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# Classical Upper Bounds for Grand Unified Monopole Masses

A.N. SCHELLEKENS and COSMAS K. ZACHOS
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

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ABSTRACT

In addition to being bounded from below by the Prasad-Sommerfield limit, monopole masses are also bounded from above by the limit in which the scalar field variables are frozen to their vacuum values. This upper bound is close to the lower one: the single, double, and triple strength SU(5) monopoles are found to have their masses bound in: $m \leq M \leq m \times 1.7867$, $2m \leq M \leq 2m \times 2.0741$, and $3m \leq M \leq 3m \times 2.3155$, respectively, where $m = 3M_X/8\alpha$.

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It has long been recognized[1-3] that, even though the classical masses of magnetic monopoles vary above their lower bound[4] with the mass of the relevant Higgs particles, nonetheless their fractional variation is slow. This slowness finds expression in an upper bound, which, in general, lies fairly close to the lower bound. The upper bound for the mass of each monopole may be directly obtained by freezing the Higgs variables to their vacuum value, through the limit of infinite Higgs self-couplings (and masses), and then simply solving the classical equations for the gauge field variables numerically.[5] As in the case of the lower bound, the upper one does not depend on particular details of the Higgs potential.

We study here the sequence of monopoles of magnetic charge 1, 2, and 3, respectively, which are present in the prototype SU(5) model.[6-8] We consider the standard SU(5) Lagrangian[9] which contains gauge fields and scalars in the adjoint and fundamental representations:

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}W_{\mu\nu}W^{\mu\nu} + \mathrm{Tr}(\mathcal{D}_\mu\Phi)^2 + |D_\mu H|^2 - V(\beta_i,\Phi,H)$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu]$$

$$D_\mu H = \partial_\mu H + igW_\mu H$$

$$\mathcal{D}_\mu\Phi = \partial_\mu\Phi + ig[W_\mu,\Phi] \qquad (1)$$

where $W_\mu \equiv W_\mu^a F^a$, $F^{a+}=F^a$, $\mathrm{Tr} F^a F^b = 1/2\ \delta^{ab}$. The canonical symmetry breaking pattern $SU(5) \to SU(3)\times SU(2)\times U(1) \to SU(3)\times U(1)$ is effected by the $\underset{\sim}{5}$ and $\underset{\sim}{24}$ Higgs vacuum values $\langle H\rangle = v(0,0,0,0,1)^T$, and $\langle\Phi\rangle = \nu$ diag $(1,1,1,-3/2,-3/2)$, respectively. The electric charge is $Q=e$ diag $(-1/3,-1/3,-1/3,1,0)$, and $e=\sqrt{3/8}\ g$; the mass of the leptoquark is $M_X=(5/2)\ g\nu$; for further convenience, we define: $m \equiv (4\pi/g^2) M_X = 3M_X/8\alpha$.

We consider positive semidefinite Higgs potentials with the above vacuum values at the absolute minimum of every potential, normalized to zero. In the limit of all self-couplings going to infinity, all scalar functions are frozen to their vacuum values, except (for some of them) in a vanishing region around the origin.[5] Constraining these variables provides an upper bound for the variational problem, and the potential drops out, just as it does in the opposite limit (Prasad-Sommerfield:PS).[4,10]

We investigate the simplest family of "spherically symmetric"[11] SU(5) monopoles[9] with asymptotic color fields stable against classical perturbations.[12-13] They have magnetic charges 1/2e, 1/e, and 3/2e; and are characterized, respectively, by the spin 1/2, 1, and 3/2 representations of the crucial SU(2) embeddings in the quintuplet space of SU(5): $\underline{5} \to \underline{1}+\underline{1}+\underline{2}+\underline{1}$, $\underline{5} \to \underline{1}+\underline{3}+\underline{1}$, and $\underline{5} \to \underline{4}+\underline{1}$. Note that the last, triple strength monopole is colorless.[7]

i) SINGLE STRENGTH MONOPOLE: The specifying representation of SU(2) is the two-dimensional one ($\vec{T}=\vec{\tau}/2$), embedded in the subspace of $\underline{5}$ that mixes the 3rd and 4th components. The appropriate Ansatz[7] is:

$$\Phi(r) = \frac{1}{g} \begin{bmatrix} \phi_2 & & & & \\ & \phi_2 & & & \\ & & \phi_0 & & \\ & & & \left.\phi_0\right\} & +\phi_1 \hat{r}\cdot\vec{T} \\ & & & & -2(\phi_2+\phi_0) \end{bmatrix} \tag{2}$$

$$\vec{W}(r) = \hat{r} \times \vec{T}\,\frac{K(r)-1}{gr} \quad , \quad H(r) = \frac{1}{g}\,(0,0,0,0,h(r))^T .$$

