Fermi National Accelerator Laboratory

FERMILAB-Pub-83/60-THY
July, 1983

The ρ-$\pi\pi$ Coupling Constant in Lattice Gauge Theory

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ABSTRACT

We present a method for studying hadronic transitions in lattice gauge theory which requires computer time comparable to that required by recent hadron spectrum calculations. This method is applied to a calculation of the decay $\rho\to\pi\pi$.

PACS Numbers: 12.40.Bb, 13.25.+m.

The valence [1] (quenched [2]) approximation to lattice QCD has been used so far in the calculation of hadron masses and decay constants [1,2,3], magnetic moments [4], and the properties of QCD at finite temperature [5]. A qualitative argument [1] suggests that this approximation may be fairly reliable for the properties of flavor non-singlet hadrons, and a quantitative estimate [6] implies that at least for the calculation of masses at inverse lattice spacing up to 1000 MeV, the valence approximation introduces errors of less than 20±20%. In the present article we use the valence approximation to evaluate the $\rho\pi\pi$ coupling constant.

The most important result of our calculation of $g_{\rho\pi\pi}$ is a demonstration that, using our method, the investigation of hadron decay processes can be carried out in reasonable amounts of time on presently available computers.

The prediction we obtain for $g_{\rho\pi\pi}$ is 3.0±1.0. The experimental value of $g_{\rho\pi\pi}$ is 6.11±0.11. The quoted error on our result is purely statistical. It was obtained by dividing our data into subensembles and reevaluating quantities independently on each subensemble. There are several sources of systematic error which may be of equal or greater significance. In particular, our value for $g_{\rho\pi\pi}$ has actually been obtained from a $\rho\pi\pi$ vertex with one of the pions somewhat off mass shell ($E^2-q^2-m^2=-0.29$ GeV^2) and thus may be suppressed by a form factor. A measurement of the $\rho\pi\pi$ coupling constant with all particles on shell could be carried out for perhaps twice the computer time of the present calculation.

The theory is defined on a four-dimensional hypercubic periodic lattice with lattice spacing a and periodicity N_μ in the μ direction. On each nearest neighbor lattice link (x,y) is defined a variable U(x,y) in SU(3) with $U(x,y)=U^\dagger(y,x)$. On each lattice site are defined two sets of Grassmann variables $\bar{\psi}^f_{sc}(x)$ and $\psi^f_{sc}(x)$ with s a spin index running from 1 to 4, c a color index running from 1 to 3, and f a flavor index taking values, in the present calculation, either u or d. We define the gauge action S to be the usual sum over plaquette contributions multiplied by g_0^{-2} for bare gauge coupling g_0, and let the quark action be $\sum\bar{\psi}(x)C_{xy}\psi(y)$ where the quark coupling matrix C_{xy} is -1 for x equal to y, and $K(r\mp\gamma_\mu)U(x,y)$ for y displaced from x by one link in the $\pm\mu$-direction. The hopping parameter K is $(8r+2m_0a)^{-1}$, where m_0 is the bare quark mass and the chirality parameter r is chosen to be 0.5 for reasons discussed in ref. [1]. The vacuum expectation value of a product of quark fields, which is all we need for the eventual evaluation of $g_{\rho\pi\pi}$, is

$$\langle\Pi\psi(x_i)\bar{\psi}(y_i)\rangle = Z^{-1}\int d\mu_G\int d\mu_Q\Pi\psi(x_i)\bar{\psi}(y_i)\ \exp(S_G+S_Q) \tag{1}$$

where x_i and y_i are multi-indices combining flavor, spin, color, and position, Z is defined by the condition $\langle 1\rangle=1$, $\int d\mu_G$ is an integral over one copy of Haar measure on each independent U(x,y), and $\int d\mu_Q$ is a Grassman integral over the quark fields ψ and $\bar{\psi}$.

