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SHORT-TIME PERTURBATION THEORY AND NONRELATIVISTIC DUALITY

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ABSTRACT

We give a simple proof of the nonrelativistic duality relation $\langle W^2\sigma_{\text{bound}}\rangle \simeq \langle W^2\sigma_{\text{free}}\rangle$ for appropriate energy averages of the cross sections for $e^+e^- \to$($q\bar{q}$ bound states) and $e^+e^- \to$(free $q\bar{q}$ pair), and calculate the corrections to the relation by relating $W^2\sigma$ to the Fourier transform of the Feynman propagation function and developing a short-time perturbation series for that function. We illustrate our results in detail for simple power-law potentials and potentials which involve combinations of powers. In the following paper, we use our results to study the nonrelativistic version of the Shifman-Vainshtein-Zakharov method for determining the nature of the $q\bar{q}$ interaction and calculating energies of bound $q\bar{q}$ systems, and suggest some improvements in the method.

I. INTRODUCTION

There is an assumed duality between the observed (bound state) cross section for $e^+e^- \to$(confined $q\bar{q}$ system) $\to$ hadrons and the (free) cross section for $e^+e^- \to$(free $q\bar{q}$ pair) calculated in perturbative QCD: if both cross sections are appropriately averaged over energy, the averages are approximately equal,

$$\langle W^2\sigma_{\text{bound}}\rangle \simeq \langle W^2\sigma_{\text{free}}\rangle \quad , \tag{1}$$

where W is the total energy of the e^+e^- pair. This duality has been used extensively in the analysis of heavy quark data.

Until recently, the duality relation had been demonstrated to hold only in the nonrelativistic case, and then only in the JWKB approximation[1] or in numerical calculations in specific potential models.[2] The corrections to the relation were not known. In two earlier papers, we gave proofs of nonrelativistic duality for the single channel[3] and coupled channel[4] problems, and investigated the corrections to Eq. (1). (The corrections were also investigated by Pasupathy and Singh[5] using an extension of the JWKB approximation.) We subsequently extended the JWKB proof of duality to the relativistic Bethe-Salpeter problem and investigated the relativistic-nonrelativistic connection in detail.[6]

Our method of proof of the nonrelativistic results was based on a short-time expansion of the Feynman propagation function and required, as presented in Refs. 3 and 4, that the $q\bar{q}$ potential be analytic in r^2 at the space origin. We have since extended our results to general potentials, and have used them to investigate the extent to which the Shifman-Vaǐnshteǐn-Zakharov[7] (SVZ) program of determining bound state parameters from perturbation theory (as modeled by potential theory) can be improved by including higher order effects. We report that work in this and the following[8] paper.

In Sec. IIA, we review the connection between the cross section for $e^+e^- \to$hadrons in confining potential models and the Fourier transform of the Feynman propagation function, and use the result to give a precise definition of the duality relation. In Sec. IIB, we establish the series of corrections to the simple duality relation in Eq. (1) by developing the short-time Born series for the propagator. We illustrate our results on the short-time perturbation series for the case of power-law potentials in Sec. IIC, give some examples in Sec. IID, and apply the results to the duality relations for exponential moments of Shifman, Vaĭnshteĭn, and Zakharov[7] in Sec. IIE. In Sec. IIF, we consider the case of the Coulomb-plus-linear potential, and show how the duality relations can be improved by extracting Coulomb corrections and treating them exactly, as was discussed in Ref. 3. We conclude with some comments in Sec. III.

In an Appendix, we derive exact expansions for $K(0,0,-i\tau)$ (the Euclidean propagator $K(\vec{r}\,',\vec{r},-i\tau)$ evaluated at the origin $\vec{r}\,' = \vec{r} = 0$) for the harmonic oscillator, linear, and Coulomb potentials. This function determines the Shifman-Vaĭnshteĭn-Zakharov exponential moments of the nonrelativistic e^+e^- annihilation cross section. Our results for the linear and Coulomb propagators are to our knowledge new.

In the following paper,[8] we use the results obtained here to study the nonrelativistic version of the Shifman-Vaĭnshteĭn-Zakharov[7] method for determining the nature of the (relativistic) $q\bar{q}$ interaction and the energies of $q\bar{q}$ bound states from duality. Our work extends the earlier analysis of the SVZ method given by Bell and Bertlmann,[9] and we propose some improvements of the method.

II. SHORT TIME PERTURBATION THEORY AND DUALITY

A. Duality and the Feynman propagator

Our derivation of the duality relation in Ref. 3 was based on two observations, first that the free and bound cross sections for $e^+e^- \to q\bar{q}$ can be expressed in terms of Fourier transforms of the corresponding Feynman propagators, and second that the two propagators are approximately equal at short times. We begin by reviewing and extending these results.

The nonrelativistic cross section for e^+e^- annihilation into a $q\bar{q}$ pair bound in a confining potential $V(r)$ is given for three quark colors by

$$\sigma_{\rm bound} = 24\pi^3\,\alpha^2 e_q^2 m_q^{-2} W^{-2} \sum_n |\psi_{nS}(0)|^2\,\delta(E-E_{nS})\ . \tag{2}$$

Here α is the fine structure constant, e_q is the quark charge in units of e, m_q is the quark mass, $W = 2m_q+E$ is the total energy in the center-of-mass system, and $\psi_{nS}(\vec{r})$ is the $q\bar{q}$ wave function for the nth S state. The sum in Eq. (2) is just the Fourier transform $\tilde{K}(0,0,E)$ of the Feynman propagator $K(\vec{r}\,',\vec{r},t)$ for the $q\bar{q}$ system in the confining potential, evaluated for zero quark separation,

$$K(\vec{r}\,',\vec{r},t) = \sum_{n\ell m} \psi_{n\ell m}(\vec{r}\,')\, e^{-iE_{n\ell}t}\, \psi^*_{n\ell m}(\vec{r})\ , \tag{3}$$

$$\begin{aligned}\tilde{K}(\vec{r}\,',\vec{r},E) &= \int_{-\infty}^{\infty} dt\, e^{iEt}\, K(\vec{r}\,',\vec{r},t) \\ &= 2\pi \sum_{n\ell m} \psi_{n\ell m}(\vec{r}\,')\,\psi^*_{n\ell m}(\vec{r})\,\delta(E-E_{n\ell})\ .\end{aligned} \tag{4}$$

Since only S states contribute to $\tilde{K}$ for $\vec{r}\,' = \vec{r} = 0$, the cross section in Eq. (2) is simply proportional to $\tilde{K}(0,0,E)$,

$$\sigma_{bound} = 12\pi^2\alpha^2 e_q^2 m_q^{-2}\ W^{-2}\ \tilde{K}(0,0,E)\ . \tag{5}$$

Similar results hold for the free cross section,

$$\begin{aligned}\sigma_{free} &= 6\pi\ \alpha^2 e_q^2\ v\ W^{-2}|\psi_E(0)|^2 \\ &= 12\pi^2\ \alpha^2 e_q^2\ m_q^{-2}\ W^{-2}\ \tilde{K}_o(0,0,E)\ ,\end{aligned} \tag{6}$$

where $K_o(\vec{r}\,',\vec{r},t)$ is the free propagator,

$$K_o(\vec{r}\,',\vec{r},t) = \left(\frac{m_q}{4\pi(it+\epsilon)}\right)^{3/2} \exp\left(-\frac{m_q}{4}\frac{(\vec{r}\,'-\vec{r})^2}{it+\epsilon}\right)\ , \tag{7}$$

$\tilde{K}_o$ is its Fourier transform,

$$\tilde{K}_o(0,0,E) = m_q^2\ v/2\pi\ \ , \tag{8}$$

and $v = (E/m_q)^{1/2}$ is the velocity of either quark in the center-of-mass system.