At infinity, the radial functions go to: $h \to gv$; $\phi_2 \to g\nu$; $\phi_0 \to -g\nu/4$; $\phi_1 \to 5g\nu/2$; $K \to 0$. In addition, the Ansatz is well defined only for $\phi_1$ vanishing at the origin. Moreover $K \to 1$ at the origin, for the energy to be finite. The mass of the monopole is just the static energy $M=-\int d^3x\,\mathcal{L}$, obtainable by substituting this Ansatz into (1):

$$M = \frac{4\pi}{g^2} \int_0^\infty dr \left[ K'^2 + \frac{(K^2-1)^2}{2r^2} + K^2\phi_1^2 + \frac{r^2}{2}\ \phi_1'^2 + g^2r^2V + \right.$$

$$\left. + \; r^2\left(h'^2 + 6\phi_0'^2 + 8\phi_0'\phi_2' + 6\phi_2'^2\right)\right] . \tag{3}$$

The kinetic terms to the right of the potential terms in Eq. (3) contain variables that would only couple to the rest (here $\phi_1$) through the potential. Hence, in the two limits considered here, they will decouple from the problem, and setting them equal to their vacuum value and dropping them furnishes a bonafide solution. For instance, in the two limits discussed, the electroweak breaking scale v simply does not enter into the equations at all, and therefore it is not reflected in the monopole solutions--this will also be true in the two higher Ansäze below.

In consequence, the particular problem at hand reduces to a solved one--the original 't Hooft monopole.[1] We complete the standard squares[4] to find:

$$M = \frac{4\pi}{g^2} \int_0^\infty dr \left[ (K'+K\phi_1)^2 + \frac{1}{2}\left( r\phi_1' - \frac{(1-K^2)}{r} \right)^2 + \partial_r(\phi_1(1-K^2)) \right] \tag{4}$$

In the PS limit, the squares may be nullified, and the surface term gets a contribution from infinity only: $M=4\pi/g^2(5/2\ g\nu)=m$. In the upper limit, after some rescaling, the variational problem reduces to Eq. (5):

$$M = m \int_0^\infty dr \left[ K'^2 + \frac{(K^2-1)^2}{2r^2} + K^2 \right], \text{ with } K(0)=1,\ K(\infty)=0. \tag{5}$$

Numerical solution of this variational problem yields $M=1.7867m$.

ii) DOUBLE STRENGTH MONOPOLE: We now use the triplet representation of SU(2), embedded in the subspace mixing the 2nd, 3rd, and 4th components.[9] Ansatz Eq. (2), extends to:

$$g\,\Phi = \left.\begin{bmatrix} \phi_3 & & & & \\ & \phi_0 & & & \\ & & \phi_0 & & \\ & & & \phi_0 & \\ & & & & -(\phi_3+3\phi_0) \end{bmatrix}\right\} + \phi_1\ \hat{r}\cdot\vec{T} + \phi_2\left((\hat{r}\cdot\vec{T})^2 - \frac{2}{3}\right) \tag{6}$$

$$\vec{W} = \frac{1}{gr}\ (\hat{r}\times\vec{T}(K-1) + K_1\{\hat{r}\cdot\vec{T},\hat{r}\times\vec{T}\}) \quad ,H = \frac{1}{g}\ (0,0,0,0,h)^T$$

The coefficient of $\phi_2$ is traceless, and both $\phi_1$ and $\phi_2$ must vanish at the origin. At infinity, $h\to g\nu$, $\phi_3\to g\nu$, $\phi_0\to g\nu/6$, $\phi_1\to 5g\nu/4$, $\phi_2\to -5g\nu/4$; $K$, $K_1\to 1/2\sqrt{2}$. As before, for $V\to 0$, the variables $\phi_0$ and $\phi_3$ will decouple and are dropped. We define new variables $K\equiv(v_1+v_2)/\sqrt{2}$, and $K_1\equiv(v_1-v_2)/\sqrt{2}$. Thus, at infinity, $v_1\to 1/2$, $v_2\to 0$. We again complete the squares:

$$M = \frac{4\pi}{g^2}\int_0^\infty dr\left[4(v_1'+v_1(\phi_1+\phi_2))^2 + 4(v_2'+v_2(\phi_1-\phi_2))^2 + \right.$$
$$+ 2r^2\left(\phi_1' + \frac{v_1^2+v_2^2-1}{r^2}\right)^2 + \frac{2r^2}{3}\left(\phi_2'+3\ \frac{(v_1^2-v_2^2)}{r^2}\right)^2 +$$
$$\left. + 4\partial_r(\phi_1(1-v_1^2-v_2^2) + \phi_2(v_2^2-v_1^2))\right]. \tag{7}$$