We now integrate out the quark fields in (1), replace the Matthews-Salam determinant which the integration produces by 1, and make a compensating shift in the bare gauge coupling constant. This pair of modifications are together the valence approximation. The vacuum expectation for a product of quark fields becomes

$$\langle \Pi\psi(x_i)\bar{\psi}(y_i)\rangle = Z^{-1}\int d\mu_G \ \det_{ij}|C^{-1}_{x_i y_j}| \ \exp(S_G) \tag{2}$$

For any choice of x_i and y_j, (2) can be evaluated numerically [1-6] by the combination of a Monte Carlo algorithm [7] to perform the integral over link fields and a Gauss-Seidel iteration [8] to determine the matrix elements of C^{-1} needed for each Monte Carlo link configuration.

Let us consider how the $\rho\pi\pi$ coupling constant could be extracted, at least in principle, from expectations of the form (2). For continuum QCD, define $g_{\rho\pi\pi}$ by the condition that the truncated on-shell ρ-$\pi\pi$ vertex for normalized ρ and π fields be given by the effective interaction

$$S_{eff} = -g_{\rho\pi\pi}\int d^4x \ \rho^\mu_i(x)\pi_j(x)\overleftrightarrow{\partial}_\mu\pi_k(x)\varepsilon^{ijk} \tag{3}$$

Now for lattice QCD let the (unnormalized) charged rho field $\rho^-_\mu(x)$ be $\bar{\psi}^u(x)\gamma_\mu\psi^d(x)$ and charged and neutral pi fields $\pi^+(x)$ and $\pi^0(x)$ be $\bar{\psi}^d(x)\gamma^5\psi^u(x)$ and $[\bar{\psi}^u(x)\gamma^5\psi^u(x) - \bar{\psi}^d(x)\gamma^5\psi^d(x)]/\sqrt{2}$ respectively. For any of these fields $f(x)$ let the spatial Fourier transform $\tilde{f}(q,t)$ with momentum in the three-direction be

$$\tilde{f}(q,t) = \sum_{\substack{\vec{x} \\ x^4=t}} \exp(iqx_3) \ f(x) \tag{4}$$

Then for large values of t and its periodic reflection, N_4a-t, the two-point functions for any of the fields f has asymptotic behavior

$$\langle \tilde{f}(q,t)^{\dagger} f(0)\rangle \longrightarrow Z_f(q)\{\exp[-E_f(q)t] + \exp[-E_f(q)(N_4 a-t)]\} \tag{5}$$

where $Z_f(q)$ is a product of vacuum-to-one-particle matrix elements of f, and $E_f(q)$ is the energy of the one-particle state with momentum q. Similarly, for large separation of the time arguments with t_1, $t_2 \ll N_4 a$, the three-point function becomes

$$G_3(t_1,t_2,0;\vec{q}) = \langle \bar{\rho}_3^-(0,t_2)\tilde{\pi}^0(q,t_1)\pi^+(0)\rangle$$

$$\longrightarrow Z_{\rho\pi\pi}(q)\ \exp[-E_\rho(0)(t_2-t_1)-E_\pi(q)t_1] \tag{6}$$

with a t_1 and t_2 independent parameter $Z_{\rho\pi\pi}(q)$.

The quantities $Z_\pi(q)$, $Z_\rho(q)$, and $Z_{\rho\pi\pi}(q)$ are given by the expectation values

$$Z_\pi(q) = (N_1N_2N_3)^{-1}|\langle\Omega|\tilde{\pi}^+(q,0)|\pi^+\rangle|^2 \tag{7}$$

$$Z_\rho(q) = (N_1N_2N_3)^{-1}|\langle\Omega|\tilde{\rho}_3^-(q,0)|\rho^-\rangle|^2 \tag{8}$$

$$Z_{\rho\pi\pi}(q) = [Z_\pi(q)Z_\rho(0)]^{1/2}\langle\rho^+|\tilde{\pi}^0(q,0)|\pi^+\rangle \tag{9}$$

