Fermi National Accelerator Laboratory

FERMILAB-Pub-82/23-THY

BROWN-HET-469

REDUCTION OF TENSOR PRODUCTS WITH DEFINITE PERMUTATION SYMMETRY: EMBEDDINGS OF IRREDUCIBLE REPRESENTATIONS OF LIE GROUPS INTO FUNDAMENTAL REPRESENTATIONS OF SU(M) AND BRANCHINGS

A. N. Schellekens
Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, IL 60510

In-Gyu Koh
Department of Physics
Sogang University
CPO Box 1142, Seoul, Korea[1]

and

Department of Physics
Brown University
Providence, RI 02912

and

Kyungsik Kang[2]
Department of Physics
Brown University
Providence, RI 02912

ABSTRACT

We consider tensor products made out of a number of identical copies of the defining representations of Lie groups that are asymptotically free and complex. Decomposition of the tensor products into the terms with definite permutation symmetry is made by using the index sum rules and the congruence class. The results can also be used to find the branchings of SU(M) into a Lie Group G where M is equal to the dimension of the defining representation of G. Application of our results to preon dynamics is indicated in two examples.

[1]Permanent address

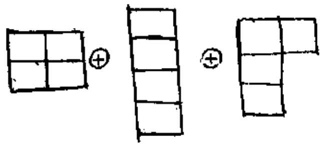
[2]Supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ER03130.A009 - Task A.

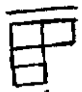
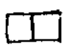
I. Introduction.

Gauge theories are generally regarded as the theories of elementary particle interactions. In a gauge theory, whether it is a grand unified theory or "preon dynamics", one generally starts out with a certain non-Abelian gauge group and writes down an invariant Lagrangian in terms of particle fields which transform as certain representations of the given gauge group. The fermion representations are usually required to satisfy additional conditions.[1] For instance, in many models the fermion representations should be complex[2] to prevent large masses for the known particles. Another requirement is that the representation should be free of triangle anomalies,[3] otherwise the theory will be unrenormalizable. The third condition often adopted is that the representations should be asymptotically free in the full gauge degree of freedom,[4] not just in the SU(3) color subgroup. Recently there have been efforts to obtain complete lists[5] of both irreducible and reducible representations that are complex, anomaly-free and asymptotically free. We use these requirements only to get a natural limit on the representations considered in this paper.

Having chosen the representations under due conditions, one has to construct a gauge invariant form of the Lagrangian. Here, one generally needs to know the properties of tensor products of the representations. Not only does one then need to specify how the tensor products can be computed, i.e., obtain the Clebsch-Gordan series, but also how they reduce to a direct sum of irreducible representations, each of which exhibits a definite permutation symmetry. The method of the decomposition[6] of the tensor product of n identical representations into the component with definite permutations property is called the algorithm of "plethysm".

For example, consider a Yukawa coupling of the form $(f_L \otimes f_L)\phi$ in an SU(N) gauge theory, where f_L is the fermion field which belongs to the irreducible representation  . We then have

$f_L \otimes f_L =$  (1)

of which the first two terms are symmetric under interchange of f_L while the third term is antisymmetric with respect to the interchange of f_L. Hence if, for example, ϕ belongs to  , then the corresponding Yukawa coupling must be antisymmetric under the interchange of other labels such as the family indices. The alluded permutation properties of each of the three components under the interchange of f_L can be understood as follows: Consider the fundamental representation  of SU(M) where M is equal to the dimension of the defining representation  in SU(N), i.e., the dimension of the fermion representation f_L. The group SU(M), in the fundamental representation, consists of all special unitary transformations of the M components of f_L; the SU(N) transformations on f_L form a subgroup, embedded in SU(M). This is referred to as a nonregular embedding of SU(N) into SU(M) in this paper. The symmetric part of $(f_L \times f_L)$ i.e., the first and second terms of Eq. (1), corresponds to  in SU(M), whereas the third term of Eq. (1) corresponds to the antisymmetric representation  of SU(M). Such a method of decomposition with given permutation properties is what we call the algorithm of phethysm. Thus the computation of plethysm is equivalent to the direct computation of branching of SU(M) $\rightarrow$ SU(N).

In this paper, we present the computation of plethysm for the complex and asymptotically free representations of Lie groups, SU(N) (type A_{N-1}), SO(4N + 2)(type D_{2N+1}), and E_6. The results of this paper have been applied to preon dynamics[7] for each of these groups with the correct Fermi statistics constraints and reported in a separate paper.[8] In particular, for SU(N) the results are given for the seven of nine irreducible, complex and asymptotically free representations;[1] for SO(4N+2), the results are given for the lowest-dimensional spinor representations of D_5, D_7 and D_9 (note that D_3 is isomorphic to A_2); for E_6, we give the computation of plethysm for the fundamental representation $\underset{\sim}{27}$. We have considered the direct products of at least two copies of the representation in all of these groups and in some cases the direct products up to ten copies of the representation. In fact, the recent suggestion[7] that the quarks and leptons are bound states of certain fundamental "preons" requires in general such computation of plethysm in conjunction with the requirement of Fermi statistics in the ground states.[9] Two examples showing the relevance of the results of this paper for preon dynamics will be discussed in Section IV. The computation of plethysm involving tensor products of several copies of the representations is also needed in tumbling gauge theories.[10]

The paper is organized as following: In Section II, we present the method of plethysm based on the index sum rules and congruence numbers. Section III contains the results of the decomposition of tensor products with definite permutation symmetry. The results can also be used to find the SU(M) branching into a Lie group G for the case of nonregular embedding of G into SU(M). Then we give two examples of application of our results to preon dynamics in Section IV. Appendix A contains continuation of SU(N)-indices to real N which allows us to use large values of N without the need to calculate large sums. Finally we present in Appendix B transposition rules for SU(N)-plethysm.

