a}$$

$$z^-|k\rangle = gk|k\rangle, \; -\infty<k<\infty, \tag{14b}$$

$$Z_0^-|k\rangle = gk|k\rangle. \tag{14c}$$

Note the spectrum of z^- is continuous, nondegenerate, and includes $k = 0$. Note also that it was not necessary to choose $|k\rangle$ to be a nonzero-mode vacuum. This was done only to simplify the analysis to follow.

The zero-mode solutions in (14b) may be explicitly obtained in the position representation, where $p = -i \frac{d}{dq}$ and where

$\mathcal{r} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $i\ell\mathcal{r} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We find

$$\langle q|k\rangle = \left(\frac{x \cosh \pi k}{2\pi^2}\right)^{1/2} \begin{pmatrix} K_{\frac{1}{2} - ik}(x) + K_{\frac{1}{2} + ik}(x) \\ iK_{\frac{1}{2} - ik}(x) - iK_{\frac{1}{2} + ik}(x) \end{pmatrix} \tag{15}$$

where K_ν is a modified Bessel function and $x \equiv \frac{4\pi M}{g^2} e^{gq}$. The normalization of the states is as appropriate for a continuum with

$$\langle k'|k''\rangle = \frac{1}{g}\, \delta(k'-k''). \tag{16}$$

Note that the $|k=0\rangle$ state is particularly simple in this representation.

$$\langle q|k=0\rangle = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(- \frac{4\pi M}{g^2} e^{gq}\right). \tag{17}$$

The state $|k=0\rangle$ is in fact the ground state of the unperturbed system governed by Z_0^-, in the usual sense that it has lowest energy (zero) and it is translationally invariant (with zero momentum). This can be understood from

$$\frac{1}{4\pi} (Z_0^-)^2 = h + 2 \sum_{n=1}^{\infty} (b_{-n}b_n + nu_{-n}u_n) \tag{18}$$

with

$$h \equiv \frac{1}{4\pi} (z^-)^2 = \frac{p^2}{4\pi} + \frac{4\pi M^2}{g^2} e^{2gq} + Mi\ell\mathcal{r} e^{gq}. \tag{19}$$

The nonzero-mode operators in (18) annihilate $|k\rangle$, by construction, while $h|k\rangle = \frac{1}{4\pi} g^2k^2|k\rangle$. The operator h in (19) is easily seen to be the

Hamiltonian for a supersymmetric quantum mechanical model with an exponential interaction, as previously discussed in [11]. The supersymmetric model has a zero-energy solution, unlike the pure Liouville theory, because $i\ell\hbar|k = 0\rangle = -1|k = 0\rangle$, and the last potential term in h is therefore attractive. Since $\bar{\psi}\psi = \frac{1}{2\pi} i\ell\hbar +$ nonzero modes, another statement of the above is that the ground state of the supersymmetric Liouville theory on a circle consists of a negative fermion-pair condensate.

The exact ground state $\|k=0\rangle$ defined by the series in Eq. (13) also has the above properties of translational invariance (zero momentum) and zero energy. This follows from the usual definition of the momentum operator.

$$P = \sum_{n=1}^{\infty} (b_{-n}b_n - a_{-n}a_n) + n(u_{-n}u_n - d_{-n}d_n), \tag{20}$$

which commutes with $Z_0^{\pm}$ and $Z_1^{\pm}$, and from the fact that the Hamiltonian for the full theory, H, is defined for supersymmetric models by

$$H \mp P = \frac{1}{4\pi}(Z^{\pm})^2. \tag{21}$$

Note that H + P is trivially finite when acting on the states defined by (13).