for rho and pi states normalized to one. Thus, according to the LSZ reduction formula, $Z_{\rho\pi\pi}(q)$ may also be thought of as the $\rho\pi\pi$ three-point function with the ρ line and one π line truncated by removing factors of appropriately normalized free field lattice propagators. If we divide $Z_{\rho\pi\pi}(q)$ by a normalized free lattice propagator for the remaining pion and adjust the two pion momenta to place the remaining pion on its mass shell, we obtain a quantity whose continuum limit, up to kinematic factors, is the coupling constant $g_{\rho\pi\pi}$ of eqn. (3):

$$Z_{\rho\pi\pi}(q) = 2qa \sqrt{\frac{Z_\pi(q)^2 Z_\rho(0)}{(2E_\pi(q)a)^2 2E_\rho(0)a}} \; P\big(E_\rho(0)a-E_\pi(q)a, E_\pi(q)a\big) g_{\rho\pi\pi} \tag{10}$$

$$P(x,y) = \frac{\sinh y}{\cosh y - \cosh x} \tag{11}$$

It is perhaps worth noting that combining (6) and (10) yields the result that the three-point function, for appropriate t_1 and t_2, is proportional to a convolution of Fourier transformed propagators for the three particles:

$$G_3(t_1,t_2,0;\vec{q}) =$$

$$= 2qg_{\rho\pi\pi} \sqrt{Z_\rho 2m_\rho Z_\pi 2E_\pi Z_\pi 2E_\pi} \; \sum_t \frac{e^{-m_\rho|t_2-t|}}{2m_\rho} \frac{e^{-E_\pi|t_1-t|}}{2E_\pi} \frac{e^{-E_\pi|t|}}{2E_\pi} \tag{12}$$

where $q=|q_3|$, the ρ is polarized in the 3 direction, and $E_\pi=\sqrt{m_\pi^2+q^2}$. Ultimately we will evaluate $g_{\rho\pi\pi}$ by equating the lattice result for the three-point function with eq.(12).

In practice the direct calculation of the three-point function from products of quark propagators cannot be carried out in a reasonable amount of computer time. To calculate directly values of the $\rho\pi\pi$ three-point function at all the points needed to form the Fourier transforms in (6) would require running a Gauss-Seidel iteration not O(1) times on each Monte Carlo gauge configuration, as done in previous calculations [1-6], but rather $O(N_1N_2N_3)$ times. This would cost far too much computer time. An alternative procedure is to obtain the three-point function (6) by differentiating the ρ-π two-point function with respect to an external pion field. The expectation value we need can be written

$$\langle \tilde{\rho}_3^-(0,t_2)\tilde{\pi}^0(q,t_1)\pi^+(0)\rangle = \frac{\partial}{\partial\alpha}\langle\tilde{\rho}_3^-(0,t_2)\pi^+(0)\rangle_\alpha\Big|_{\alpha=0} \tag{13}$$

where $\langle\ldots\rangle_\alpha$ is a vacuum expectation with quark action modified by adding an external pion source

$$\begin{aligned} S_{Q\alpha} &= S_Q + \frac{\alpha}{2i}[\tilde{\pi}^0\ (q,t_1) - \tilde{\pi}^0\ (-q,t_1)] \\ &= \sum_{xy}\overline{\psi}(x)C_{xy}\psi(y) + \alpha\sum_{xy}\overline{\psi}(x)D_{xy}\psi(y) \end{aligned} \tag{14}$$

The valence approximation for $\langle\ldots\rangle_\alpha$ gives

$$\langle\prod_i\psi(x_i)\overline{\psi}(y_j)\rangle_\alpha = Z^{-1}\int d\mu_G\ \det_{ij}|(C+\alpha D)^{-1}_{x_iy_j}|\ \exp S_G \tag{15}$$

The procedure we adopt to measure $\langle\tilde{\rho}_3^-(0,t_2)\tilde{\pi}^0(q,t_1)\pi^+(0)\rangle$ is simply to evaluate $\langle\tilde{\rho}_3^-(0,t_2)\pi^+(0)\rangle_\alpha$ from (15) using Monte Carlo and Gauss-Seidel, and then to differentiate with respect to α. We obtain $\langle\tilde{\rho}_3^-(0,t_2)\pi^+(0)\rangle_\alpha$ using (15) in the same way propagators of the form (5) are obtained in the valence approximation without sources [1-6]. This calculation requires only O(1) Gauss-Seidel iterations.