II. Method of Plethysm Computation.

In this section we describe the method of the decomposition of the tensor products into the components with definite permutation property. This involves two steps: the first step is to calculate the tensor products of some copies of representations and the second step is to reduce the tensor products to a direct sum of components, each of which has definite permutation property.

There are several ways of handling these steps. For example, one may use the complete weight systems of the representation[11] to obtain the tensor products and find the highest weight terms successively. This method, however, is not only cumbersome when the dimension or rank is large but is not complete to identify the terms with definite permutation property. One may, on the other hand, use the method of the projection operators.[12] As the projection operator takes an irreducible representation of SU(M) into the representations of a group G directly where M is the dimension of the defining representation of the fermion fields in G, the identification of the terms with given permutation properties are achieved automatically without recourse to the reduction of tensor products. But this method too becomes rapidly impractical as the dimension or rank becomes large.

The method we adopt in this paper is based on the properties of the zeroth, second, third and fourth indices of representations as well as classification of the representations by congruence number. It is known that[13] the indices of the representations of Lie groups provide useful clues in the search of Clebsch-Gordan series as well as branching rules. In addition, the congruence number [14] reduces further the search problem by classifying the representations. Generally speaking, a representation and its complex conjugate do not have the same congruence number, thus the congruence class is useful in groups like D_{2N+1} and E_6 where the indices alone can not distinguish a representation from its complex conjugate. Note that D_{2N+1} and E_6 are anomaly-free while in A_N the triangle anomaly, i.e., the third index, of a representation has opposite sign of that of the complex conjugate representation. It should be emphasized that the indices and congruence number satisfy certain elegant relationships that can be used easily even when the dimension of the representation or rank of the group becomes huge.

Now we proceed to introduce the indices of the representations of a semisimple Lie algebra. In general, the indices are defined differently depending on whether the order of the representation is even or odd.

The index of order 2m of a representation R is defined by

$$I_{2m}(R) = \sum_{M \varepsilon W(R)} (M,M)^m \tag{2}$$

where W(R) is the weight system of R and m = 0,1,2,$\cdots$. Though the indices of higher orders are known, we will use in this paper only up to the fourth index as these low-order indices can be computed simply[13] from the use of the highest weight of R. It should be obvious that $I_o(R)$ is the dimension of R as every weight contributes 1 to the sum (2). The third order index of R is defined[13] by

$$I_3(R) = a \sum_{M \varepsilon W(R)} (PM)^3 \tag{3}$$

where PM is the projection of the weight $W(R)$ on a properly chosen direction in weight space. The third order index I_3 is trivial, i.e., zero from the property of $W(R)$ for all Lie algebras except for SU(N) with $N \geq 3$. As noted before, I_3 for SU(N) is the triangle anomaly number. The normalization constant a of Eq. (3) can be fixed in such a way that $I_3 = 1$ for the fundamental representation of SU(N) ($N \geq 3$). The first index of any representation of any Lie group is zero and thus plays no useful role. Thus indices can be used both to calculate tensor products and to determine branching rules.

To calculate the tensor product of the representations R and R' one has to determine the multiplicities ℓ_i of representations R_i which appear as a direct sum:

$$R \otimes R' = \sum_{\oplus i} \ell_i R_i \tag{4}$$

The total indices of $R \otimes R'$ are related to the individual indices of R and R' in the following way[13]:

$$I_o(R \otimes R') = I_o(R)\ I_o(R') \tag{5}$$

$$I_2(R \otimes R') = I_2(R)\ I_o(R') + I_2(R')\ I_o(R) \tag{6}$$

$$I_3(R \otimes R') = I_3(R)\ I_o(R') + I_3(R')\ I_o(R) \tag{7}$$

$$I_4(R \otimes R') = I_4(R)\ I_o(R') + I_4(R')\ I_o(R) + \frac{2(r+2)}{r} I_2(R) I_2(R') \tag{8}$$

Here r is the rank of the Lie algebra. The total indices of the right-hand side of Eq. (4) are given by sums of individual indices of the representations R_i

$$I_\ell(R \otimes R') = \sum_i \ell_i I_\ell(R_i)\ , \qquad (\ell = 0,2,3,4) \tag{9}$$

Combining Eqs. (5) - (9) one obtains four relations which will be referred to as the index sum rules for tensor products henceforth. These relations form four linear equations for the multiplicities ℓ_i, which determine the ℓ_i's completely if there is a unique integer solution. For sufficiently small representations (which turn out to be sufficiently large for all our purposes) there is only one source of ambiguity, the complex anomaly-free representations. Since the indices I_o, I_2 and I_4 are identical for R_i and R_i^* they can only be distinguished by the I_3 relation, which for anomaly-free representations is identically zero. Since complex anomaly-free irreducible representations are extremely large in SU(N)[1] this problem occurs in practice only in the groups SO(4n+2) and E_6. This ambiguity can easily be settled by means of the congruence class.[14]

The congruence class is the generalization of "N-ality" for SU(N) to any simple Lie algebra. All representations of a Lie algebra can be assigned to such a class. This class is identified by one or two numbers C(R), defined modulo a certain integer n_c. For the details of this assignment we refer to Ref. 14. For our purpose, the important properties are the fact that complex conjugate representations have different congruence numbers, and that each representation R_i in Eq. (14) has the same congruence number, related to the congruence classes of R and R' in the following way:

$$C(R_i) = C(R) + C(R') \qquad (\text{modulo } n_c) \tag{10}$$

This additional relation resolves the ambiguity.