The surface term for the PS limit gets a contribution 2m from infinity only. The upper bound limit, with $v_1^2(0)=v_2^2(0)=1/2$, $v_1(\infty)=1/2$, $v_2(\infty)=0$, is:

$$M = 2m\int_0^\infty dr\left[2(v_1'^2+v_2'^2) + 2v_2^2 + \frac{1}{r^2}\ (3(v_1^2-v_2^2)^2+(v_1^2+v_2^2-1)^2)\right] \tag{8}$$

This is solved numerically to yield $M=2m \times 2.0741$

iii) TRIPLE STRENGTH MONOPOLE: The quartet representation of SU(2) is embedded in the upper four components of the SU(5) space. Ansatz Eq. (6), now extends to:

$$g\Phi = \left\{\begin{bmatrix} \phi_0 & & & & \\ & \phi_0 & & & \\ & & \phi_0 & & \\ & & & \phi_0 & \\ & & & & -4\phi_0 \end{bmatrix}\right\} + \phi_1 \hat{r}\cdot\vec{T} + \phi_2\left((\hat{r}\cdot\vec{T})^2 - \frac{5}{4}\right) + \phi_3(\hat{r}\cdot\vec{T})^3$$

$$\vec{W} = \frac{1}{gr}\left(\hat{r}\times\vec{T}(K-1) + \left\{K_1\hat{r}\cdot\vec{T} + K_2\left((\hat{r}\cdot\vec{T})^2 - \frac{5}{4}\right), \hat{r}\times\vec{T}\right\}\right) \qquad H=\frac{1}{g}(0,0,0,0,h)^T. \tag{9}$$

The coefficient of $\phi_2$ is traceless, and $\phi_1$, $\phi_2$, $\phi_3$ must vanish at the origin. We drop the superfluous variables h and $\phi_0$, and define new linear combination variables: $v_1=\sqrt{6}(K_1+K/2)$, $v_2=\sqrt{6}((K_1-K/2)$, $v_3=2\sqrt{2}(K_2-K/2)$, $\psi_1=2\phi_2$, $\psi_2=\phi_1+13\phi_3/4$, $\psi_3=2\phi_1+14\phi_3/4$. At infinity, $v_1\to 1$, $v_2\to 0$, $v_3\to 1$, $\psi_1\to -5g\nu/4$, $\psi_2\to 5g\nu/4$, $\psi_3\to 5g\nu/4$. The mass of the monopole is:

$$M = \frac{4\pi}{g^2} \int_0^\infty dr \Bigg[ 2(v_1' + v_1(\psi_1+\psi_2))^2 + 2(v_2'+v_2(\psi_2-\psi_1))^2 + $$
$$+ 2(v_3'+v_3(\psi_3-\psi_2))^2 + \left( r\psi_1' + \frac{v_1^2-v_2^2}{r} \right)^2 +$$
$$+ \left( r\psi_2' + \frac{v_1^2+v_2^2-v_3^2-1}{r} \right)^2 + \left( r\psi_3' + \frac{v_3^2-2}{r} \right)^2$$
$$+ 2\partial_r (\psi_1(v_2^2-v_1^2) + \psi_2(-v_1^2-v_2^2+v_3^2+1) + \psi_3(-v_3^2+2)) \Bigg] . \tag{10}$$

The surface term yields the PS value 3m. The upper bound variational problem:

$$M = 3m \int_0^\infty \frac{dr}{3} \Bigg[ 2({v_1'}^2+{v_2'}^2+{v_3'}^2) + 2v_2^2 +$$
$$+ \frac{1}{r^2} \left( (v_1^2-v_2^2)^2 + (v_3^2-2)^2 + (v_1^2+v_2^2-v_3^2-1)^2 \right) \Bigg] \tag{11}$$

with $v_3^2(0)=2, v_1^2(0)=v_2^2(0)=3/2$, $v_2(\infty)=0$, $v_1(\infty)=v_3(\infty)=1$ is solved to yield $M=3m \times 2.3155$.

Our preceding results have broader applicability. Consider spherically symmetric monopoles[11] in an SU(N) theory with an adjoint representation of scalars. If the monopole is in an embedding with SU(2) spin $\ell/2$, we expect the essential variables to be--at most--$\ell$ scalar and $\ell$ gauge field radial functions, extending Eqs. (2), (6), and (9) in the regular pattern exhibited above. We expect the ratio of the upper ~~and~~ to the lower limit to be a function of $\ell$-1 variables: the frozen values of the $\ell$ relevant Higgs variables minus an overall scale. At r=0 the relevant boundary conditions of the gauge fields are fixed by finiteness of the energy; normally, for $r\to\infty$, only part of their asymptotic values are constrained by finiteness requirements (i.e. $D_\mu\Phi\to 0$ for $r\to\infty$), but the others are irrelevant.[14] As a result, expressions (3), (7), and (10) can be used to calculate the masses of any SU(N) monopoles with $\ell$=1,2,3. If, moreover, the values of the scalar variables can be transformed into ours, by addition of a constant and/or rescaling, the same numbers are obtained.