The derivative in (13) we extract by evaluating $\langle\tilde{\rho}_3^-(0,t_2)\pi^+(0)\rangle_\alpha$ at different values of α, using a single Monte Carlo ensemble of gauge field configurations. If different ensembles of gauge configurations are used at different values of α, statistical fluctuations will occur in $\langle\tilde{\rho}_3^-\ (0,t_2)\pi^+(0)\rangle_\alpha$ from one α value to the next, and will be hard to extract in a reliable way. On the other hand, using the same gauge configurations we have found that for small values of α ($\lesssim$0.05) the dependence on α is linear to within about 1%, and the derivative can be measured very accurately.

We now present our results. Our calculations are done on a lattice 6^2x12x18 with 12 taken as the three-direction (decay plane) and 18 taken as the four-direction (time). The large transverse size of the lattice is required to make possible sufficiently small values of pion momenta to place the external pion close to mass shell. We use a g_0^{-2} of .95, the same value as ref. [1]. Adjusting the lattice spacing by setting the string tension to its physical value then gives an inverse lattice spacing of 1000±150 Mev. Averages were calculated from an ensemble of 10 gauge configurations. The first was obtained after 1000 Monte Carlo sweeps of the lattice with 10 Metropolis updates on each link. Successive configurations were spread over an additional 5800 sweeps. The rho and pi propagators of eq.(5) were measured and energies and renormalization constants were extracted by fitting the observed propagators to eq.(5) plus an additional contribution for the first radially excited state. Calculations of the pion propagator were done both at zero momentum and at q of 521 MeV, which is the smallest momentum permitted by our transverse lattice size of 12 links. The value of q for the pions in the $\rho\pi\pi$ three-point function was also chosen to be 521 MeV/c. We extracted the derivative of $\langle\bar{\rho}_3^-(0,t_2)\pi^+(0)\rangle_\alpha$ by evaluating it at α of 0 and 0.025. Our result at α of 0 was statistically consistent with zero, as required by charge conjugation invariance.

In fig. 1 we show our data for the three-point function with the pions at fixed time slices 0 and 4 as a function of the ρ time. The solid line is the analytic evaluation of eq. (12). Its shape agrees very well with our data. Our value of $g_{\rho\pi\pi}$ comes from comparing the normalization of the two curves at the trough, where the ρ is farthest from the two pions; it is nice to see that about the same value for $g_{\rho\pi\pi}$ is obtained even when the ρ is quite close to the pions.

Values of energies, renormalization constants and $g_{\rho\pi\pi}$ as a function of the hopping constant K are shown in Table 1. We have also included for reference values of the renormalized quark mass, $(2K_c)^{-1}-(2K)^{-1}$, where K_c is the critical K at which the pion mass is zero. Based on ref. [1] we expected convergence of the Gauss-Seidel to fail for sufficiently large K and therefore chose to work at a range of K smaller than the value corresponding to physical pion mass. Table 1 strongly suggests that an extrapolation of $g_{\rho\pi\pi}$ to the physical value of K can be done reliably.

To test the consistency of our procedure, we have also evaluated the $\rho\pi\eta$ three-point function, which must be zero by G-parity conservation. We found it to be smaller by an order of magnitude than the $\rho\pi\pi$ three-point function, fluctuating in sign, and statistically consistent with zero. Another check was done to determine the contribution of radially excited pion states. For the external pion field we replaced the local operator by a nonlocal one formed from the gauge invariant product of quark fields at nearest neighbor sites connected by a link matrix. By examining the two-point functions for these operators, we determined the ratio of the renormalization constants connecting the local and nonlocal operators to the pion state. The ratio of three-point functions obtained using the local and nonlocal operators agreed with that predicted from the ratio of renormalization constants to an accuracy of 3±4%, indicating little contamination from excited states.