Having determined the right-hand side of Eq. (4) we now have to identify terms with definite permutation properties. This problem is equivalent to finding the branching rules for $SU(M) \supset G$, when a representation D of G (hereafter referred to as the "defining representation") is embedded in the fundamental representation of SU(M). Of course M must be equal to the dimension of D in G.

Irreducible representations of SU(M) can be specified by Young-diagrams. On the other hand, Young-diagrams also have an interpretation as representations of the permutation group. This dual interpretation is the basis of our results.

A representation R of SU(M), given by a Young-diagram Y_m with m boxes, branches into a direct sum of representations of the subgroup G:

$$R \rightarrow \sum_{\oplus j} \ell_j R_j \tag{11}$$

where the ℓ_j's are integer multiplicities. The permutation group interpretation tells us that the left-hand side of Eq. (11) corresponds to those terms in the m^{th} tensor power of D which have symmetry properties given by Y_m. To determine the multiplicities we use the SU(M)-interpretation of (11). The branching is governed by the following index sum rules [8,13]

$$I_o(R) = \sum_j \ell_j \, I_o(R_j) \tag{12}$$

$$I_2(R) = \rho_2 \sum_j \ell_j I_2(R_j) \tag{13}$$

$$I_3(R) = \rho_3 \sum_j \ell_j I_3(R_j) \tag{14}$$

The scale factors ρ_2 and ρ_3 are only dependent on the way the subgroup is embedded in SU(M). We emphasize that they do not depend on R. Therefore we can calculate them by choosing R equal to the fundamental representation of SU(M), which branches to the representation D of G:

$$\rho_2^{-1} = I_2(D) \tag{15}$$

$$\rho_2^{-1} = I_3(D) \tag{16}$$

Here we have used the standard normalization, $I_2 = I_3 = 1$ for the fundamental representation of SU(M). A relation similar to (13) and (14) for I_4 holds only for a few special cases and can not be applied to SU(M)-branchings.[13] Notice that relation (14) is only nontrivial for SU(N).

The tensor product is used only to limit the number of representations R_j on the right-hand side of Eq. (11). The most effective way of doing this is to use recursion in the number of boxes m of Y_m. When the results for all Young-diagrams Y_{m-1} are known one can multiply each of them with D and use relations (12) to (14) to decompose the tensor product into the terms with permutation properties defined by m-box Young-diagrams. With this procedure the direct sums belonging to all m-box Young-diagrams, with the exception of totally symmetric and totally antisymmetric ones, are determined several times, which can be used either as a consistency check or as additional information to rule out possible ambiguities which might arise from Eqs. (12) - (14) alone. In practice we did not encounter any persistent ambiguities.

In the special case G = SU(N) the procedure can be made much more effective in the following way. The results we are calculating can be expressed entirely in terms of SU(N)-tensors, without any reference to the rank of the group. Therefore all branchings can be generalized to arbitrary N, with the understanding that Young-diagrams with more than N rows should be ignored. Thus one can use the index relations with arbitrary N. For an n-box Young-diagram the formula for I_o is an n^{th} order polynomial in N, and those for I_2

and I_3 have order n-1. Therefore, if n and m are the number of boxes of SU(N) and SU(M) diagrams one gets roughly 3nm equations for the multiplicities from Eqs. (12) - (14), a few of which turn out to be dependent. Since the number of terms in the tensor product grows faster than the number of equations with increasing n or m, this method has its limitations. In general, even with the help of a computer, it turned out to be very hard to go beyond nm = 12.

In Appendix A we derive formulas for the indices which proved to be very useful for our calculations, since they are continuous in the rank of the group. This allowed us to exploit the N-independence property of the index sum rules more effectively. In Appendix B we derive rules which relate the plethysms for a Young-diagram Y_n of SU(N) to those for the transposed Young-diagram.

III. Results.

Generally grand unified theories and preon dynamics require fermion representations which are anomaly-free, complex and asymptotically free. Complete list of these representations has already been compiled. In SU(N), there is no complex irreducible representation that satisfies the requirement of both anomaly-freedom and asymptotic freedom.[1] Thus one considers the reducible complex representations formed out of the anomaly-free combinations of the irreducible complex representations which are asymptotically free. It has been shown that there are nine such irreducible representations in SU(N). Of all of the representations of Lie algebras, the only complex irreducible representations which are both anomaly-free and asymptotic free, are the following: the 16-, 126-, 144-dimensional representation of SO(10); the lowest dimensional spinorial representations of SO(14) and SO(18); and the 27-dimensional representation of E_6.

Now we proceed to present the results of computations of plethysm for the seven asymptotically free and complex representations $[1^4]$, $[2^2]$, $[1^3]$, $[2,1]$, $[2,1^2]$, $[2]$ and $[1^2]$ of SU(N) (see the notation for the Young-diagrams below); the five irreducible representations of SO(4N+2) mentioned above; and the lowest dimensional representation of E_6.

A. SU(N)

Since irreducible representations for an SU(N) can be represented by simple Young-diagrams, the use of the Young-diagrams is convenient. We will denote a typical Young-diagram that has a boxes in each of the first n rows followed by b boxes in each of the next m rows and so on by $[a^n,b^m,\cdots]$. These Young-diagrams are used for both the defining representations of SU(N) and the representations of SU(M), M being the dimension of the defining representation.