This can be applied to some of the monopoles we have not discussed: e.g. the double strength monopole of Ref. 7 (embedded in components 3-5 of SU(5)) turns out to have the same mass upper limit as the one derived from Eq. (8), apart from $SU(2)_W$ breaking effects. All other possible triple strength monopoles have the same mass upper limit as the purely electromagnetic one.[15]

We care to conclude with the following remarks:

1) No fundamental understanding of the precise empirical values 1.7867, 2.0741, 2.3155 is available.

2) Saturating the upper bound (infinite scalar self-coupling) is as physically inapposite as saturating the lower bound. However, if the actually prevailing, radiatively corrected, effective potentials fall within

the general class investigated, the physical answer will fall within the above bounds. (The bounds will only be meaningful if the quantum situation is not criticaly different than the classical background solutions). As a result, experimental determination of either $M_x$ or M would allow constraining the other within roughly a factor of two. Order-of-magnitude calculations relying on the PS limit, common e.g. in cosmology, may thus be supported.

3) The higher strength monopoles considered have sufficient energy for decaying to single strength monopoles, while preserving the overall topological number. The decay would be at threshold at the lower bound, while there is surplus energy at the upper bound.[16] We have no statements to make, however, concerning <u>local</u> stability beyond Refs. 12-13.

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## REFERENCES AND FOOTNOTES


1. G. 't Hooft, Nucl. Phys. B79, 276 (1974);
   A.M. Polyakov, JETP Lett. 20, 194 (1974).
2. E.B. Bogomol'nyi and M.S. Marinov, Sov. J. Nucl. Phys. 23, 355 (1976);
   F.A. Bais and J. Primack, Phys. Rev. D13, 819 (1976).
3. D. Wilkinson, Nucl. Phys. B125, 423 (1977).
4. E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24, 449 (1976).
   S. Coleman et al., Phys. Rev. D15, 544 (1977).
5. T. Kirkman and C. Zachos, Phys. Rev. D24, 999 (1981).
6. D.M. Scott, Nucl. Phys. B171, 95 (1980).
7. C. Dokos and T. Tomaras, Phys. Rev. D21, 2940 (1980).
8. M. Daniel et al., Nucl. Phys. B170 [FS1], 156 (1980).
   I.G. Koh et al., J. Math. Phys. 23, 1210 (1982).
   Zhong-qi Ma, Academia Sinica preprint BIHEP-TH-82-13, July 1982.
9. To the extent possible, we follow Ref. 7. However, we correct minor errors and omit redundant symmetry breaking parameters and variables, e.g. in view of identities like $\{\hat{r}\times\vec{T},(\hat{r}\cdot\vec{T})^2\}=\hat{r}\times\vec{T}$ for spin 1, and $\{\hat{r}\times\vec{T},(\hat{r}\cdot\vec{T})^3\}=7/4\ \{\hat{r}\times\vec{T},\hat{r}\cdot\vec{T}\}$ for spin 3/2. We further omit superfluous terms of the form $L(r)\{\vec{T}-\hat{r}\hat{r}\cdot\vec{T}\ ,(\hat{r}\cdot\vec{T})^n\}$, since they do not lower the mass of the monopole;[11] they are connected to the ones present by global gauge transformations $\exp i\theta(\hat{r}\cdot T)^n$. For the double strength monopole we choose a different embedding, which has

stable asymptotic color fields in the sense of Ref. 12. Moreover our embedding belongs to the same sequence as the single and triple strength monopoles.

10. M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett 35, 760 (1975).

11. D. Wilkinson and A. Goldhaber, Phys. Rev. D16, 1221 (1977).

12. R. Brandt and F. Neri, Nucl. Phys. B161, 253 (1979); S. Coleman 1981 Erice Lectures, HUTP-82/A032, unpublished.

13. E. Weinberg, Nucl. Phys. B203, 445 (1982).

14. If the action is minimized with different boundary conditions for the unconstrained radial functions at a finite (large) radius R, the differences in mass vanish for $R\to\infty$. This does not mean that, in this limit, solutions exist for any choice of boundary conditions: usualy only a subset of the possible Ansätze admit solutions. See Ref. 3 for illustrations.

15. However, closer examination reveals that a solution exists only for the purely electromagnetic monopole. For all others, the equations of motion do not admit acceptable solutions for $r\to\infty$ (see footnote 14 and Ref. 3).

16. For a double strength monopole below threshold, see J. Harvey (in preparation).