A variety of other interesting measurements can be performed by analyzing our data in more detail. Comparing values of hadron masses with those reported for a 6^3x14 lattice [1] provides a measure of the sensitivity of masses to lattice volume. In partial answer to the doubts raised in ref.[9], a preliminary examination showed these effects to be of the order of a few percent for our lattice sizes. For significantly smaller lattices [2-4], these effects have been found to be large however [10]. Our data can also be used to obtain a variety of other coupling constants. These will be reported elsewhere. Finally, it is perhaps worth pointing out that by an extension of our method to four-point functions it may be possible to measure scattering amplitudes.

We are grateful to W.A. Bardeen and R. Raja for helpful discussions. We would also like to thank the Colliding Detector group and the Fermilab Computer Department for the use of their VAX's. This work was supported in part by the United States Department of Energy.

FOOTNOTES AND REFERENCES

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1. D. Weingarten, Phys. Lett. 109B, 57 (1982); Nucl. Phys. B215[FS7], 1 (1983)

2. H. Hamber and G. Parisi, Phys. Rev. Lett. 47, 1792 (1981); E. Marinari, G. Parisi, and C. Rebbi, Phys. Rev. Lett. 47, 1795 (1981).

3. C. Bernard, T. Draper, and K. Olynyk, Phys. Rev. D27, 227 (1982); R. Gupta and A. Patel, Caltech preprint (1982); K. Bowler, G. Pawley, D. Wallace, E. Marinari, and F. Rapuano, Edinburg preprint (1982); M. Fukugita, T. Kaneko and A. Ukawa, KEK preprint (1982).

4. C. Bernard, T. Draper, K. Olynyk, and M. Rushton, Phys. Rev. Lett. 49, 1076 (1982); G. Martinelli, G. Parisi, R. Petronzio, and F. Rapuano, Phys. Lett. 116B, 434 (1982).

5. J. Kogut, M. Stone, H. Wyld, J. Shigemitsu, S. Shenker, and D. Sinclair, Phys. Rev. Lett. 48, 1140 (1982); I.M. Barbour, J.P. Gilchrist, H. Schneider, G. Schierholz, and M. Tepper, DESY preprint (1983).

6. W. Duffy, G. Guralnik, and D. Weingarten, Phys. Lett. 125B, 311 (1983)

7. M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. Lett. 42, 1390 (1979); K. Wilson, Cornell Preprint (1979).

8. D. Weingarten and D. Petcher, Phys. Lett. 99B, 333 (1981).

9. P. Hasenfratz and I. Montvay, Phys. Rev. Lett. 50, 309 (1983).

10. H. Lipps, G. Martinelli, R. Petronzio, and F. Rapuano, Phys. Lett. 126B, 250 (1983).

TABLE I. Energies, decay constants and $g_{\rho\pi\pi}$ for three values of the hopping parameter K. The pions have momentum $2\pi/12a \approx 520$ MeV.

K	E_π	Z_π	M_ρ	Z_ρ	$g_{\rho\pi\pi}$	$(2K)^{-1}-(2K_c)^{-1}$
.325	1.09±.04	.56±.14	1.07±.03	.202±.032	3.24 ±.17	.240
.34	.95±.10	.47±.30	.94±.05	.132±.035	3.16±.20	.172
.355	.83±.20	$.37^{+.35}_{-.20}$	.79±.03	.072±.010	3.07±.60	.110

FIGURE CAPTION

Fig. 1: Data for the ρ-$\pi\pi$ three-point function G_3 as a function of the time slice of the ρ. The pions are fixed at times 0 and 4. The solid curve is the theoretical expectation for G_3 based on eqn. (12) with parameters fixed by the two-point function results. It is normalized by fixing $g_{\rho\pi\pi}$ so that the solid curve fits the monte carlo data where the ρ is far from the pions.