Table I shows the terms up to the tensor product of ten copies with the definite permutation properties under the interchange of defining representations $[2]$ and $[1^2]$ of SU(N). In other words, the results contained in Table I correspond to Young-diagrams of SU(M) having up to ten boxes. Table II summarizes the results for the four representations $[1^4]$, $[2^2]$, $[1^3]$ and $[2,1]$ of SU(N) up to three boxes in the SU(M) Young diagrams.

B. SO(4N+2)

Table III gives the results for the spinorial representation of SO(10) up to five boxes in the SU(16) Young diagrams. Tables IV and V contain the results for the spinorial representations of SO(14) and SO(18) up to four and two boxes in the SU(M) Young diagram respectively. The results for the 126- and 144-dimensional representations of SO(10) are summarized in Table VI up to three boxes.

C. E_6

Table VII summarizes the results up to six boxes. As we mentioned before, the 27-dimensional representation is the only E_6 representation which satisfies asymptotic freedom.

IV. Application of the Results to Preon Dynamics.

The results of this paper can be applied to preon dynamics in which quarks and leptons are viewed as the bound states of the elementary preons. Here, we give two such examples.[8]

(A) $(5 + 10^*)_L$ of SU(5) metacolor group as the preon representation.

The SU(5) representation $(5 + 10^*)_L$ is anomaly-free where L denotes the left-handed chiral state. In order for the preons to be confined, the preon representations should satisfy asymptotic freedom. The anomaly-free representation can then be repeated up to 13 times without loosing asymptotic freedom. Suppose that we allow the representation $5 + 10^*$ to repeat N times where N is an integer less than 14. Such repetition then introduces the metaflavor group U(N) x U(N) which is broken to SU(N) x SU(N) x U(1) taking into account the instanton effects due to the metacolor group SU(5).

Let us denote the metacolor representation as

$$(5 + 10^{*})_L = \alpha + \beta \tag{17}$$

The transformation properties of α and β under metacolor group SU(5) and metaflavor group SU(N) x SU(N) x U(1) as well as the spin group $SU_L(2)$ x $SU_R(2)$ are summarized as follows:

	SU(5)	SU(N)	SU(N)	U(1)	$SU_L(2)$	$SU_R(2)$	5-ality
α	$\square$	$\square$	•	Q_1	$\square$	•	1
β	$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$	•	$\square$	Q_2	$\square$	•	3
$\bar{\alpha}$	$\overline{\square}$	$\overline{\square}$	•	$-Q_1$	•	$\square$	4
$\bar{\beta}$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	•	$\overline{\square}$	$-Q_2$	•	$\square$	2

Here, 5-ality is the congruence number and Q_1 and Q_2 are chosen in such a way that the $[SU(5)]^2$ U(1) anomaly vanishes. There are four candidates for the massless bound states coming from the four singlet states of metacolor SU(5):

$$\alpha^5,\ \alpha\bar{\beta}^2,\ \alpha^2\beta,\ \beta^5 \tag{18}$$

The representations for the bound states are to be constructed from these candidates by imposing further Fermi statistics so as to preserve total antisymmetricity under metacolor-metaflavor-spin transformation. The transformation under orbital angular momentum can be assumed to be symmetric. The metacolor singlet states constrained by Fermi statistics actually lead to the definite metaflavor wave function as we will see below.

In order to construct the ground state wave function consistent with Fermi statistics, it is convenient to use the antisymmetric representation of one SU(I) group where I = 2MN, i.e., the product of the dimensions of the spin-, metacolor-, and metaflavor representations of the defining states α, β, $\bar{\alpha}$ or $\bar{\beta}$. Note M = 5 and 10 for $\alpha(\bar{\alpha})$ and $\beta(\bar{\beta})$ respectively.

Of the four candidates of Eq. (18), we take the state β^5 by way of explanation. Since there are five identical β's to form a fermion bound state, we take the totally antisymmetric representation $[1^5]$ of SU($\mathfrak{I}$) = SU(20N) and consider its branching to $SU_{mc}(5) \times SU_{mf}(N) \times SU(2)_{spin}$. This branching consists of three steps:

$$SU(20N) \underset{\text{step 1}}{\rightarrow} SU(10N) \times SU(2) \underset{\text{step 2}}{\rightarrow} SU(10) \times SU(N) \times SU(2)$$

$$\underset{\text{step 3}}{\rightarrow} SU(5) \times SU(N) \times SU(2) \tag{19}$$

The first and second steps are special cases of the branching type $SU(pq) \supset SU(p) \times SU(q)$ which have already been discussed extensively in the literature.[16] Our results apply to the third stage, i.e., SU(10) → SU(5). This type of branching, i.e., SU(M) → the defining group particularly when M is large can not be found in the existing literature to our knowledge.

The branching of $[1^5]$ under the first step is[16]

$$\begin{aligned}[1^5] \rightarrow &[3,1^2] \otimes [3,1^2] + [5] \otimes [1^5] + [1^5] \otimes [5] \\ &+ [4,1] \otimes [2,1^3] + [2,1^3] \otimes [4,1] + [3,2] \otimes [2^2,1] \\ &+ [2^2,1] \otimes [3,2]\end{aligned} \tag{20}$$

where the first factor in each term is the representation of SU(10N) and the second factor is that of SU(2). Since we know that the fermion bound state β^5 must have spin $\frac{1}{2}$ and left-handed chirality, only the last term is permissible, so that the SU(10N) representation is uniquely determined to be $[2^2,1]$. Now we proceed to observe the branching of SU(10N) → SU(10) x SU(N):