Although the energy dependence of the cross sections σ_{bound} and σ_{free} (or equivalently, of the functions $\tilde{K}(0,0,E)$ and $\tilde{K}_o(0,0,E)$) is drastically different, the propagators $K(0,0,t)$ and $K_o(0,0,t)$ are nearly equal at short times. In particular, in the presence of a potential $V(r)$, K is related to K_o by the integral equation

$$K(\vec{r}\,',\vec{r},t) = K_o(\vec{r}\,',\vec{r},t) - i\int_0^t dt'\int d^3r''\ K_o(\vec{r}\,',\vec{r}\,'',t-t')V(\vec{r}\,'')K(\vec{r}\,'',\vec{r},t'). \tag{9}$$

The integral term vanishes relative to K_o as $t \to 0$ for potentials $V(r)$ less singular than r^{-2} at the origin. To make use of this information and obtain a duality relation connecting σ_{bound} and σ_{free}, we average the

cross sections over a range of energies by convoluting $W^2\sigma$ with a smooth function $f(E'-E)$,[10] and use Eqs. (5) and (6) and the convolution theorem for Fourier transforms to write the results in terms of $K(0,0,t)$,

$$\begin{aligned}\langle W^2\sigma\rangle &\equiv \int_{-\infty}^{\infty} dE'\ f(E'-E)W^2\sigma(E') \\ &= 12\pi^2\ \alpha^2 e_q^2 m_q^{-2}\int_{-\infty}^{\infty} dE'\ f(E'-E)\tilde{K}(0,0,E') \\ &= 12\pi^2\ \alpha^2 e_q^2 m_q^{-2}\int_{-\infty}^{\infty} dt\ \tilde{f}(t)K(0,0,t)e^{iEt}\ .\end{aligned} \tag{10}$$

If $f(E'-E)$ is chosen so that its Fourier transform $\tilde{f}(t)$ is sharply peaked around $t = 0$, we may use the approximate equality of K and K_o at short times to obtain the simple duality relation in Eq. (1). This relation corresponds physically to our expectation that a $q\bar{q}$ pair produced at $r=0$ is unaffected by the potential for a short period of time, i.e. until the quarks encounter the confining potential barrier. We will next make this assertion more precise, and obtain a corrected version of Eq. (1).

B. Short-time perturbation expansion

In Ref. 3, we estimated the corrections to the duality relation by using the operator expression for the full propagator $K(\vec{r}\,',\vec{r},t)$,

$$K(\vec{r}\,',\vec{r},t) = e^{-iH(\vec{r}\,')t}\ \delta(\vec{r}\,'-\vec{r})\ \ , \tag{11}$$

$H(\vec{r}\,') = -\nabla^2_{\vec{r}\,'}/m_q + V(r')$, and making a (Wigner-Kirkwood[11]) expansion in terms of derivatives of the potential evaluated at the origin. This procedure fails for potentials which are singular or have singular derivatives at the origin, and we have since found it much more convenient to e the integral equation in Eq. (9) directly, and solve for K by eration. This gives the Born series

$$K(0,0,t) = K_o(0,0,t) + K_1(0,0,t) + K_2(0,0,t) + \dots \tag{12}$$

ith

$$K_n(0,0,t) = (-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \int d^3r_n \cdots \int d^3r_1 \tag{13}$$

$$\times\ K_o(0,\vec{r}_n,t-t_n)V(r_n)K_o(\vec{r}_n,\vec{r}_{n-1},t_n-t_{n-1})V(r_{n-1})\dots V(r_1)K_o(\vec{r}_1,0,t_1).$$

It is straightforward to perform the time integrations in Eq. (13) re- ursively using the explicit form of K_o in Eq. (7) and the identity[12]

$$\int_0^t dt' \frac{1}{[(t-t')t']^{3/2}} e^{-x^2/(t-t')-x'^2/t'} = \frac{\sqrt{\pi}}{t^{3/2}} \frac{x+x'}{xx'} e^{-(x+x')^2/t} \quad . \tag{14}$$

e find that

$$K_n(0,0,t) = \left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2} \left(-\frac{m_q}{4\pi}\right)^n \int d^3r_n \cdots \int d^3r_1\ V(r_n)\dots V(r_1) \tag{15}$$

$$\times \frac{r_n+r_{n,n-1}+\dots+r_{21}+r_1}{r_n r_{n,n-1}\dots r_{21} r_1}\ e^{-m_q(r_n+r_{n,n-1}+\dots+r_1)^2/4(it+\varepsilon)} \, ,$$

here $r_{ij} = |\vec{r}_i - \vec{r}_j|$. The remaining spatial integrals can be simplified omewhat by referring the polar angle of $\vec{r}_i$ to $\vec{r}_{i-1}$, $\cos\theta_{i,i-1} = \hat{r}_i \cdot \hat{r}_{i-1}$, nd the azimuthal angle to the plane defined by $\vec{r}_{i-1}$ and $\vec{r}_{i-2}$. The

integrations over the n-2 azimuthal angles and the Euler angles which specify the orientation of $\vec{r}_1$ and $\vec{r}_2$ are then trivial, and give a factor $2\cdot(2\pi)^n$. The integrations over the angles $\theta_{i,i-1}$ can be replaced, finally, by integrations over the lengths $r_{i,i-1}$, and we find that $K_n(0,0,t)$ is given by

$$K_n(0,0,t) = 2\left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2} \left(-\frac{m_q}{2}\right)^n \int_0^\infty dr_n \cdots \int_0^\infty dr_1\ V(r_n)\dots V(r_1)$$

$$\times \int_{|r_n-r_{n-1}|}^{r_n+r_{n-1}} dr_{n,n-1} \cdots \int_{|r_2-r_1|}^{r_2+r_1} dr_{21} (r_n+r_{n,n-1}+\dots+r_1) e^{-m_q(r_n+r_{n,n-1}+\dots+r_1)^2/4(it+\varepsilon)} \, . \tag{16}$$

Equation (16) gives $K_n(0,0,t)$ as a weighted average of $V(r_n)\dots V(r_1)$ over the region with the r_i and $r_{i,i-1} \lesssim (t/m_q)^{1/2}$, $i=1,\dots n$. (This is the region which can be sampled by the quarks in a random walk in time t.) Successive terms in the Born series in Eq. (12) therefore differ in magnitude by a factor $\sim t\langle V\rangle_t$, where $\langle V\rangle_t$ is an average of $V(r)$ for $r \lesssim (t/m_q)^{1/2}$, and the series will give a useful expansion of the full propagator for $t\langle\bar{V}\rangle_t$ sufficiently small. We conclude that Eqs. (12) and (16) give a short-time perturbation expansion for $K(0,0,t)$.