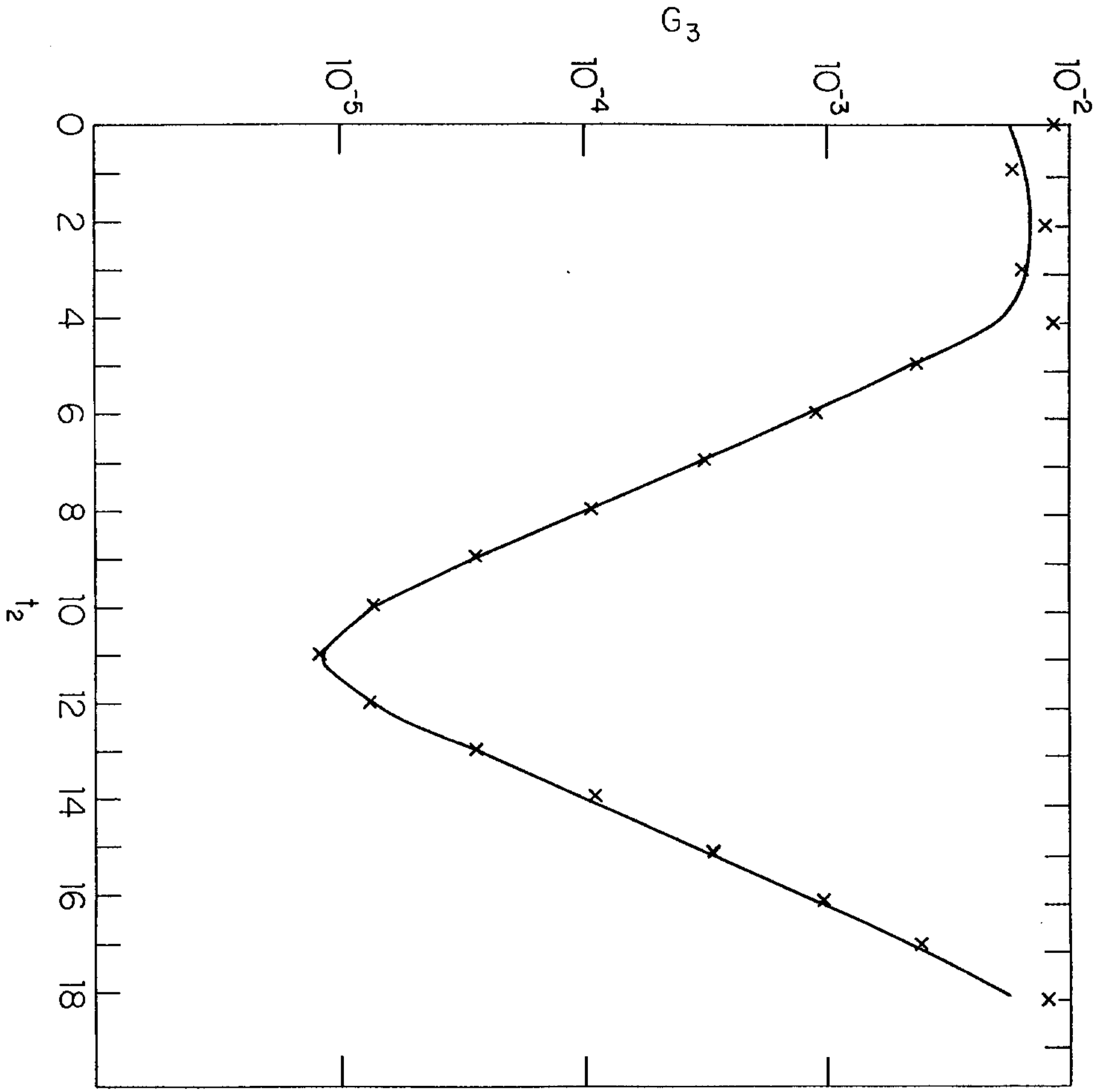

Fig. 1