$$
\begin{aligned}
[2^2,1] \rightarrow\ & [4,1] \otimes [3,2] + [3,2] \otimes [4,1] + [2,1^3] \otimes [2^2,1] \\
& + [2^2,1] \otimes [2,1^3] + [4,1] \otimes [3,1^2] + [3,1^2] \otimes [4,1] \\
& + [2,1^3] \otimes [3,1^2] + [3,1^2] \otimes [2,1^3] + [3,2] \otimes [3,2] \\
& + [2^2,1] \otimes [2^2,1] + [3,2] \otimes [3,1^2] + [3,1^2] \otimes [3,2] \\
& + [2^2,1] \otimes [3,1^2] + [3,1^2] \otimes [2^2,1] + 2[3,1^2] \otimes [3,1^2] \\
& + [1^5] \otimes [3,2] + [3,2] \otimes [1^5] + [5] \otimes [2^2,1] + [2^2,1] \otimes [5] \\
& + [4,1] \otimes [2,1^3] + [2,1^3] \otimes [4,1] + [2,1^3] \otimes [3,2] \\
& + [3,2] \otimes [2,1^3] + [4,1] \otimes [2^2,1] + [2^2,1] \otimes [4,1] \\
& + [3,2] \otimes [2^2,1] + [2^2,1] \otimes [3,2]
\end{aligned} \tag{21}
$$

Again the first factor in each term of Eq. (21) is the SU(10) representation and the second factor is the SU(N) representation.

Now we come to the most important stage of identifying the metacolor, i.e., SU(5) singlets from the branching SU(10) $\rightarrow$ SU(5), for which our results of Section III play a crucial role. Of all representations of SU(10) in Eq. (21), we see from Table I that only the terms containing the representation $[3,1^2]$ of SU(10) can give the metacolor singlet states:

$$
\begin{aligned}
[3,1^2] \rightarrow\ & [5,3,1^2] + [4^2,2] + [4,3,2,1] + [4,3,1^3] \\
& + 2[3^2,2,1^2] + [4,2^3] + [4,2^2,1^2] + [3^3,1] \\
& + [3,2^3,1] + 2[3,2^2,1^3] + [4,2,1^4] \\
& + [3,2,1^5] + [2^5] + [2^3,1^4] + [3,1^7]
\end{aligned} \tag{22}
$$

where $[2^5]$ is obviously the SU(5) singlet. In this way, the metaflavor representations of SU(N) are determined to be:

$$
2[3,1^2]\ ,\quad [2,1^3]\ ,\quad [3,2]\ ,\quad [2^2,1]\ ,\quad [4,1] \tag{23}
$$

Similarly, the metaflavor representations of α^5, $\alpha\bar{\beta}^2$ and $\alpha^2\beta$ can be obtained. These and other related subjects of preon dynamics are presented elsewhere.[8]

(B) 27_L of E_6 metacolor group as the preon representation.

The second example is E_6 metacolor group with preon α in the 27-dimensional (000010) representation. This representation belongs to congruence class 2 and can be repeated up to 22 times without losing asymptotic freedom. Such repetition introduces the metaflavor group SU(N) in addition.

The bound state α^3 is a singlet of E_6 and is a candidate massless bound state. The metaflavor representation of the bound state α^3 is determined by Fermi statistics in the similar way as in case (A).

The branching necessary for satisfying Fermi statistics in the bound state α^3 is again through several steps:

$$SU(54N) \underset{\text{step 1}}{\rightarrow} SU(27N) \times SU(2) \underset{\text{step 2}}{\rightarrow} SU(27) \times SU(N) \times SU(2) \qquad (24)$$

In the first step, the totally antisymmetric $[1^3]$ of SU(54N) have the following SU(27N) x SU(2) branching

$$[1^3] \rightarrow [2,1] \otimes [2,1] + [3] \otimes [1^3] + [1^3] \otimes [3] \qquad (25)$$

Only the first term can give spin ½, and hence we find that the SU(27N) representation is [2.7] Under step 2 this branches into $[2.1] \otimes [2.1] + [3] \otimes [2.1] + [2.1] \otimes [3] + [1^3] \otimes [2.1] + [2.1] \otimes [1^3]$. Finally, Table VII shows that the E_6 singlet (0,0,0,0,0,0) belongs to the [3] of SU(27), and the metaflavor representation of α^3 is determined as [2,1] of SU(N).

Appendix A. Continuation of SU(N)-indices to real N.

The indices of a representation, defined by Eqs. (2) and (3) can be calculated from the knowledge of the complete weight system, for any representation. For the lower indices explicit expressions for arbitrary representations have been obtained in many papers.[3,13] These formulas usually contain sums up to the rank of the group, which makes a continuation to real N impossible.

One may wonder why we are interested in such a continuation, since the groups themselves cannot be continued in the rank in a sensible way. For our purpose these formulas are advantageous in three ways. First of all they allow us to use large values of N without the need to calculate large sums. In the procedures of Section II large values of N are unavoidable if N is restricted to integer values. For the expressions derived in this Appendix, the computing time depends only on the structure of the Young-diagram, not on N. Secondly, since the index sum rules (12) - (14) can be generalized to arbitrary integer N if G = SU(N), they can also be generalized to real N, when continuous functions for the indices can be found. This provides another way to avoid large values of N, since the N-dependence can now be probed by small, non-integer values of N. Finally, there exists an intimate relation between a continuation of N to negative values and transposition of the Young-diagram, which will be exploited in Appendix B.

Continuous formulas are already known for the dimension of an SU(N) representation and for the second and third order index of a few small representations.[1,3] We will present a general formula for the second and third index and indicate how the results generalize to indices of arbitrary order.

