FERMILAB-Pub-82/54-THY
August

Ref.TH.3402-CERN

FRACTIONAL CHARGES : GLOBAL AND LOCAL ASPECTS

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A B S T R A C T

The properties of topological objects with fractional charge are studied. We first transform to new variables, in terms of which the lowest order contribution gives the fractional charge. We then gauge that fractional charge. Effects of the θ angle in two dimensions are studied. In particular arbitrary periods may be obtained. Induced extra charge due to θ turns out to be proportional to $\sqrt{\theta}$. C and P violations due to θ may be arbitrarily small in some cases.

*) Supported in part by the Israel-United States binational Science Foundation and the Israeli Academy of Science.

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Ref.TH.3402-CERN

3 September 1982

1. - INTRODUCTION

In both solid state physics and particle physics it has been realized that the particle spectrum of a theory may consist not only of the excitations manifest in perturbation theory, but also of particles of a non-perturbative topological nature, termed solitons [1].

While in solid state phenomena the existence of soliton excitations has been demonstrated, in particle physics one is still at the beginning of the process of unravelling the impact of solitons on physical phenomena.

The structure of the soliton sector has become even more rich by the discovery that solitons may carry fractional charges [2)-4)]. This possibility has caused great interest in both solid state and particle physics.

In this note we discuss some aspects of fractionally charged particles. The fractional charge may correspond to either a global symmetry or to a gauged local symmetry. The global charge may form when light fermions are added to a system containing topological excitations. The propagation of light fermions in a solitonic background causes an instability in the Dirac sea of fermions resulting in the topological excitations acquiring a global fractional charge. In this paper we study in some detail the properties of these hybrid objects. In two dimensions one can go further by gauging the global charge and investigating its electromagnetic interactions.

In Section 2 we review the formation of fractional global charges. Their fate, once they are gauged, is followed. We investigate in particular the case in which a background electric field is applied to the system. A rich structure emerges, notably models with an arbitrary period can be constructed. The models studied are two-dimensional. A new scenario for C and P conservation in these models emerges.

The second type of fractional charge is obtained by adding a topological θ term to a theory in which solitons appear. In four dimensions it was shown by Witten that monopoles turn into dyons [5] ; in Section 3 we construct a similar phenomenon in two dimensions and study it under various circumstances. Massless fermions are added, and so are chirality-violating Yukawa couplings. In particular, we get models in which the dependence on the θ parameter is not periodic.

In Section 4, we summarize the somewhat wide range of results and try to draw some four-dimensional lessons from them.

2. - GLOBAL FRACTIONAL CHARGE : A REVIEW

Slowly varying field approximation

Goldstone and Wilczek [4] have proposed a general method for calculating the global charges residing on solitons. We recast their results in a slightly different language. In fact, we shall transform to new field variables, in terms of which the result for the fractional charge is contained in the lowest non-zero order. In the case of slowly varying fields the results are useful in both two and four dimensions. We start by solving a Yukawa model [4]. Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - g\bar{\psi}\left(\phi_1 + i\gamma_5\phi_2\right)\psi \tag{2.1}$$

Defining the new variables ρ, θ and χ by

$$\phi_1 = \rho\cos\alpha \quad ; \quad \phi_2 = \rho\sin\alpha \quad ; \quad \psi = \exp\left(-\frac{i\alpha\gamma_5}{2}\right)\chi \tag{2.2}$$

one obtains, in two dimensions :

$$\mathcal{L} = i\bar{\chi}\not{\partial}\chi - g\bar{\chi}\rho\chi + \frac{1}{2}\bar{\chi}\gamma_\mu\gamma_5\chi\partial_\mu\alpha + \frac{1}{8\pi}\left(\partial_\mu\alpha\right)^2 \tag{2.3}$$

Note that the field χ is a chiral singlet and that under a chiral transformation angle β the field α is shifted to $\alpha + \beta$. We rewrite (2.3) as

$$\mathcal{L} = i\bar{\chi}\not{\partial}\chi - g\bar{\chi}\rho\chi + \bar{\chi}\not{A}\chi + \frac{1}{8\pi}\left(\partial_\mu\alpha\right)^2 \tag{2.4}$$

where

$$A_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu}\partial^\nu\alpha \tag{2.5}$$

In order to calculate the fermionic charge one defines :

$$j_\mu^{(\psi)}(x) = \frac{1}{2}\left[\bar{\psi}(x), \gamma_\mu\psi(x)\right] \tag{2.6}$$

Thus

$$j_\mu^{(\psi)}(x) = j_\mu^{(\chi)}(x) - \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\alpha(x) \tag{2.7}$$

$j_\mu^{X}(x)$ is the gauge invariant current (defined by point splitting and a line integral). This result is obtained as follows

$$\bar{\psi}(x+\varepsilon)\gamma_\mu\psi(x) = \bar{\chi}(x+\varepsilon)\gamma_\mu\chi(x) + \frac{i}{2}\bar{\chi}(x+\varepsilon)\gamma_\mu\gamma_5[\varepsilon^\alpha\partial_\alpha\alpha(x)]\chi(x) + o(\varepsilon) =$$

$$j_\mu^{(X)}(x) - \frac{i}{2}\bar{\chi}(x+\varepsilon)[\gamma_\mu\varepsilon_{\alpha\beta}\varepsilon^\alpha\partial^\beta\alpha) + \varepsilon_{\mu\beta}\gamma^\beta\varepsilon_\alpha(\partial^\alpha\alpha)]\chi(x) + o(\varepsilon)$$