When we substitute the series for $K(0,0,t)$ in the expression for $\langle W^2\sigma_{bound}\rangle$ in Eq. (10), we obtain our corrected duality relation,

$$\langle W^2\sigma_{bound}\rangle = 12\pi^2\,\alpha^2\,e_q^2\,m_q^{-2} \int_{-\infty}^{\infty} dt\ e^{iEt}\,\tilde{f}(t)[K_o(0,0,t)+K_1(0,0,t)+\dots]$$

$$= \langle W^2\sigma_{free}\rangle + 12\pi^2\,\alpha^2\,e_q^2\,m_q^{-2} \int_{-\infty}^{\infty} dt\ e^{iEt}\,\tilde{f}(t)K_1(0,0,t)+\dots \, . \tag{17}$$

By the arguments above, we can make the correction terms in Eq. (17) small by making use of the short-time convergence of the perturbation series, and choosing $\tilde{f}(t)$ to be sharply peaked around $t = 0$, that is, by using a broad, smooth smearing function $f(E'-E)$ in the convolution in Eq. (10). We will use power-law potentials in the next section to illustrate the short-time nature of the perturbation series, and explicitly calculate the corrections in Eq. (17).

C. Perturbation series for power-law potentials

General power-law potentials of the form

$$V(r) = V_o \int_{\nu_{min}}^{\nu_{max}} d\nu\, \rho(\nu)\, (r/a)^{\nu} \ , \quad -2 < \nu_{min} \ , \tag{18}$$

have been used extensively in the analysis of quarkonium systems, and are flexible enough to be of broad interest. For example, the popular Coulomb-plus-linear potential,[13] the Martin potential $V = A+Br^{\nu}$,[14] and the logarithmic potential considered by Quigg and Rosner[15] are all in this class. (In the last case, $\rho(\nu) = -\delta'(\nu)$.) The short-time character of the perturbation series for $K(0,0,t)$ is also particularly clear for power-law potentials, so we will consider them in detail.

The nth order term in the perturbation series is given for a power-law potential by

$$\begin{aligned} K_n(0,0,t) = {} & 2^{n+1}(-(it+\epsilon)V_o)^n \left(\frac{m_q}{4\pi(it+\epsilon)}\right)^{3/2} \\ & \times \int d\nu_n\, \rho(\nu_n) \ldots \int d\nu_1\, \rho(\nu_1) \left(\frac{4(it+\epsilon)}{m_q a^2}\right)^{1/2(\nu_1+\ldots+\nu_n)} \\ & \times \int_0^{\infty} dx_n\, x_n^{\nu_n} \ldots \int_0^{\infty} dx_1\, x_1^{\nu_1} \int_{|x_n-x_{n-1}|}^{x_n+x_{n-1}} dx_{n,n-1} \ldots \int_{|x_2-x_1|}^{x_2+x_1} dx_{21} \\ & \times (x_n+x_{n,n-1}+\ldots+x_1)\, e^{-(x_n+x_{n,n-1}+\ldots+x_1)^2} \ , \end{aligned} \tag{19}$$

where we have introduced dimensionless variables $\vec{x}_i = (m_q/4(it+\epsilon))^{1/2}\, \vec{r}_i$. The factor $(m_q/4\pi(it+\epsilon))^{3/2}$ in this expression is just $K_o(0,0,t)$, Eq. (7). The leading t dependence of $K_n(0,0,t)$ is clearly determined by the minimum power in the potential $V(r)$,

$$K_n(0,0,t)/K_o(0,0,t) \propto t^{n(1+\frac{1}{2}\nu_{min})} \ . \tag{20}$$

In the case of a single power, $V = V_o(r/a)^{\nu}$, $K(0,0,t)/K_o(0,0,t)$ is given by a power series in $t^{1+\frac{1}{2}\nu}$. From Eq. (19), we can identify the nth term in the series for K with the nth power of $tV((t/m_q)^{\frac{1}{2}})$, where $(t/m_q)^{\frac{1}{2}}$ is the characteristic distance discussed after Eq. (16).

It is straightforward to calculate K_1 and K_2 for the general potential in Eq. (18). A simple calculation for K_1 gives

$$K_1(0,0,t) = -K_o(0,0,t)(it+\epsilon)V_o \int d\nu_1\, \rho(\nu_1) \left(\frac{it+\epsilon}{m_q a^2}\right)^{\nu_1/2} \Gamma(1+\frac{\nu_1}{2}) . \tag{21}$$

The calculation for K_2 involves a triple integration on the spatial variables. Integrating first on x_{21}, we find that

$$K_2(0,0,t) = K_o(0,0,t)\ 4((it+\epsilon)V_o)^2 \int d\nu_2\ \rho(\nu_2)\int d\nu_1\ \rho(\nu_1)\times\left(\frac{4(it+\epsilon)}{m_q a^2}\right)^{\frac{1}{2}(\nu_1+\nu_2)}$$

$$\times \int_0^\infty dx_2 \int_0^\infty dx_1\ x_2^{\nu_2}\ x_1^{\nu_1}\left[e^{-4x_>^2} - e^{-4(x_1+x_2)^2}\right] . \tag{22}$$

where $x_>$ is the greater of x_1,x_2. The remaining integrals can be evaluated in terms of gamma and beta functions,with the result

$$K_2(0,0,t) = K_o(0,0,t)((it+\epsilon)V_o)^2 \int d\nu_2\ \rho(\nu_2) \int d\nu_1\rho(\nu_1)\left(\frac{it+\epsilon}{m_q a^2}\right)^{\frac{1}{2}(\nu_1+\nu_2)}$$

$$\times \frac{1}{2}\,\Gamma\left(1+\frac{\nu_1+\nu_2}{2}\right)\left[\frac{\nu_1+\nu_2+2}{(\nu_1+1)(\nu_2+1)} - \frac{\Gamma(\nu_1+1)\Gamma(\nu_2+1)}{\Gamma(\nu_1+\nu_2+2)}\right] . \tag{23}$$

We will use the general results in Eqs. (21) and (23) later to discuss the physically interesting case of the Coulomb-plus-linear potential. For the special case of the simple power laws $V(r) = V_o(r/a)^\nu$, $\rho(\nu_i) = \delta(\nu_i-\nu)$, these results reduce to

$$K_1(0,0,t) = -K_o(0,0,t)(it+\epsilon)V_o\left(\frac{it+\epsilon}{m_q a^2}\right)^{\nu/2}\Gamma\left(1+\frac{\nu}{2}\right) , \tag{24}$$

and

$$K_2(0,0,t) = K_o(0,0,t)((it+\epsilon)V_o)^2\left(\frac{it+\epsilon}{m_q a^2}\right)^{\nu}\Gamma(\nu+1)\left[\frac{1}{\nu+1} - \frac{\Gamma^2(\nu+1)}{2\Gamma(2\nu+2)}\right] . \tag{25}$$

We have not evaluated $K_3(0,0,t)$ for a general power-law potential. However, one of us (JBW[16]) has performed the rather lengthy calculation for a single power with the result

$$K_3(0,0,t) = -K_o(0,0,t)\ ((it+\epsilon)\ V_o)^3\ \left(\frac{it+\epsilon}{m_q a^2}\right)^{3\nu/2}$$

$$\times \frac{\Gamma(\frac{3}{2}\nu+3)}{(3\nu+4)(3\nu+3)}\left[\frac{\Gamma^3(\nu+1)}{\Gamma(3\nu+3)} - \frac{4\Gamma(\nu+1)\Gamma(2\nu+2)}{(\nu+1)\Gamma(3\nu+3)}\right. \tag{26}$$

$$\left. + \frac{2}{(\nu+1)^2} + \frac{1}{(2\nu+2)^2}\ {}_3F_2\left(\begin{matrix}3\nu+3,\ 1,\ 1\\ 2\nu+3,\ 2\nu+3\end{matrix}\middle|\,1\right)\right] ,$$

where ${}_3F_2(\cdot)$ is a generalized hypergeometric function.