Our starting point will be the integer-N formulas for the indices. We will use the results of Perelomov and Popov.[17] The relation between the symmetrized Casimir operator J_α (in Ref. 17 denoted as I_α) with the indices defined by Eqs. (2) and (3) is:

$$I_2(R) = \frac{I_o(R)}{N^2-1} J_2(R) \tag{A-1}$$

$$I_3(R) = \frac{2N}{(N^2-1)(N^2-4)} I_o(R) J_3(R) \tag{A-2}$$

The symmetrized Casimir operators can be expressed in terms of the quantities

$$S_\alpha = \sum_{i=1}^{N} (L_i^\alpha - R_i^\alpha) \tag{A-3}$$

where

$$L_i = f_i - \frac{f}{N} + N - i \tag{A-4}$$

$$R_i = N - i , \tag{A-5}$$

and f_i is the length of the i^{th} row of the Young-diagram representing R. The number of boxes of this Young-diagram is f. We take from Ref. 17 the following formulas for the symmetrized Casimir operators

$$J_2 = S_2 \tag{A-6}$$

$$J_3 = S_3 - 3/2(N - 1) S_2 \tag{A-7}$$

The sums we want to avoid appear in (A-3). Continuous expressions can be obtained by summing the parts of the summand which do not depend on f_i explicitly, and using the fact that the f_i's vanish for $i > p$, where p is the number of rows of the Young-diagram. The structure of the Young-diagram appears in the results in the form of the following "moments"

$$M_{nm} = \sum_{j=1}^{p} j^n [f_j]^m \qquad (m \geq 1, \quad n \geq 0) \tag{A-8}$$

Straightforward computation yields then the following expressions

$$J_2(R) = (1 + N)\, M_{01} - \frac{1}{N^2} M_{01}^2 + M_{02} - 2M_{11} \tag{A-9}$$

$$\begin{aligned} J_3(R) = {} & (-1/2 + 3/2N + 2N^2)\, M_{01} - (9/2 + \frac{3}{2N})\, M_{01}^2 + \frac{2}{N^2} M_{01}^3 \\ & + (M_{02} - 2M_{11})(3N - 3\, \frac{M_{01}}{N}) + M_{03} - 3M_{12} + 3M_{21} \\ & - 3/2(N - 1)\, J_2(R) \end{aligned} \tag{A-10}$$

The crucial point is, that the moments M_{nm} depend only on the Young-diagram, but not on M.

Although (A-9) and (A-10) have the desired properties, they can be simplified by means of the transposed moments, defined as

$$T_{nm} = \sum_{j=1}^{q} j^n [g(j)]^m \tag{A-11}$$

where q is the number of columns and g(j) the length of the j^{th} column of the Young-diagram. Several relations between the moments and the transposed moments can be derived. We will only give the ones for $n + m \leq 3$:

$$M_{01} = T_{01} = f \tag{A-12}$$

$$T_{11} = 1/2(M_{02} + M_{01}) \tag{A-13}$$

$$3T_{21} - T_{11} = M_{03} + M_{02} \tag{A-14}$$

$$T_{12} + T_{11} = M_{12} + M_{11} \tag{A-15}$$

Additional relations are obtained by interchanging M and T. These relations can be proved by induction: they are trivial for the single-box Young-diagram, and when a box is added to an arbitrary Young-diagram the left-hand sides and right-hand sides of (A-12) to (A-15) change by the same amounts.

We use these relations to express the indices in terms of the symmetric and antisymmetric moments:

$$S_{nm} = 1/2(M_{nm} + T_{nm}) \tag{A-16}$$

$$A_{nm} = 1/2(M_{nm} - T_{nm}) \tag{A-17}$$

Then we obtain the following expressions[18]

$$J_2 = f(N - \frac{f}{N}) + 2\, A_{02} \tag{A-18}$$

$$J_3 = -\frac{1}{2}\ f\ (1 - N^2) - 3f^2 + 2\ \frac{f^3}{N^2}$$

$$+(3N - \frac{6f}{N})A_{02} + S_{03} - 3S_{12} + 3S_{21} \tag{A-19}$$

These formulas reveal the transformation of the indices under transposition of the Young-diagram. Transposition is defined as an interchange of rows and columns, or equivalently a reflection of the Young-diagram with respect to the diagonal. The moments M_{nm} and T_{nm} are interchanged by transposition, so that S_{nm} is unchanged and A_{nm} changes sign. The effect of this is equivalent to a replacement of N by -N, apart from an overall sign. More precisely when $I_p(Y_m,N)$ denotes the p^{th} order index of the representation of SU(N) defined by the m-box Young-diagram Y_m, then the following relations hold

$$I_o(Y_m^T,N) = (-1)^m\ I_o(Y_m,-N) \tag{A-20}$$

$$I_2(Y_m^T,N) = (-1)^{m-1}\, I_2(Y_m,-N) \tag{A-21}$$

$$I_3(Y_m^T,N) = (-1)^{m-1}\, I_3(Y_m,-N) \tag{A-22}$$

where "T" denotes transposition. Relation (A-21) is a consequence of the well-known dimension formula

$$I_o = \frac{1}{H} \prod_{i,j} (N + i-j) \tag{A-23}$$

where the product is over all boxes of the Young-diagram, located in the j^{th} row and i^{th} column; H (the product of the "hook-lengths") is just a numerical factor.

The extension of our results to higher indices is straightforward, but becomes rapidly complicated.

Appendix B. Transposition rules for SU(N)-pletyhsms.