The two terms in the square brackets, one coming from the ε separation in the X fields and one from the line integral needed to reach the gauge invariant $j_\mu^{(X)}$, contribute equally in the limit $\varepsilon \to 0$, resulting in $-1/2\pi\ \varepsilon_{\mu\nu}\partial^\nu\alpha$. In an external α field (a background field A_μ)

$$\langle j_\mu^{(X)}(x)\rangle_\alpha = i\int\langle T\, j_\mu^{(X)}(0)\, j_\nu^{(X)}(x)\rangle_0 \left[\tfrac{1}{2}\varepsilon^{\nu\rho}\partial_\rho\alpha(x)\right] d^2x \tag{2.8}$$

$$+ \text{ higher order terms in } \partial\alpha.$$

It is here that one invokes the slowly varying field approximation. Thus

$$\langle j_\mu^{(X)}(x)\rangle_\alpha = \varepsilon_{\mu\nu}\partial^\nu F(x) \tag{2.9}$$

with

$$F = \left\{(\text{const.})\frac{\Box}{g^2\rho^2} + \left[\text{higher order terms in } \frac{\Box}{g^2\rho^2}\right]\right\}\alpha + \left[\text{second and higher order terms in } \partial\alpha\right] \tag{2.10}$$

This leads to

$$\left\langle \int dx\, j_0^{(X)}(0)\right\rangle_\alpha = 0 \tag{2.11}$$

Leaving, to leading order in derivatives (as compared with $g\rho$)

$$\langle j_\mu^{(\psi)}(x)\rangle_\alpha \simeq -\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\alpha \tag{2.12}$$

and

$$Q_F = \left\langle\int j_0^{(\psi)}(x)\,dx\right\rangle_\alpha = \frac{1}{2\pi}\left[\alpha(\infty) - \alpha(-\infty)\right] \tag{2.13}$$

Thus a fractional charge is induced on a solitonic background. As pointed out in Ref. 4), Q_F may obtain any value. In four dimensions, the model of Eq. (2.1) produces no induced current, since both scalar and pseudoscalar densities are even under charge conjugation. However, we get an induced current for the analogous model

$$\mathcal{L} = i\bar\psi \partial\!\!\!/ \psi - g\bar\psi\,(\phi_0 + i\vec\tau\cdot\vec\phi\,\gamma_5)\psi \tag{2.14}$$

Now we define

$$\phi_0 + i\vec\tau\cdot\vec\phi\,\gamma_5 = \rho\, e^{i\gamma_5\vec\tau\cdot\vec\alpha} \;;\; \psi = \exp\left(-\frac{i\gamma_5\vec\alpha\cdot\vec\tau}{2}\right)\chi \equiv S\chi \tag{2.15}$$

and obtain

$$\mathcal{L} = i\bar\chi\partial\!\!\!/\chi - g\bar\chi\rho\chi + i\bar\chi\gamma_\mu S^{-1}(\partial_\mu S)\chi \tag{2.16}$$

We do not compute extra terms [analogous to the $(\partial_\mu\alpha)^2$ term of (2.3)] since they do not contribute to the induced current. It turns out, that the answer of Goldstone and Wilczek[4)] is reproduced by using only the axial part $(S^{-1}\gamma_\mu S)_A$ of the interaction term. In this case, the first non-vanishing contribution comes from the box diagram (the triangle diagram vanishes due to charge conjugation). Also, the answer is reproduced with the unregulated box. Thus

$$\langle j_\mu^\psi(x)\rangle_\alpha = \frac{1}{12\pi^2}\,\mathrm{Tr}\left[\gamma_\mu(\gamma_\alpha S^{-1}\partial^\alpha S)_A(\gamma_\beta S^{-1}\partial^\beta S)_A(\gamma_\lambda S^{-1}\partial^\lambda S)_A\right] \tag{2.17}$$

(+ higher derivatives)

The charge is given by the contribution of the first term only. The vector part of the interaction must therefore be cancelled by the terms coming from the ε separation. Substituting (2.15) into (2.17) one obtains

$$\langle j_\mu^{(\psi)}(x)\rangle_\alpha = \frac{1}{12\pi^2}\,\varepsilon_{\mu\beta\gamma\delta}\,\varepsilon_{abc}(\partial_\beta\alpha_a)(\partial_\gamma\alpha_b)(\partial_\delta\alpha_c)\left(\frac{\sin\alpha}{\alpha}\right)^2 \tag{2.18}$$

which is automatically conserved. As stated in the beginning of this paragraph, all results here agree with those obtained in Ref. 4) by direct methods.