Next we consider the effects of the Z_1 terms in (13) by examining specific operator matrix elements, as was done for the pure Liouville theory in [10]. We have calculated $\langle k''\|e^{\alpha g\Phi}\|k'\rangle$ to lowest nontrivial order in the limit $g \to 0$, with k" and k' fixed. The basic methods are straightforward extensions of those in [10] and only the results of the calculation will be given. It is necessary to normal-order the components of $e^{\alpha g\Phi}$ to remove ultraviolet divergences. We find

$$\langle k''\| :e^{\alpha g\phi}: \|k'\rangle = e^{-i\chi(k'')} \langle k''|e^{\alpha gq}|k'\rangle e^{i\chi(k')} f(\alpha;k'',k') \tag{22}$$

$$\langle k''\| \psi_R :e^{\alpha g\phi}: \|k'\rangle = e^{-i\chi(k'')} \langle k''|\frac{r}{\sqrt{4\pi}} e^{\alpha gq}|k'\rangle e^{i\chi(k')} f(\alpha;k'',k') \tag{23}$$

$$\langle k''\| \psi_L :e^{\alpha g\phi}: \|k'\rangle = e^{-i\chi(k'')} \langle k''|\frac{\ell}{\sqrt{4\pi}} e^{\alpha gq}|k'\rangle e^{i\chi(k')} f(\alpha;k'',-k') \tag{24}$$

where the phase $\chi(k)$ is

$$\chi(k) = \frac{7g^6\zeta_3}{768\pi^3}(k + 4k^3) + O(g^8) , \quad \zeta_3 = \sum_{n=1}^{\infty} \frac{1}{n^3} , \tag{25}$$

and where f is a polynomial in α,k", and k' when calculated to any finite order. For example,

$$f(\alpha;k'',k') = 1 - \frac{g^6\zeta_3}{256\pi^3}\Big[4(k''^2-k'^2)^2 + (\alpha-1)(4\alpha+3)(k''-k')^2$$

$$+ \alpha(4\alpha-7)(k''+k')^2 + \alpha(\alpha-1)(7+16\alpha+12\alpha^2)/3\Big] + O(g^8). \tag{26}$$

It is necessary to consider terms up to and including n=3 in the series of Eq. (13) in order to obtain these results.

The net effect of nonzero-mode perturbations on the matrix elements in (22-24) is to multiply the zero-mode (supersymmetric QM) matrix elements by simple polynomials. Exactly the same effect appeared in the pure Liouville case [10]. Also, the arbitrariness of the $i\varepsilon$ prescription in Eq. (13) is completely expressed by the phases $\chi(k'')$ and $\chi(k')$ in Eqs. (22-24), since the QM matrix elements are real.

Note that the structure of the polynomial in (26), in particular the presence of α and $(\alpha-1)$ factors, implies that matrix elements of the spinor field obey the appropriate equations of motion. That is,

$$i\partial\!\!\!/\psi = 2M :e^{g\phi}: \psi \tag{27}$$

The change of sign of k' in the polynomials in Eqs. (23) and (24) is crucial to verify Eq. (27) for matrix elements of that equation. Also, the results Eqs. (22) and (23) are compatible with the supersymmetry transformation obtained from using canonical commutation relations, $[Z^-, :e^{\alpha g\phi}:] = -i\alpha g\sqrt{4\pi}\ \psi_R :e^{\alpha g\phi}:$.

Next we comment on matrix elements of the $\bar{\theta}\theta$ component of the superfield $e^{\alpha g\Phi}$. To be compatible with supersymmetry (cf. $\{Z^-, \psi_L :e^{\alpha g\phi}:\}$) we must obtain

$$\langle k''\| :e^{\alpha g\phi}: (\tfrac{1}{2}\alpha g\bar{\psi}\psi - F)\|k'\rangle$$

$$= e^{-i\chi(k'')}\langle k''|e^{\alpha gq}(\frac{1}{4\pi}\alpha g i\ell h + \frac{2M}{g} e^{gq})|k'\rangle e^{i\chi(k')} f(\alpha;k'',-k') \tag{28}$$