In this Appendix we will formulate and derive relations between tensor products with definite permutation properties of a Young-diagram and its transpose.

To simplify the notation we introduce an operation of a Young-diagram Y_m on a Young-diagram Y_n, denoted as $Y_m * Y_n$. This operation is defined as the m^{th} tensor power of Y_n, symmetrized according to Y_m. The result of this operation is a direct sum of Young-diagrams with nm boxes:

$$Y_m * Y_n = \sum_i \ell_i \, Y^i_{mn} \tag{B-1}$$

where i labels different Young-diagrams. The multiplicities ℓ_i can be read off from Tables I and II, for example

$$[1^3] * [2] = [3^2] \oplus [4,1^2] \tag{B-2}$$

$$[2] * [1^3] = [2^3] \oplus [2,1^4] \tag{B-3}$$

This should clarify our notation.

The transposition rules can now be formulated as follows:

(i) if n is even

$$Y_m * Y_n^T = (Y_m * Y_n)^T \tag{B-4}$$

(ii) if n is odd

$$Y_m^T * Y_n^T = (Y_m * Y_n)^T \tag{B-5}$$

To illustrate this we apply these rules to (B-2) and (B-3)

$$[1^3] * [1^2] = [1^3] * [2]^T = ([1^3] * [2])^T = [2^3] + [3,1^3] \qquad \text{(B-6)}$$

$$[1^2] * [3] = [2]^T * [1^3]^T = ([2] * [1^3])^T = [3^2] + [5,1] \qquad \text{(B-7)}$$

The first expression can be checked with Table I. The representation [3] is not included in the table, but one can easily check that the index sum rules (12) - (14) are satisfied. This example also illustrates the main application of the transposition rules, namely, to supplement the tables in such a way that all defining representations with not more than four boxes are included.

The validity of these rules can be demonstrated by a continuation of the rank of the group to negative values. Consider the index sum rules satisfied by (B-1)

$$I_o(Y_m, M) = \sum_i \ell_i \, I_o(Y^i_{mn}, N) \qquad \text{(B-8)}$$

$$I_p(Y_m, M) \; I_p(Y_n, N) = \sum_i \ell_i \, I_p(Y^i_{mn}, N) \quad , \qquad (p = 2,3) \qquad \text{(B-9)}$$

where

$$M = I_o(Y_n, N) \qquad \text{(B-10)}$$

These rules follow from Eqs. (12) - (16); we use the notation introduced in Appendix A for the indices of SU(N). Since the multiplicities ℓ_i satisfy (B-8) - (B-10) for any value of M, we can replace N by -N. Then we use relations (A-21) - (A-23) to obtain

$$I_o(Y_m, (-1)^n \, I_o(Y^T_n, N)) = (-1)^{mn} \sum_i \ell_i \, I_o(Y_{mn}, N) \qquad \text{(B-11)}$$

$$I_p(Y_m, (-1)^n\ I_o(Y_n^T,N))\ \ (-1)^{n-1}\ I_p(Y_n^T,N)$$

$$= (-1)^{mn-1} \sum_i \ell_i\ I_p(Y_{mn}^{iT},N) \tag{B-12}$$

If n is even, (B-11) and (B-12) reduce to

$$I_o(Y_m,M^T) = \sum_i \ell_i\ I_o(Y_{mn}^{iT},N) \tag{B-13}$$

$$I_p(Y_m,M^T)\ \ I_p(Y_n^T,N) = \sum_i \ell_i\ I_p(Y_{mn}^{iT},N) \tag{B-14}$$

where

$$M^T = I_o(Y_n^T,N)$$

For odd n we obtain the same results, but with Y_m^T instead of Y_m.

These are exactly the equations for ℓ_i which one would have to solve to calculate the direct sums which are equal to the left-hand sides of (B-4) and (B-5). We conclude that the ℓ_i's appearing in (B-1) satisfy these equations when they are associated with the transpose of Y_{mn}^i. This is exactly the content of the transposition rules (B-4) and (B-5). This derivation is not completely rigorous because we have not shown that the sum rules have a unique solution. In fact, although we have not found such a case, we expect that at some stage ambiguities will exist, since for large representations Y_m and Y_n the number of equations becomes much smaller than the number of variables ℓ_i. Such an ambiguity invalidates the derivation, but of course not the result.

ACKNOWLEDGEMENTS

We would like to thank C. Albright, W. Bardeen, E. Eichten, F. W. Lemire, W. G. McKay, J. McKay and P. Winternitz for helpful discussions. Special thanks are due to J. Patera for teaching us the plethysm method used in this paper. One of us (I.G.K.) wishes to thank members of the Fermilab Theory Group and Brown University High-Energy Theory Group for warm hospitality extended to him during the course of this work. The same author is supported in part by the Korea Traders Foundation, the Korea Science and Engineering Foundation, and the Senior Fulbright Fellowship.

REFERENCES

1. E. Eichten, K. Kang and I. G. Koh, Fermilab-Pub-81/83-THY and Brown-HET-464 (1981), in which other relevant references can be found.

2. H. Georgi, Nucl. Phys. B156, 126 (1979).

3. S. L. Adler, Phys. Rev. 177, 2425 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969); J. Banks and H. Georgi, Phys. Rev. D14, 1159 (1976); S. Okubo, Phys. Rev. D16, 3528 (1977); J. Patera and R. T. Sharp, J. Math. Phys. 22, 2352 (1981); A. M. Perelomov and V. S. Popov, Sov. J. Nucl. Phys. 3, 676 (1966).