The classical approximation

The formation of fractionally charged solitons can be visualized by bosonizing the system and treating classically the soliton configuration[4)]. We shall demonstrate this for the system (2.1) with $\phi_2 = 0$. Then the two dimensional Lagrangian is :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + i\bar\psi\partial\!\!\!/\psi - \lambda(\phi^2-a^2)^2 + G\bar\psi\phi\psi \tag{2.19}$$

The Lagrangian can be studied by bosonizing the system. A real field σ is introduced such that the following relations hold,

$$
\begin{aligned}
i\bar{\psi}\not{\partial}\psi &= \tfrac{1}{2}(\partial_\mu\sigma)^2 \\
\bar{\psi}\psi &= -C\mu N_\mu \cos 2\sqrt{\pi}\sigma \\
\bar{\psi}\gamma_\mu\psi &= \frac{1}{\sqrt{\pi}}\,\varepsilon_{\mu\nu}\partial^\nu\sigma
\end{aligned}
\tag{2.20}
$$

Thus the bosonized version of the Lagrangian of Eq. (2.19) is given by

$$
\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \lambda(\phi^2-a^2)^2 - G\mu\,(\cos 2\sqrt{\pi}\sigma)\,\phi \tag{2.21}
$$

It will turn out to be useful to write also the fermionic charge Q_F in terms of σ

$$
Q_F = \int_{-\infty}^{\infty}\bar{\psi}\gamma_0\psi\,dx = \frac{1}{\sqrt{\pi}}\,\Delta\sigma \tag{2.22}
$$

In order to identify the particle excitations of the system one should identify the various classical vacua of the system. They consist of the configurations (ϕ,σ) which minimize the potential $\tilde{V}(\phi,\sigma)$, where $\tilde{V}(\phi,\sigma)$ is given by

$$
\tilde{V}(\phi,\sigma) = \lambda(\phi^2-a^2)^2 + C\,G\mu\,\phi\cos(2\sqrt{\pi}\sigma) \tag{2.23}
$$

The minimal action configurations for $8\lambda a^3 \gg G\mu C$ are

$$
\left\{\phi = \bar{a} \simeq a + \frac{G\mu}{8\lambda a^2}\,,\ \sigma = \left(n+\tfrac{1}{2}\right)\sqrt{\pi}\right\} \quad \text{and} \quad \left\{\phi = -\bar{a}\,,\ \sigma = n\sqrt{\pi}\right\}
$$

The minima are shown in Fig. 1. Actually, one can show that in general the minima are at $\sigma_m = \frac{1}{2}\sqrt{\pi}k$ and $\phi_m = (-1)^{k+1}\bar{a}$, where $\bar{a}$ is the positive solution of the equation $4\lambda\bar{a}(\bar{a}^2-a^2) = CG\mu$ $(\bar{a}>a)$. The breakdown of the discrete chiral symmetry $\sigma \to \sigma+(\sqrt{\pi}/2)$ is reflected in two dimensions by the existence of solitons.

A drastic change has occurred with the introduction of the fermions into the system. For a zero Yukawa coupling, G, the solitons in the system would have $\phi(\infty) - \phi(-\infty) = 2a$ and, of course, zero $\Delta\sigma$ ($\Delta\sigma = k\sqrt{\pi}$ correspond to the original fermions). Once G is non-zero, this soliton has an infinite (infra-red infinity) energy, the Dirac sea becomes unstable, and a change in σ following the change in ϕ is required to stabilize the system. In this example, $\Delta\sigma$ is $\pm\sqrt{\pi}/2$, the two solitons acquire a $\pm\frac{1}{2}$ fermionic charge (the corresponding antisoliton will have as well $Q_F = \pm\frac{1}{2}$). The two solitons are shown in Fig. 1.

The fact that $\Delta\sigma \neq 0$ signifies the spontaneous breaking of the discrete chiral symmetry possessed by the Lagrangian, and $\bar{\psi}\psi$ obtains an expectation value [as the broken symmetry is discrete $(\phi \to -\phi,\ \psi \to \gamma_5\psi)$ no Goldstone boson is necessary].

Gauging the fractional charge

It has been shown that the lowest energy state in the soliton sector is a collective excitation of both the original soliton and the fermionic degrees of freedom. This excitation has a fractional global charge. We would like to find out to what extent the soliton will behave as a charged object. In order to test this we gauge the fermionic charge by adding a term $j_\mu A_\mu$ to the Lagrangian.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + i\bar{\psi}\not{\partial}\psi + G\bar{\psi}\phi\psi + e\bar{\psi}\gamma_\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}^2 \tag{2.24}$$

It is again convenient to study the energetics of the system in terms of its bosonized version. Using the notation displayed in Eq. (2.20), we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + N_\mu\left\{\frac{1}{2}(\partial_\mu\sigma)^2 - CG\mu\,\phi\cos(2\sqrt{\pi}\sigma) + \frac{e}{\sqrt{\pi}}\sigma\,\varepsilon_{\mu\nu}\partial^\mu A^\nu\right\} - \frac{1}{4}F_{\mu\nu}^2 \tag{2.25}$$

The redundant gauge field A_μ can be integrated out to give :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + N_\mu\left\{\frac{1}{2}(\partial_\mu\sigma)^2 - Gm\phi\cos(2\sqrt{\pi}\sigma) - \frac{e^2\sigma^2}{2\pi}\right\} \tag{2.26}$$

where $m = C\mu$. The classical vacua are the minima of the potential $W(\phi,\sigma)$, where $W(\phi,\sigma)$ is given by :

$$W(\phi,\sigma) = V(\phi) + Gm\phi\cos(2\sqrt{\pi}\sigma) + \frac{e^2\sigma^2}{2\pi} \tag{2.27}$$

We can now check the various regions. Let us start in the region $e^2 \ll Gma$ where a is the minimum of the potential $V(\phi)$. For simplicity, we assume that $V(\phi) = V(-\phi)$.