D. Examples: The oscillator, linear, logarithmic and Coulomb propagators

It is interesting to examine the expansions of $K(0,0,t)$ more closely for some familiar cases,for some of which exact results are known. For the oscillator potential $V(r) = \frac{1}{4}\,m_q\omega^2 r^2$, Eqs. (7), (24), and (25) give the expansion (see Eq. (A.4) in the Appendix with $\tau \to it$)

$$K_{osc}(0,0,t) = \left(\frac{m_q}{4\pi(it+\epsilon)}\right)^{\frac{3}{2}}\left[1+\frac{1}{4}\,\omega^2\,t^2+\frac{19}{480}\,\omega^4\,t^4+\ldots\right], \tag{27}$$

in agreement with the expansion of the exact result for $\nu=2$ [12]

$$K(0,0,t) = (m_q\omega/4\pi i\ \sin\omega t)^{3/2} . \tag{28}$$

The expansion of $K(0,0,t)/K_o(0,0,t)$ converges in this case for $|\omega t| < \pi$. It could also be obtained as in Ref. 3 using the Wigner-Kirkwood expansion.[11]

For the linear potential $V(r) = br$, our expansion gives

$$K_{linear}(0,0,t) = \left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2}\left[1 - \frac{\sqrt{\pi}}{2}(\lambda(it+\varepsilon))^{3/2} + \frac{5}{12}(\lambda(it+\varepsilon))^3 + \ldots\right],$$

$$\lambda = (b^2/m_q)^{1/3} \quad , \tag{29}$$

again in agreement with the exact result discussed in the Appendix, Eqs. (A.10) and (A.12). In this case, $V(r)$ is not analytic at the origin in three-dimensional space, and the Wigner-Kirkwood expansion fails as noted in Ref. 3. The present methods are clearly superior to those used there.

The logarithmic potential $V = V_o \ln(r/a)$ discussed by Quigg and Rosner[15] is a special case of Eq. (18) with $\rho(\nu) = -\delta'(\nu)$. The approximate expansion of $K(0,0,t)$ can be obtained from our general results in Eqs. (21) and (23) by differentiating with respect to the independent indices ν_1 and ν_2 and then setting both equal to zero. We find that

$$K_{log}(0,0,t) = \left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2}\Big\{1 + \frac{1}{2}(it+\varepsilon)V_o[\gamma - \ln((it+\varepsilon)/m_q a^2)]$$
$$+ \frac{1}{8}((it+\varepsilon)V_o)^2\,\{[\gamma - \ln((it+\varepsilon)/m_q a^2)]^2 + \frac{5\pi^2}{6} - 8\} + \ldots\Big\}. \tag{30}$$

Finally, the expansion of $K(0,0,t)$ for the Coulomb potential can be obtained by taking the limit $\nu \to -1$ in Eqs. (24) and (25). The singularities in the gamma functions in Eq. (25) cancel, and we find that for $V(r) = -\alpha/r$,

$$K_{Coul}(0,0,t) = \left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2}\left[1 + [\pi\alpha^2 m_q(it+\varepsilon)]^{1/2} + \frac{1}{6}\pi^2\alpha^2\, m_q(it+\varepsilon) + \ldots\right]. \tag{31}$$

This result agrees with the expansion of the exact Coulomb propagator derived in the appendix,

$$K_{Coul}(0,0,t) = \left(\frac{m_q}{4\pi(it+\varepsilon)}\right)^{3/2} \cdot 4\sqrt{\pi} \sum_{n=0}^{\infty} \frac{\zeta(n)}{\Gamma(\frac{n-1}{2})} \left(\frac{\alpha^2 m_q}{4}(it+\varepsilon)\right)^{n/2} , \tag{32}$$

where $\zeta(n)$ is the Riemann zeta function.

E. Duality for simple power-law potentials

The duality relation in Eq. (17) can be restated conveniently in terms of the (convolution) average of $\tilde{K}(0,0,E)$,

$$\langle\tilde{K}(0,0,E)\rangle \equiv \langle\tilde{K}(E)\rangle \quad .$$

Using this notation, and the result in Eq. (21), we find for a simple power-law potential $V(r) = V_o(r/a)^\nu$ that

$$\langle\tilde{K}(E)\rangle = \int_{-\infty}^{\infty} dt\, e^{iEt}\, \tilde{f}(t)\, K_o(0,0,t)\left[1 - (it+\varepsilon)V_o\left(\frac{it+\varepsilon}{m_q a^2}\right)^{\nu/2}\Gamma(1+\frac{\nu}{2}) + \ldots\right]$$
$$= \langle\tilde{K}_o(E)\rangle + \langle\tilde{K}_1(E)\rangle + \ldots \; . \tag{33}$$

The sizes of the correction terms $\langle\tilde{K}_1(E)\rangle$, etc., depend on both the potential and the choice of the function $\tilde{f}(t)$, or equivalently of the smearing function $f(E'-E)$ in Eq. (10). Several choices for f have been used frequently in recent work, e.g., the Gaussian smearing used by Barnett et al.[17] and the inverse power moments and so-called exponential moments used in Ref. 7. The corrections were investigated in Ref. 3 for Gaussian smearing and power moments using our earlier methods. We will now illustrate the content of Eq. (33) using exponential moments, thus connecting with the work of Shifman, Vaǐnshteǐn, and Zakharov[7] which we will examine in detail in the following paper.[8]

Exponential moments are defined by the smearing function

$$f(E'-E) = \theta(E'-E)e^{-\tau(E'-E)}, \qquad \tau > 0, \tag{34}$$

$$\tilde{f}(t) = \frac{1}{\tau - it} \quad . \tag{35}$$

With this choice of f, $\langle\tilde{K}(E)\rangle$ is proportional to the Euclidean or imaginary time propagator for the $q\bar{q}$ system,

$$\begin{aligned} \langle\tilde{K}(E)\rangle &= 2\pi \sum_n |\psi_{nS}(0)|^2 \, e^{-(E_{nS}-E)\tau} \\ &= 2\pi \; e^{E\tau} K(0,0,-i\tau) \quad . \end{aligned} \tag{36}$$

We can develop a perturbation expansion for $K(0,0,-i\tau)$ by iterating the Euclidean version of the integral equation for K, Eq. (9), or equivalently and more directly by evaluating the time integrals in Eq. (33) using the Cauchy residue theorem. (This procedure is valid for $E < 0$. The general result follows by analytic continuation. It is important in this calculation to know the branch of the $(it+\epsilon)^{-3/2+n+\frac{1}{2}n\nu}$ in Eq. (19), hence our retention throughout of the ϵ.) The result for the single power potential $V(r) = V_o(r/a)^\nu$ is

$$\begin{aligned} K(0,0,-i\tau) = \left(\frac{m_q}{4\pi\tau}\right)^{\frac{3}{2}} \Bigg\{ & 1 - \tau V_o \left(\frac{\tau}{m_q a^2}\right)^{\nu/2} \Gamma\left(1+\frac{\nu}{2}\right) \\ & + (\tau V_o)^2 \left(\frac{\tau}{m_q a^2}\right)^{\nu} \Gamma(\nu+1) \left[\frac{1}{\nu+1} - \frac{\Gamma^2(\nu+1)}{2\Gamma(2\nu+2)}\right] + \ldots \Bigg\} \; . \end{aligned} \tag{37}$$

This result was obtained to first order by Bell and Bertlmann[9] in their nonrelativistic study of the Shifman-Vaǐnshteǐn-Zakharov[7] program by applying a Borel transform to the energy Green function for the $q\bar{q}$ system. While that method of derivation appears natural in the field-theoretic context considered by SVZ, in our view it obscures the simple connection of the exponential moments to the convolution averaging basic to duality.