and we indeed find this result. However, some care is needed in removing ultraviolet divergences to obtain (28) by direct calculation. The problem is that $\langle k''\| :e^{\alpha g\phi}: \bar{\psi}\psi\|k'\rangle$ and $\langle k''\| :e^{\alpha g\phi}: F\|k'\rangle$ are not separately UV finite. A proper definition of $:e^{\alpha g\phi}: F$ must be given to cancel the UV divergence in $:e^{\alpha g\phi}: \bar{\psi}\psi$ and subsequently give the finite result in (28). We have found that the naive choice, $:e^{\alpha g\phi}: F = -\frac{2M}{g} :e^{\alpha g\phi}: :e^{g\phi}:$, with the mode sums for ϕ and ψ cut-off symmetrically, is <u>not</u> an acceptable definition. A prescription which does yield (28) is to symmetrically cut-off the mode sums for ϕ, ψ, <u>and</u> F. Hence F differs from $:e^{g\phi}:$ at order g^2 and beyond [12]. This acceptable cut-off treats all components of the superfield Φ on a more equal footing, and for this reason we believe it respects the supersymmetry of the model.

We also note that Eqs. (22) and (28) imply that these matrix elements are consistent with the bosonic equations of motion

$$\partial^2\phi = M :e^{g\phi}: (2F - g\bar{\psi}\psi), \tag{29}$$

where the RHS is defined as the limit as the aforementioned cut-off is removed.

Finally, we consider the conformal covariance of the supersymmetric Liouville theory. A formal, canonical operator calculation leads to the super conformal algebra [13]

$$\{G_k^{\pm}, G_n^{\pm}\} = L_{k+n}^{\pm} + (\tfrac{1}{4} + 2\pi C^2)\ k^2\ \delta_{k+n,0} \tag{30a}$$

$$[G_k^{\pm}, L_n^{\pm}] = (k - \tfrac{1}{2} n)\ G_{k+n}^{\pm} \tag{30b}$$

$$[L_k^{\pm}, L_n^{\pm}] = (k-n)\ L_{k+n}^{\pm} + (\tfrac{1}{8} + \pi C^2)\ k^3\ \delta_{k+n,0} \tag{30c}$$

where $L_n^{\pm}$ is defined as the spatial Fourier transform of the conformally improved [10] energy-momentum tensor, $L_n^{\pm} = \frac{1}{2}\int d\sigma\ e^{\pm in\sigma}\ (T_{00} \pm T_{01})$. To arrive at Eq. (30), one may naively take $F = -\frac{2M}{g} :e^{g\phi}:$ and ignore cut-off subtleties. One then finds that the algebra does not close unless the condition in Eq. (5) is imposed. A careful investigation of cut-off schemes which preserve the algebra in (30) is in progress [14].

Here we report on a check of the algebra Eq. (30), and the relation in Eq. (5), using the weak-coupling analysis defined above. An immediate consequence of Eq. (30b) is that

$$G_N^{\pm} \| k\rangle = 0, \text{ for } N > 0, \tag{31}$$

or else the positivity of the Hamiltonian (cf. Eq. (21)) for the supersymmetric Liouville theory would fail. To test this, we compute directly for $N > 0$

$$\begin{aligned} &\langle k''| :e^{\alpha g\phi}: G_N^{-} \| k'\rangle \\ &= \frac{\alpha g M}{N}\left\{\left(1 + \frac{g^2}{4\pi} - gC\right) + \frac{\alpha g^2}{4\pi}\left(\frac{1}{N} + \sum_{n>N}\frac{N}{n^2} - \sum_{n=1}^{N}\frac{2}{n}\right)(gC-1)\right. \\ &\qquad\left. + O(g^5)\right\} \langle k''|e^{(\alpha+1)gq}\ell|k'\rangle \\ &+ \frac{\alpha g M}{2N^2}\left\{(gC-1) + O(g^2)\right\} \langle k''|e^{\alpha gq}[h,\ell e^{gq}]|k'\rangle \end{aligned} \tag{32}$$

To the order calculated (again, up to and icluding n=3 in Eq. (13)) we find that the matrix element vanishes as expected, given Eq. (5). Thus the weak-coupling expansion reveals the conformal covariance of the model, as was also the case for the pure Liouville theory [10].

In conclusion, we have developed a weak-coupling expansion for the supersymmetric Liouville theory which preserves translational invariance and conformal covariance, and which exhibits the spectrum of the theory. It remains to develop an exact quantum solution to the model, and to find strong coupling approximations.

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