In the (ϕ,σ) plane, only one absolute minimum now survives, the one with $\sigma = 0$, $\phi = -\bar{a}$. Discrete chiral symmetry $(\psi \to \gamma_5\psi, \phi \to \phi)$ is explicitly broken. If e is small, many unstable minima survive, allowing the existence of pairs of both almost integrally and almost half-integrally charged particles. As e is increased, the maximally allowed almost integer charge, n_{max}, decreases until no unstable minima exist for a large enough coupling. What does this imply for the fractionally charge solitons ? We gauged the fermion charge, it thus interacts with A_μ, and in two dimensions a charged particle gets confined. The potential has only one absolute minimum and without degeneracy no solitons can exist. The fractionally charged soliton got confined and, of course, so did the fermions. The string which confines two objects has a tension proportional to the product of their charges. In our semi-classical picture the tension between two solitons is determined by the energy difference between the two minima which they bridge. The fractionally charged solitons are indeed confined by a tension which is weaker than the tension confining ordinary fermions by a factor 1/4 as required for a particle of charge $\frac{1}{2}$.

In this model one envisages also bound states composed of three particles. One fermion and a soliton-antisoliton pair, all these have together a total fermion charge of zero. This configuration in the (ϕ,σ) plane is shown in Fig. 2. These states are not stable topologically and may at best exist as metastable states. Another test to the way in which the soliton is really charged is to switch on a background electric field. The behaviour of charged particles in a background electric field has been studied extensively [6)]. We wish to verify that the fractionally charged solitons behave in a similar manner.

A constant electric field is introduced by adding a term

$$\frac{1}{2}\frac{\theta e}{2\pi}\varepsilon_{\mu\nu}F_{\mu\nu}$$

to the Lagrangian of Eq. (2.24). The bosonized form turns out to be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - V(\phi) - Gm\phi\cos(2\sqrt{\pi}\sigma) - \frac{e^2}{2\pi}\left(\sigma + \frac{\theta}{2\sqrt{\pi}}\right)^2 \qquad (2.28)$$

and the potential to be minimized is

$$\tilde{W}(\phi,\sigma) = V(\phi) + Gm\phi\cos(2\sqrt{\pi}\sigma) + \frac{e^2}{2\pi}\left(\sigma + \frac{\theta}{2\sqrt{\pi}}\right)^2 \qquad (2.29)$$

In this case, the system will still have one absolute minimum at that value of σ (picked from the extrema of the cosine term) which minimizes $(\sigma + \theta/2\sqrt{\pi})^2$. However, the degeneracy in σ is resurrected for $\theta = \frac{\pi}{2}$ which is equidistant from $\sigma = 0$ and $\sigma = -\sqrt{\pi}/2$; thus at $\theta = \frac{\pi}{2}$ the configuration $(\phi = \tilde{a}, \sigma = -\sqrt{\pi}/2)$ and the configuration $(\phi = -\tilde{a}, \sigma = 0)$ are degenerate, and a soliton with $\Delta\sigma = \pm\sqrt{\pi}/2$ (fermion number $\pm\frac{1}{2}$) reappears in the same manner as Coleman's half asymptotic states appear in the massive Schwinger model for $\theta = \pi$. The fractional charged soliton has been liberated. Note that for large e ,

$\sigma = -\theta/2\sqrt{\pi}$ is the sole minimum and no liberation seems to occur. These results are periodic in θ (with a period π), in the sense that for $\theta = \pi/2 + n$ ($n = 0,\pm1,\pm2, \ldots$) solitons are liberated.

The fact that the period has been reduced to π is evident from the invariance of the Lagrangian under the following transformations

$$\theta \to \theta + n\pi \ ; \ \sigma \to \sigma - \frac{\sqrt{\pi}}{2}n \ ; \ \phi \to -\phi \tag{2.30}$$

Obviously, on the background of a soliton with well-defined "ϕ number", the periodicity is still 2π. In the massive Schwinger model where the periodicity in θ is 2π, the θ term generally breaks C and P, however, for $\theta = n\pi$ one can redefine the parity and charge conjugation symmetries such that the theory is C and P invariant.

These modified transformations are :

$$\begin{aligned} P \ ; \ & \sigma(x) \to -n\sqrt{\pi} - \sigma(-x) \quad ; \ F_{01}(x) \to -F_{01}(-x) \\ C \ ; \ & \sigma(x) \to -n\sqrt{\pi} - \sigma(x) \quad ; \ F_{01}(x) \to -F_{01}(x) \end{aligned}$$

One expects, however, that these symmetries will be spontaneously broken at $\theta = (2n+1)\pi$. In fact, this violation is realized by the appearance of the "half asymptotic" states.