The leading correction term in Eq. (37) has the magnitude

$$\langle\tilde{K}_1(E)\rangle/\langle\tilde{K}_o(E)\rangle = K_1(0,0,-i\tau)/K_o(0,0,-i\tau) = -\tau \, V_o\left(\frac{\tau}{m_q a^2}\right)^{\nu/2} \Gamma\left(1+\frac{\nu}{2}\right) , \tag{38}$$

where we note that

$$K_o(0,0,-i\tau) = (m_q/4\pi\tau)^{3/2} \; . \tag{39}$$

More generally Eq. (38) gives a reasonable estimate of the leading correction for any smooth smearing function $f(E'-E)$ with a width τ^{-1} in energy space. The condition that this correction be small then determines the minimum allowable width for $f(E'-E)$.

F. The Coulomb-plus-linear potential

It is important to recognize that (for fixed V_o and a) the series in Eq. (37) converges less rapidly for singular than for nonsingular potentials. This becomes important for potentials which combine more than one power of r. For example, the Coulomb potential gives a series in powers of $\tau^{1/2}$, Eq. (31), while the linear potential gives a series in $\tau^{3/2}$, Eq. (29). The first order corrections for the physically interesting Coulomb-plus-linear potential therefore differ by a factor of τ, and it is quite possible for the linear correction to be negligible while the Coulomb correction is still significant.

This quite different behavior of the correction terms in Eq. (37) for different powers is illustrated in Fig. 1, where we have plotted the ratio $K(0,0,-i\tau)/K_0(0,0,-i\tau)$ as a function of the dimensionless variable

$$x = V_0^{2/(2+\nu)} (m_q a^2)^{-\nu/(\nu+2)} \tau \tag{40}$$

for the (attractive) linear and Coulomb potentials using the exact results discussed in the appendix and also for the Coulomb-plus-linear potential. The corrections are dramatically different for the different potentials.

As we noted in Ref. 3, we can greatly improve the duality relation for the Coulomb-plus-linear potential by extracting the slowly convergent Coulomb series and treating it exactly. (The same technique can be used in principle for other singular potentials.) Thus, using the results of Eqs. (21) and (23) for $V(r) = -\alpha/r + br$ (and exponential smearing), we find that

$$\begin{aligned} \langle\tilde{K}(E)\rangle &= 2\pi e^{E\tau}\left(\frac{m_q}{4\pi\tau}\right)^{3/2}\left[1+\alpha(\pi m_q\tau)^{\frac{1}{2}}+\frac{1}{6}\pi^2\alpha^2 m_q\tau^{\frac{3}{2}}-\frac{1}{2}(\pi/m_q)^{\frac{1}{2}}b\tau^{\frac{3}{2}}-\frac{3}{2}\alpha b\tau^2+O(\tau^{\frac{5}{2}})\right] \\ &= \langle\tilde{K}_{\mathrm{Coul}}(E)\rangle\left[1-\frac{1}{2}(\pi/m_q)^{\frac{1}{2}}b\,\tau^{\frac{3}{2}}+\frac{1}{2}(\pi-3)\alpha b\tau^2+O(\tau^{\frac{5}{2}})\right], \end{aligned} \tag{41}$$

where $\langle\tilde{K}_{\mathrm{Coul}}(E)\rangle$ sums the Coulomb terms only. In this form, the correction to the leading Coulomb contribution is of order $\tau^{3/2}$, and is easily made small.

In terms of the cross sections, Eq. (41) and its generalizations state that[3]

$$\langle W^2\sigma_{\mathrm{bound}}\rangle = \langle W^2\sigma_{\mathrm{short\ range}}\rangle + \text{small corrections}, \tag{42}$$

where $\sigma_{\mathrm{short\ range}}$ is the cross section calculated including only the singular short-range components of the interaction (e.g., the Coulomb components). This relation was used by Barnett et al.[17] in their tests of perturbative QCD in e^+e^- annihilation. Those authors compared a Gaussian average of the physical cross section for $e^+e^- \to$ hadrons with the same average of the free cross section calculated including short-range gluonic corrections. The effects of the (nonperturbative) confining interaction are suppressed by the averaging, and the success (or failure) of the comparison tests the calculated cross section.

III. COMMENTS

In the present work, we have used the connection between the Fourier transform of the Feynman propagation function at the origin and the non-relativistic e^+e^- annihilation cross section to establish a (quantitative) duality relation connecting convolution averages of the cross sections for $e^+e^- \to$ ($q\bar{q}$ bound states) and $e^+e^- \to$ (free $q\bar{q}$ pair). The convolution averaging procedure allowed us to transform the problem to one involving the short-time behavior of the Feynman propagator $K(0,0,t)$ which we could investigate using the Born expansion for K. (Other averaging procedures do not give such a simple method for calculating the corrections to the duality relation.) We illustrated the short-time nature of the duality relation in detail for the case of (simple or multiple) power-law potentials, and presented a number of examples. We emphasize the important conclusion that duality holds as usually applied if the smearing function used in the energy averaging is sufficiently broad and smooth, so that the conjugate time variable is small, and that the corrections are calculable for a given model of the interaction.

A very different and important use of duality was proposed in the relativistic context by Shifman, Vainshtein, and Zakharov.[7] Those authors use a narrow smearing function $f(E'-E)$ in Eq. (10) to isolate the contribution of a single state. This procedure leads to large corrections in Eq. (17). They then attempt to obtain information about the confining interaction of the energies of the bound states by comparing the corrected expressions with experiment. We will examine the limitations of this procedure in detail in the following paper,[8] and will suggest improvements based on our present results.

ACKNOWLEDGMENT

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APPENDIX

We collect here some exact results on the Euclidean propagators $K(0,0,-i\tau)$ for the oscillator, linear, and Coulomb potentials. These have been useful in checking our expressions and investigating the rate of convergence of the short-time perturbation series in Eq. (37) and the transition to the exponential behavior at long times implied by Eq. (36). The results for the linear and Coulomb propagator are to our knowledge new.