In our case, Eq. (2.28), due to the shorter period in θ, we can now redefine C and P symmetries for $\theta = (2n+1)\pi/2$ which are again expected to be spontaneously broken. These are:

$$\begin{aligned} P \ ; \ & \sigma(x) \to -n\frac{\sqrt{\pi}}{2} - \sigma(-x) \quad ; \ F_{01}(x) \to -F_{01}(-x) \quad ; \ \phi(x) \to -\phi(-x) \\ C \ ; \ & \sigma(x) \to -n\frac{\sqrt{\pi}}{2} - \sigma(x) \quad ; \ F_{01}(x) \to -F_{01}(x) \quad ; \ \phi(x) \to -\phi(x) \end{aligned} \tag{2.31}$$

The existence of a liberation angle emphasizes again that the fractional charge behaves as a regular charge. One may wonder if by varying the value of the fractional charge, one may also vary the value of the θ angle at which charge liberation occurs. This does happen. To see the latter, consider a Lagrangian introduced by Goldstone and Wilczek [4] whose interaction part is :

$$\mathcal{L}_I = \bar{\psi}\, e^{i\alpha\gamma_5}\psi - V(\alpha) \tag{2.32}$$

α being a real scalar field. By bosonization, one obtains that in the presence of an electric background field (after gauging first the fractional charge), the potential $\tilde{V}(\phi,\sigma)$ to be minimized is

$$\tilde{V}(\phi,\sigma) = Gm\cos(2\sqrt{\pi}\sigma-\alpha)+V(\alpha)+\frac{e^2}{2\pi}\left(\sigma+\frac{\theta}{2\sqrt{\pi}}\right)^2 \tag{2.33}$$

$V(\alpha)$ has a discrete set of degenerate minima. For $e = 0$, the soliton carries a fermionic charge given by

$$Q_F = \frac{\Delta\alpha}{2\pi}+n \tag{2.34}$$

where $\Delta\alpha$ is the difference between the α values of degenerate minima of the effective potential. By tuning $\Delta\alpha$ one may obtain any value for Q_F.

For $e^2 << 2\pi$ Gm and $\theta = 0$ the soliton, which now carries a gauged charge, gets confined. The effective potential has only one minimum. At the liberation angle, θ_ℓ , at which half-asymptotic states appear in the spectrum, the effective potential will have two degenerate minima. This occurs for

$$\theta_\ell = Q_F\pi-\alpha_o \tag{2.35}$$

where α_o is the value of α at $+\infty$. [For a symmetric $V(\alpha)$, α at $-\infty$ is $-\alpha_0$ and $\theta_\ell = 0$.] In other words, the existence of a fractional charge causes the screening to occur at a different value of the background electric field. The ratio $(\theta_\ell+\alpha_o)/Q_F$ is π. One may also enquire as to the length of the period in θ. It is clear that the effective potential (2.33) has a period 2π, independent of the form of the potential $V(\alpha)$.

If the potential $V(\alpha)$ is periodic too, an additional structure will emerge. Actually the outcome depends on $2\pi P$ the period of $V(\alpha)$. The effective potential will remain invariant under the transformation

$$\begin{aligned}
\theta &\to \theta - 2\pi P m + 2\pi n\\
\sigma &\to \sigma + \sqrt{\pi}Pm - \sqrt{\pi}n\\
\alpha &\to \alpha + 2\pi P m
\end{aligned} \tag{2.36}$$

In the case $e = 0$, the soliton carries a fractional charge $Q_F = P$ and thus the system will have a period $2\pi Q_F$. The problem can become more intriguing if P is not a rational number, that is if the periodic of $V(\alpha)$ is incommensurate with 2π. In that case, the system will have two periods. It is known that similar systems [7] have quite an erratic θ dependence. Any $\theta/2\pi$ of the form $Pm+n$ is equivalent to $\theta = 0$. This is true also for any P, but for P irrational, we have an infinite number of values of θ, arbitrarily close to $\theta = 0$, for which there is no C and P violation (they still are of measure zero). As the minima of V are discrete a would-be Goldstone boson ("axion") would not result. A concrete example of such effects may be a model with two massive charged particles, one of them also having a Thirring interaction, which makes their periods incommensurable. One may wonder whether a similar mechanism may work in four dimensions to cure the CP problem.

The four-dimensional non-Abelian gauge theory has an extra C and P violating parameter, θ, in a dilute instanton approximation the physics is periodic in θ with a period 2π. We do not understand how to describe this phenomenon in terms of screening of some "background" described by a non-zero θ. We presume that such a description exists and that some kind of charge does the screening, if such a charge can be endowed with an additional fractional charge it could increase the amount of CP conserving θ's without leading to an axion Admittedly, this scenario for the non-Abelian case is rather loose, but in view of the rich structure theories have in the presence of a θ parameter, we wish to point out this possibility.

3. - FRACTIONALIZING A GAUGED CHARGE

Having studied the gauging of a global fractional charge, let us turn to a system in which the fractional charge is gauged ab initio. It was pointed out by Witten [5] that a monopole (a solitonic configuration in non-Abelian theories) of magnetic charge M (in units of $2\pi/e$), will obtain a fractional electric charge $\theta M/2\pi$ (in units of e), in the presence of a CP violating $(\theta/32\pi^2)F\tilde{F}$ term in the Lagrangian. In two dimensions, there is only an electric field, F_{01}. In order to introduce an analogue to the magnetic charge, we will make use of another charge available in two dimensions, the topological charge. Consider thus a real scalar field coupled to the gauge field via its topological charge. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \frac{e\phi\,\varepsilon_{\mu\nu}F_{\mu\nu}}{2\sqrt{\pi}} - \frac{1}{4}F_{\mu\nu}^2 + \frac{\theta e\,\varepsilon_{\mu\nu}F_{\mu\nu}}{4\pi} \tag{3.1}$$

where $V(\phi)$ has a number of degenerate minima. Integrating out $F_{\mu\nu}$ one obtains :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) - \frac{e^2}{\pi}\left(\phi + \frac{\theta}{2\sqrt{\pi}}\right)^2 \tag{3.2}$$