Harmonic oscillator potential

The exact propagator for the oscillator potential $V(r) = \frac{1}{4} m_q \omega^2 r^2$ is well known[18] and gives

$$K(0,0,-i\tau) = \left(\frac{m_q \omega}{4\pi \sinh \omega\tau}\right)^{3/2} . \tag{A.1}$$

We obtain a series expansion for this quantity by using the Taylor series for $\sinh \omega\tau$, then expanding the result using the binomial and multinomial expansions,[19]

$$K(0,0,-i\tau) = K_o(0,0,-i\tau) \left\{ 1 + \sum_{\ell=1}^{\infty} c_\ell (\omega\tau)^{2\ell} \right\} \tag{A.2}$$

where

$$c_\ell = \sum_{m=1}^{\ell} \binom{-3/2}{m} \sum_{\substack{n_1 \dots n_\ell \\ n_1+\dots+n_\ell = m \\ n_1+2n_2+\dots+\ell n_\ell = \ell}} \frac{m!}{(3!)^{n_1}\, n_1!\,(5!)^{n_2}\, n_2! \dots ((2\ell+1)!)^{n_\ell}\, n_\ell!} . \tag{A.3}$$

Explicitly,

$$K(0,0,-i\tau) = \left(\frac{m_q}{4\pi\tau}\right)^{3/2} \left[1 - \frac{1}{4}(\omega\tau)^2 + \frac{19}{480}(\omega\tau)^4 - \frac{631}{120960}(\omega\tau)^6 + \frac{1219}{1935360}(\omega\tau)^8 - \frac{5723}{74649600}(\omega\tau)^{10} + \ldots\right] . \tag{A.4}$$

Linear potential

The bound state energies for an S-state $q\bar{q}$ pair confined in the three-dimensional linear potential $V(r) = br$ are given by

$$E_{nS} = (b^2/m_q)^{1/3} \alpha_n \tag{A.5}$$

where α_n is the nth zero of the Airy function,

$$Ai(-\alpha_n) = 0, \quad n=1,2,\ldots . \tag{A.6}$$

The square of the bound state wave function at the origin is independent of n,[20]

$$|\psi_{nS}(0)|^2 = \frac{m_q b}{4\pi} . \tag{A.7}$$

The exact Euclidean propagator for the linear potential is therefore given by

$$K(0,0,-i\tau) = \sum_n |\psi_{nS}(0)|^2 e^{-E_n\tau} = \frac{m_q b}{4\pi} \sum_{\alpha_n} e^{-\alpha_n\lambda\tau} \tag{A.8}$$

where

$$\lambda = (b^2/m_q)^{1/3} . \tag{A.9}$$

We can easily convert the sum in Eq. (A.8) into a contour integral

$$K(0,0,-i\tau) = \frac{m_q b}{4\pi} \frac{1}{2\pi i} \int_C d\alpha\, e^{-\alpha\lambda\tau} \frac{Ai'(-\alpha)}{Ai(-\alpha)} = \frac{m_q b}{4\pi} \frac{1}{2\pi i} \int_C d\alpha\, e^{-\alpha\lambda\tau} \frac{d}{d\alpha} \ln Ai(-\alpha) , \tag{A.10}$$

where the contour C encircles the zeros of $Ai(-\alpha)$ as shown in Fig. 2a, and $-\alpha = e^{-i\pi}\alpha$, $0 \leq \arg\alpha \leq 2\pi$. If we expand the contour outward to C' with $|\arg(-\alpha)| \leq \pi-\delta$ as shown in Fig. 2b, we can use the asymptotic expansion of the Airy function[21]

$$Ai(z) \sim \frac{\sqrt{\pi}}{2} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \sum_{k=0} \frac{(-1)^k\, \Gamma(3k+\frac{1}{2})}{\Gamma(k+\frac{1}{2})\Gamma(k+1)(36z^{3/2})^k} , \quad |\arg z| < \pi , \tag{A.11}$$

to calculate the logarithmic derivative in Eq. (A.10) on the contour, and find that

$$\frac{d}{d\alpha} Ai(-\alpha) = (-\alpha)^{1/2} - \frac{1}{4\alpha} - \frac{5}{32(-\alpha)^{5/2}} + \frac{15}{64\alpha^4} - \frac{1105}{2048(-\alpha)^{11/2}} + \cdots , \quad |\arg(-\alpha)| < \pi . \tag{A.12}$$

When this result is substituted in Eq. (A.10), the integrals on α can be evaluated using Hankel's representation for the reciprocal of the gamma function,[22]

$$\frac{1}{2\pi i} \int_{C'} d\alpha (-\alpha)^{-s} e^{-\alpha\lambda\tau} = -\frac{(\lambda\tau)^{s-1}}{\Gamma(s)} . \tag{A.13}$$

The result is the desired expansion of the Euclidean propagator for the linear potential,

$$K(0,0,-i\tau) = \left(\frac{m_q}{4\pi\tau}\right)^{3/2}\left[1 - \frac{1}{2}\sqrt{\pi}(\lambda\tau)^{3/2} + \frac{5}{12}(\lambda\tau)^3 - \frac{5}{64}\sqrt{\pi}(\lambda\tau)^{9/2} + \frac{221}{6048}(\lambda\tau)^6 - \ldots\right] . \tag{A.14}$$

This result is probably exact despite our use of the asymptotic expansion of $Ai(-\alpha)$ in the integral. Vaĭnshteĭn et al.[7] have obtained the first terms in this expansion by a somewhat different method, but their result contains some errors as published.[23]

Coulomb potential

The expression for the Euclidean propagator for the Coulomb potential $V(r) = -\alpha/r$ includes an integral over the continuum as well as a sum over the discrete bound states,

$$K(0,0,-i\tau) = \int_0^\infty dE\, \rho(E)|\psi_E(0)|^2\, e^{-E\tau} + \sum_{n=1}^\infty |\psi_{nS}(0)|^2\, e^{-E_n\tau} . \tag{A.15}$$

Here $\rho(E) = m_q^{3/2}\, E^{1/2}/4\pi^2$ is the usual density of states, $\psi_E(0)$ is the continuum S-state wave function at the origin,

$$|\psi_E(0)|^2 = \frac{2\pi\eta}{1-\exp(-2\pi\eta)} , \quad \eta = \frac{\alpha}{2}\left(\frac{m_q}{E}\right)^{1/2} , \tag{A.16}$$

$$|\psi_{nS}(0)|^2 = \frac{1}{8\pi}\left(\frac{\alpha m_q}{n}\right)^3 , \tag{A.17}$$

and

$$E_n = \frac{\alpha^2 m_q}{4n^2} , \quad n = 1,2,\ldots . \tag{A.18}$$

It will be convenient to use a scaled energy $E = E_0 z$, $E_0 = \frac{1}{4}\alpha^2 m_q$, in Eq. (A.15). With this convention,

$$K(0,0,-i\tau) = \frac{\alpha^3 m_q^3}{16\pi}\left[\int_0^\infty dz\, \frac{1}{1-\exp(-2\pi/\sqrt{z})}\, e^{-E_0\tau z} + 2\sum_{n=1}^\infty \frac{1}{n^3}\, e^{E_0\tau/n^2}\right]$$

$$= \frac{\alpha^3 m_q^3}{16\pi}\int_C dz\, \frac{1}{1-\exp(-2\pi/\sqrt{z})}\, e^{-E_0\tau z} . \tag{A.19}$$

In the second line we have used the fact that the entire expression in brackets can be written as a contour integral on the contour C shown in Fig. 3a. We can complete the contour as shown in Fig. 3b by adding and subtracting a segment below the real axis,[24] and find that

$$K(0,0,-i\tau) = \frac{\alpha^3 m_q^3}{16\pi}\left[\int_{C'} dz\, \frac{1}{1-\exp(-2\pi/\sqrt{z})}\, e^{-E_0\tau z} + \int_0^\infty dz\, \frac{1}{1-\exp(+2\pi/\sqrt{z})}\, e^{-E_0\tau z}\right]. \tag{A.20}$$

The second integral can itself be related to $K(0,0,-i\tau)$ by using the identity

$$\int_0^\infty dz\, \frac{1}{1-\exp(2\pi/\sqrt{z})}\, e^{-E_0\tau z} = \int_0^\infty dz\left[1 - \frac{1}{1-\exp(-2\pi/\sqrt{z})}\right] e^{-E_0\tau z}$$

$$= \frac{1}{E_0\tau} - \int_0^\infty dz\, \frac{1}{1-\exp(-2\pi/\sqrt{z})}\, e^{-E_0\tau z} \tag{A.21}$$

the first equality in Eq. (A.19). The combination of these results with (A.20) gives the expression

$$K(0,0,-i\tau) = \frac{\alpha^3 m_q^3}{32\pi}\left[\int_{C'} dz\, \frac{1}{1-\exp(2\pi/\sqrt{z})}\, e^{-E_o \tau z} + \frac{1}{E_o \tau} + 2\sum_{n=1}^{\infty} \frac{1}{n^3}\, e^{E_o \tau/n^2}\right] . \tag{A.22}$$

We can evaluate the contour integral in Eq. (A.22) by expanding the tour so that $|z| > 4$, using the Bernoulli expansion[25]

$$\frac{1}{1-\exp(-2\pi/\sqrt{z})} = \frac{\sqrt{z}}{2\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, B_n\left(\frac{2\pi}{\sqrt{z}}\right)^n , \qquad |z| > 4 , \tag{A.23}$$

integrating term-by-term. This gives

$$\int_{C'} dz\, \frac{1}{1-\exp(-2\pi/\sqrt{z})}\, e^{-E_o \tau z} = -(E_o\tau)^{-3/2}\sum_{n=0}^{\infty} e^{i\pi n/2}\, \frac{(2\pi)^n B_n}{n!}\, \frac{(E_o\tau)^{n/2}}{\Gamma(\frac{n-1}{2})}$$

$$= \frac{(E_o\tau)^{-3/2}}{2\sqrt{\pi}} + \sum_{k=1}^{\infty} \frac{2\zeta(2k)}{\Gamma(k-\frac{1}{2})}\,(E_o\tau)^{k-\frac{3}{2}} , \tag{A.24}$$

re we have used the relations[26]

$$B_n = 0,\ n = 3,5,\ldots, \qquad B_{2n} = (-1)^{n+1}\,\frac{2\zeta(2n)(2n)!}{(2\pi)^{2n}} , \tag{A.25}$$

express the Bernoulli numbers B_{2n} in terms of the Riemann zeta function.

We can convert the series in Eq. (A.22) to a power series in τ by

expanding the exponentials and then summing on the principle quantum number n. The result is

$$2\sum_{n=1}^{\infty} \frac{1}{n^3}\, e^{E_o\tau/n^2} = \sum_{k=1}^{\infty} \frac{2\zeta(2k+1)}{\Gamma(k)}\,(E_o\tau)^{k-1} \tag{A.26}$$

Combining Eqs. (A.22), (A.24), and (A.26), we obtain as our final result the remarkably simple expression

$$K(0,0,-i\tau) = \frac{\alpha^3 m_q^3}{32\pi}\left[\frac{(E_o\tau)^{-3/2}}{2\sqrt{\pi}} + \frac{1}{E_o\tau} + \sum_{n=2}^{\infty} \frac{2\zeta(n)}{\Gamma(\frac{n-1}{2})}\,(E_o\tau)^{(n-3)/2}\right]$$

$$= \left(\frac{m_q}{4\pi\tau}\right)^{3/2} \sum_{n=0}^{\infty} \frac{4\sqrt{\pi}\,\zeta(n)}{\Gamma(\frac{n-1}{2})}\,(E_o\tau)^{\frac{n}{2}} \tag{A.27}$$

$$= \left(\frac{m_q}{4\pi\tau}\right)^{3/2}\left[1 + 2\sqrt{\pi}(E_o\tau)^{1/2} + \frac{2\pi^2}{3}E_o\tau + 4\sqrt{\pi}\,\zeta(3)(E_o\tau)^{3/2} + \ldots\right],$$

$$E_o = \frac{1}{4}\alpha^2 m_q ,$$

where $\zeta(0) = -\frac{1}{2}$ and

$$\lim_{n\to 1} \frac{\zeta(n)}{\Gamma(\frac{n-1}{2})} = \frac{1}{2} . \tag{A.28}$$

This result holds also for the repulsive Coulomb potential if we insert an extra factor $(-1)^n$ corresponding to the replacement of α by $-\alpha$. One of us (JBW) obtained a particularly compact derivation of the latter result using the identity[16]

$$\frac{1}{e^z-1} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, \alpha^{-s}\,\Gamma(s)\zeta(s) \tag{A.29}$$

the second integral in Eq. (A.20) (the integral for the repulsive Coulomb
propagator), integrating over the energy variable, and then evaluating the remaining integral by closing the contour in the left half plane.

FOOTNOTES AND REFERENCES

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23. See Sec. 4.3 of the second paper in Ref. 7. A factor $\sqrt{\pi}$ is missing in their last term, and the time variable τ was omitted in the expansion.

24. The integral over this extra segment is just the propagator for the repulsive Coulomb potential $V(r) = +\alpha/r$. One of us (JBW) would like to thank Professor C.J. Goebel for a useful discussion on this calculation.

25. Ref. 19, Sec. 23.1.1.

26. Ref. 19, Sec. 23.2.

FIGURE CAPTIONS

;. 1 Plots of the ratio $K(0,0,-i\tau)/K_o(0,0,-i\tau)$ of the exact Euclidean propagator to the free Euclidean propagator for power-law potentials $V(r) = V_o(r/a)^\nu$ as functions of the dimensionless variable $x = V_o^{2/(2+\nu)}(m_q a^2)^{-\nu/(2+\nu)}\tau$. The curves show the very different rates at which the correction terms grow for the linear potential ($\nu=1$), and the Coulomb potential ($\nu=-1$). We also show the correction to second order for the realistic Coulomb-plus-linear potential for $\alpha=0.25$, $b=0.2$ GeV^2, and $m_q = 1.5$ GeV, with x scaled according to the linear term ($\nu=1$).

;. 2 (a) The contour of integration C in the exact expression for the Euclidean propagator for the linear potential, Eq. (A.10). The crosses denote poles of $Ai'(-\alpha)/Ai(-\alpha)$ at the zeros of the Airy function.

(b) The expanded contour C' used in the evaluation of the integral. We must take the overall scale of the contour to ∞ with $\delta > 0$ in order to use the asymptotic expansion of the integrand in Eq. (A.12).

;. 3 (a) The contour of integration C in the exact expression for the Euclidean propagator for the Coulomb potential, Eq. (A.19). The crosses denote the poles of the integrand at $x = 1/n^2$, $n = 1,2,\ldots$.

(b) The modified contour C' used in Eq. (A.20).