First, consider the system for $\theta = 0$ and $V(\phi) = V(-\phi)$. In that case the modified potential $V(\phi) + (e^2/\pi)\phi^2$ still has a discrete symmetry $(\phi \to -\phi)$ and for small enough e, the potential has two degenerate minima at $\phi = \pm a$, say. Solitons for which $\phi(\infty) - \phi(-\infty) \neq 0$ exist. We wish to draw an analogy between some properties of this soliton and some properties of a magnetic monopole. Let us now switch on θ. The discrete symmetry is explicitly broken. The total potential will have only one minimum as shown in Fig. 3. This situation should remind us of the gauged soliton of the former section. As the system has only one minimum, the soliton state has acquired infinite energy. Only pairs of soliton-antisolitons interpolating between the non-degenerate minima are allowed, their energy being estimated by the tension of the "string" which connects them. The string tension $2e^2 a\theta/\pi\sqrt{\pi}$ (for small e) is proportional to θ. We describe these circumstances by claiming that the soliton has acquired an electric charge whose confining properties are proportional to $\sqrt{\theta}$. In conclusion, for $\theta = 0$ the soliton, and the perturbation theory excitations are not confined. For $\theta \neq 0$, the topologically trivial perturbation theory excitations retain their neutrality, while the topological excitation gains an electric charge ("monopole") $M = e(2a\theta/\pi\sqrt{\pi})^{\frac{1}{2}}$ (and becomes a confined "dyon"). In this model $\theta = 0$ is the "liberation angle". Note that if $V(\phi)$ is non-periodic in ϕ then, in this system, one has an example of a non-periodic behaviour in the θ parameter. This behaviour follows from the absence of an infinite sequence of integer electric charges due to the finite number of minima of the potential $V(\phi)$. Once a θ dependence has been introduced, let us focus on phenomena which could conspire to eliminate it. One expects that by adding massless fermions (with chiral invariant interactions), the θ dependence disappears. If one would break the continuous chiral symmetry by adding a Yukawa coupling, G, the θ dependence should survive. Let us first set G to zero, in which case all theories for different θ should be equivalent. We recall that for $\theta = 0$, objects of non-trivial topology were neutral in the sense that they were not confined. We should now enquire how, for $\theta \neq 0$ and with massless fermions around, does the soliton ("dyon") lose its charge and gets liberated.

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \frac{e_1\phi}{2\sqrt{\pi}}\varepsilon_{\mu\nu}F_{\mu\nu} + \frac{e_2\sigma\varepsilon_{\mu\nu}F_{\mu\nu}}{2\sqrt{\pi}} + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{4}F_{\mu\nu}^2 + \frac{e_3\theta}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu} \tag{3.3}$$

where we have already bosonized the fermion bilinear. Integrating out the gauge field, one obtains

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \left[V(\phi) + \frac{1}{2\pi}\left(e_1\phi + e_2\sigma + \frac{e_3\theta}{2\sqrt{\pi}}\right)^2\right] \tag{3.4}$$

For any θ the potential $V_\theta(\phi,\sigma)$ would be minimized by the classical configurations $(\phi = \phi_\pm, \sigma = \sigma_\pm)$, where ϕ_+, ϕ_- refer to the degenerate minima of $V(\phi)$ and $\sigma_\pm = -1/e_2(e_1\phi_\pm + \{\theta e_3/2\sqrt{\pi}\})$. Solitons are thus liberated and have effectively lost their charge. We have already remarked that when σ obtains a classical expectation value, breakdown of continuous chiral symmetry is implied. No Goldstone bosons arise due to the chiral anomaly which generates the mass of a "η" particle represented by the fluctuation of σ. Another language to describe the phenomenon is the formation of a chiral condensate resulting in the complete screening of the θ background electric field. What has occurred is not unlike what occurs in the masless Schwinger model in the presence of a background field.

Let us next consider the case for which continuous chiral symmetry is explicitly broken by the addition of a Yukawa term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + i\bar{\psi}\not{\partial}\psi - \frac{1}{4}F_{\mu\nu}^2 - V(\phi) + \frac{\varepsilon_{\mu\nu}F_{\mu\nu}}{2}\left(\frac{e_1\phi}{\sqrt{\pi}} + \frac{e_3\theta}{2\pi}\right) + G\mu\phi\bar{\psi}\psi \tag{3.5}$$

The bosonized form is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \left[V(\phi) + \frac{1}{2\pi}\left(e_1\phi + e_2\sigma + \frac{e_3\theta}{2\sqrt{\pi}}\right)^2 + c\, G\mu\phi\cos 2\sqrt{\pi}\sigma\right] \tag{3.6}$$

Due to the Yukawa term, the effects of θ cannot be eliminated. For $e_3 = e_2 = 0$ and $e_1 << 2\pi G\mu$ the system consists of solitons of fermion number $\pm\frac{1}{2}$.

Next, let us switch on the background field keeping the fermion number ungauged, namely $e_2 = 0$, with e_1^2, $e_3^2 << 2\pi G\mu$. In this case, we expect the soliton with fermion number $\pm\frac{1}{2}$ to gain an electric charge related to θ and thus become confined. The confined solitons will have both electric and fermionic charge. In this case, the fermionic charges are $\pm\frac{1}{2}$ and the electric charges are $\pm[\,|(e_1e_3a\theta/\pi\sqrt{\pi})|\,]^{\frac{1}{2}}$.

The theory is not periodic in θ. When we gauge the fermionic charge $(e_2 \neq 0)$, integer charged fermions are confined and the periodicity in θ is regained. The system is invariant under

$$\sigma \to \sigma + \sqrt{\pi}\, n$$
$$\theta \to \theta - 2\pi n \frac{e_2}{e_3}$$

Thus the period in θ is $2\pi(e_2/e_3)$.

4. - DISCUSSION

We have shown that there exist new variables in terms of which a lowest order calculation gives the fractional charge residing on a soliton. We have demonstrated in two dimensions that this is a bona-fide charge. This has been established by gauging the fractional charge and studying its electromagnetic properties. In the two-dimensional models the charged soliton confined. It may be liberated in the presence of an electric background field θ. The period length of θ, and the value of θ at which liberation occurs are determined by the value of the fractional charge. Interesting phenomena occur when the scalar field is self-interacting with a periodic potential whose period is incommensurable with 2π. A scenario for having a large number of C and P conserving theories emerges.

We have also constructed two dimensional models in which fractional gauged charges are induced on a soliton by a background electric field, and are wiped out by the introduction of massless fermions.

We conclude by a four-dimensional analogue of the construction. Assume one has a Georgi-Glashow [8] model with a potential of the form $V(\phi) = \vec{\phi}^2(\vec{\phi}^2-a^2)^2$, for which $|\vec{\phi}|$ at infinity may be either zero or a. Such a theory would support monopoles, when a term which prefers $\vec{\phi} = 0$ is added to $V(\phi)$ the monopole ceases to be a stable excitation. Nevertheless one may envisage the following metastable excitation. The field ϕ in the radial direction would start from $\vec{\phi} = 0$ rise to $|\vec{\phi}| = a$ and go back to $\vec{\phi} = 0$ at infinity. This configuration looks like a monopole wrapped by an antimonopole. The monopole would behave as if it were superconfined, that is as if it had acquired some form of "colour". Unfortunately $V(\phi)$ is non-renormalizable in more than three spacetime dimensions. In the 2+1 dimensional case, vortices play the rôle of the monopoles.

ACKNOWLEDGEMENTS

W.A.B. and Y.F. would like to thank the Max-Planck Institute for Physics for its hospitality during the completion of this work. W.A.B. would also like to thank the Racah Institute of the Hebrew University for making this research possible. S.E. would like to thank the Theory Division at CERN for its hospitality. E.R. is indebted to K. Bardakci with which he collaborated on the issue of disappearing dyons. We wish to thank also David Gross for a discussion.

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FIGURE CAPTIONS

Fig. 1 : The classical vacua in the (ϕ,σ) plane.

Fig. 2 : A metastable soliton-antisoliton-fermion bound state.

Fig. 3 : The effective potential for $\theta \neq 0$.