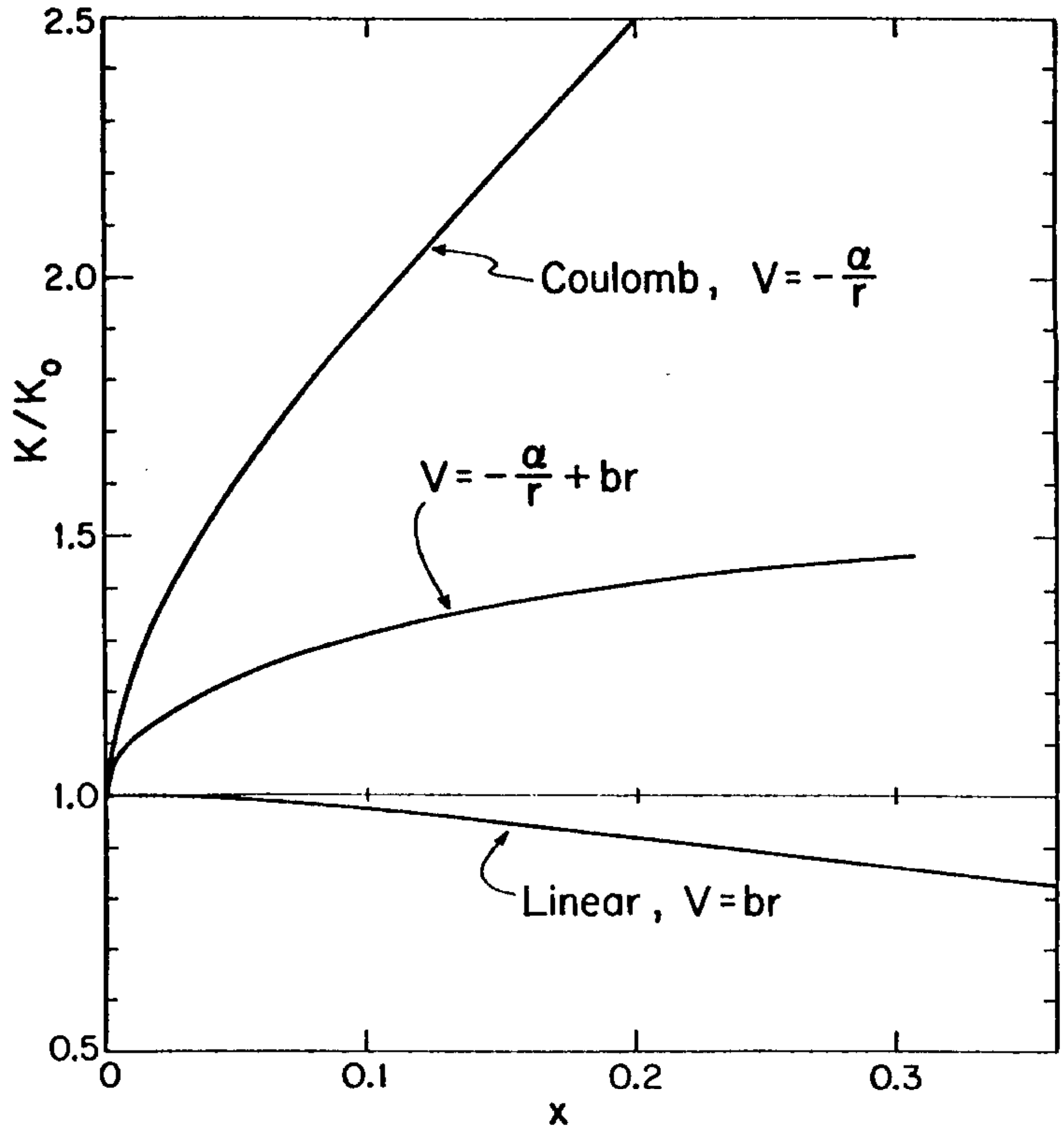

FIGURE 1

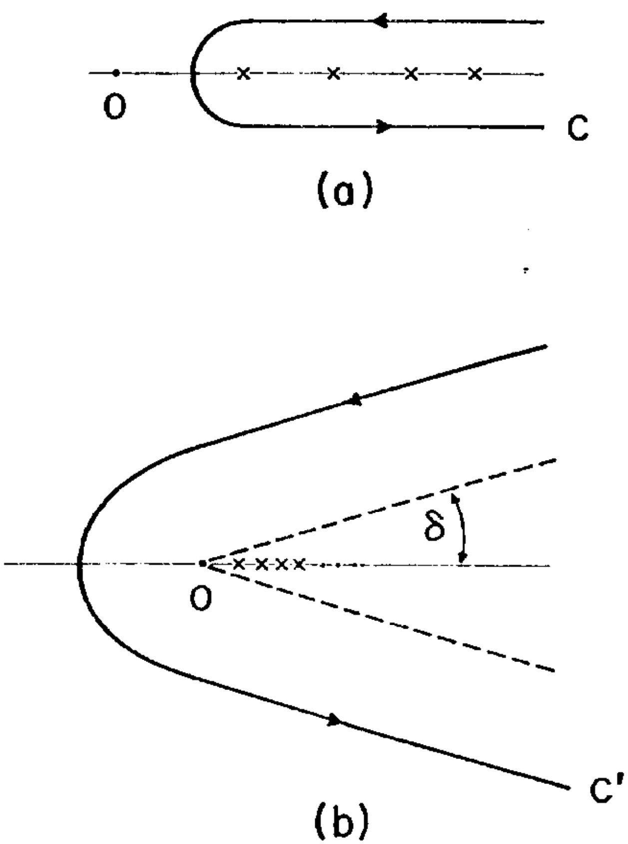

FIGURE 2

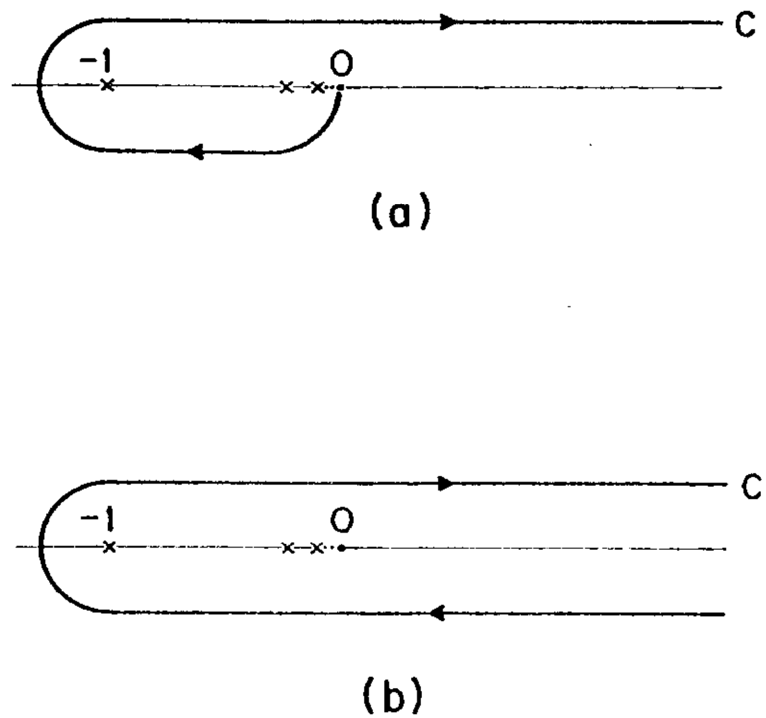

FIGURE 3