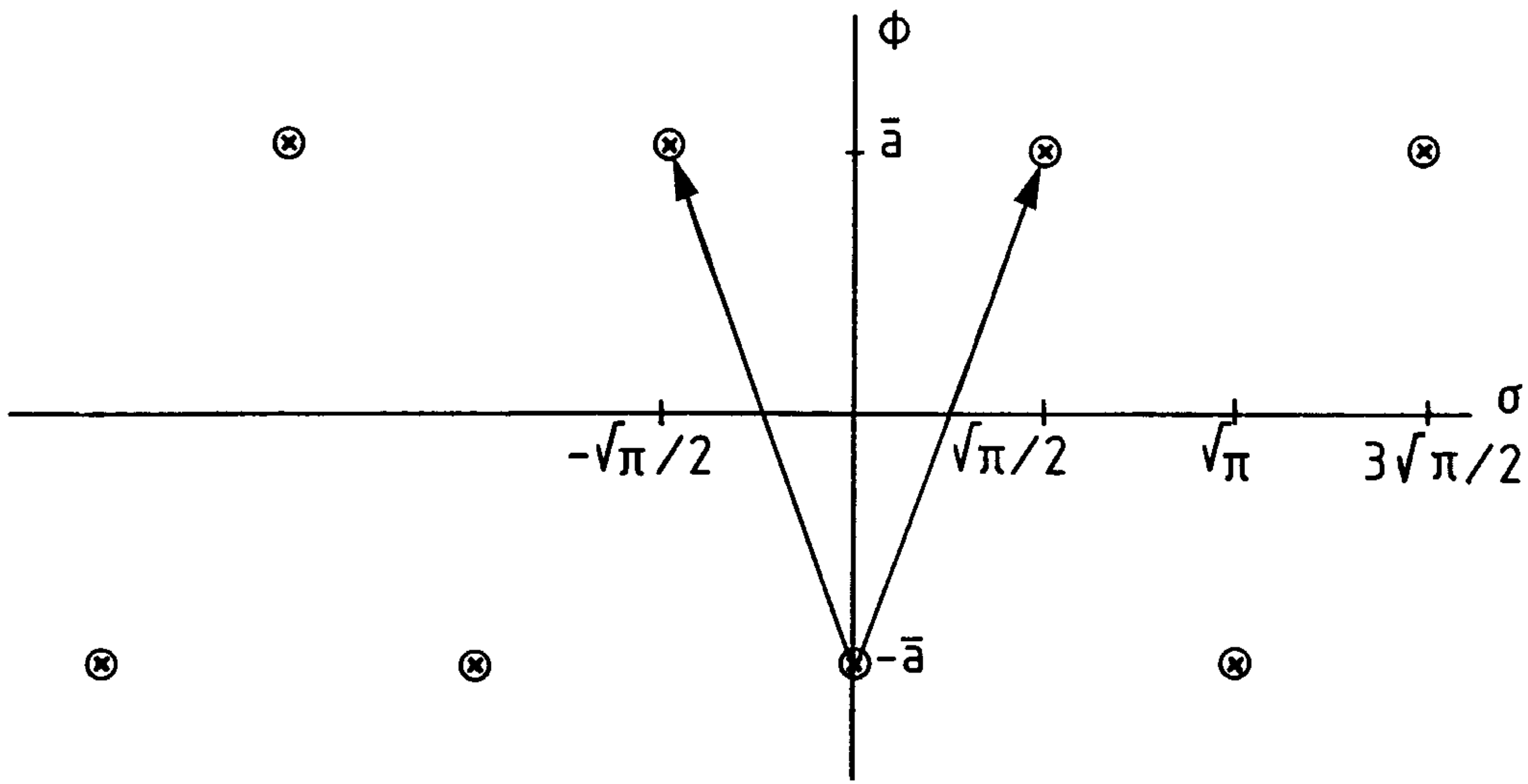

Fig.1

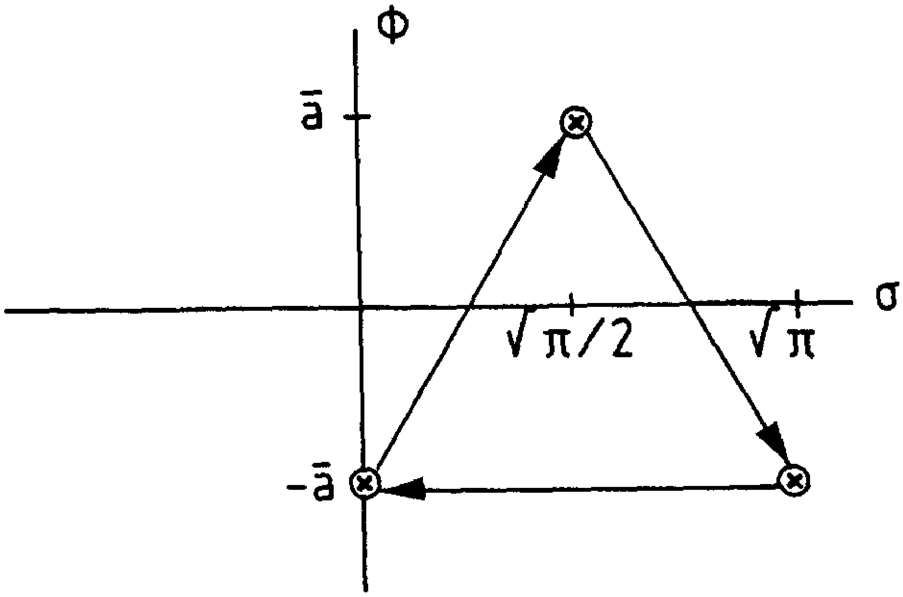

Fig.2

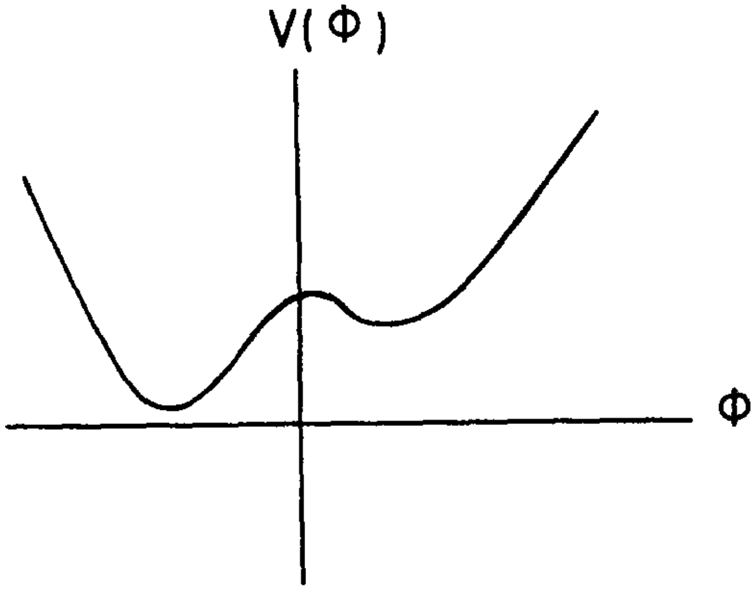

Fig.3