4. See, for example, P. H. Frampton, Phys. Lett. 88B, 299 (1979) in the case of grand unified theories; E. Eichten and F. Feinberg, Fermilab-Pub-81/62-THY (1981) for preon dynamics.

5. E. Eichten, K. Kang, and I. G. Koh, Ref. 1; one of us (A.N.S.) has also investigated this subject in an unpublished report; C. L. Ong, SLAC-Pub-2778 (to appear in Phys. Rev. D).

6. J. McKay, J. Patera, and R. T. Sharp, J. Math. Phys. 22, 2770 (1982).

7. G. 't Hooft, in "Recent Developments in Gauge Theories",; G. 't Hooft, C. Itzykson, A. Jaffe, H. Lehmann, P. K. Mitter, I. M. Singer, and R. Stora, eds. (Plenum Press N.Y., 1980), p. 135.

8. A. N. Schellekens, K. Kang, and I. G. Koh, Fermilab-Pub-82/24-THY and Brown-HET-471, 1982.

9. C. H. Albright, Phys. Rev. D24, 1969 (1981); A. N. Schellekens, K. Kang and I. G. Koh, Ref. 8.

10. S. Raby, S. Dimopoulos and L. Susskind, Nucl. Phys. B169, 373 (1980).

11. B. Wybourne, "Classical Groups for Physicists", Wiley-Interscience, New York (1974); R. Slansky, Phys. Rep. C 79, 1 (1981); Y. D. Kim, K. Y. Kim, I. G. Koh, Y. J. Park, W. S. l'Yi, Y. S. Kim, C. H. Kim, and H. T. Park, Tables of Irreducible Representations for Classical Groups; Complete Weight System, Branching Matrices with U(1) and Anomaly Table, (Sogang University Press, Seoul, Korea, 1981).

12. A. Navon and J. Patera, J. Math. Phys. 8, 489 (1967); W. McKay, J. Patera, and D. Sankoff, in Computers in Nonassociative Rings and Algebras, ed. J. Beck and B. Kolman (Academic Press, New York, 1977); G. L. Shaw and R. Slansky, Phys. Rev. D22, 1760 (1980).

13. J. Patera, R. T. Sharp, P. Winternitz, J. Math. Phys. 17, 1979 (1976), 18, 1519 (1977); J. Patera, "Higher Indices of Group Representations and their Applications", Proceedings of the Second Workshop on Current Problems in High Energy Particle Theory, ed. G. Domokos, and S. Kovesi-Domokos, Johns Hopkins University, 1978; W. McKay, J. Patera, R. T. Sharp,

J. Math. Phys. 17, 1371 (1976); W. G. McKay and J. Patera, Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras (Marcel Dekker, New York, 1981); S. Okubo, J. Math Phys. 23, 8 (1982).

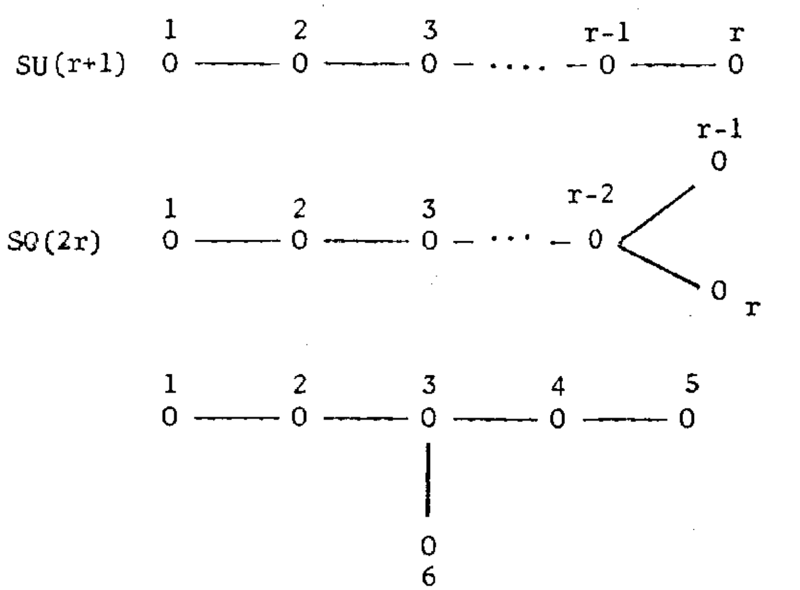
14. F. W. Lemire and J. Patera, J. Math. Phys. 21, 2026.

15. J. Patera and R. T. Sharp, J. Math. Phys. Ref. 3.

16. C. Itzykson and M. Nauenberg, Rev. Mod. Phys. 38, 95 (1966).

17. A. M. Perelomov and V. S. Popov, Ref. 3.

18. The existence of the expression (A-18) was pointed out to us by J. E. Ribeiro.



A_r

D_r ,r = odd integer

E_6

Fig. 1. Dynkin diagrams for simple Lie algebras admitting complex representations and the weight systems follows the corresponding numberings in $(a_1,a_2,\cdots,a_r)$.

TABLE I. The reduction of the tensor products up to twelve copies of [2] and $[1^2]$ of SU(N) with the given symmetry property of the Young digram in the first column. The table can also be used to obtain the branching of SU(M) into the representations of SU(N) when the defining representation of SU(N) is embedded to the fundamental representation of SU(M) such that M is the dimension of the defining representation of SU(N). For the defining representation [2] and $[1^2]$ of SU(N), M is N(N+1)/2, and N(N-1)/2 respectively.

SU(M)	Defining Representations of SU(N)	
[1]	[2]	$[1^2]$
[2]	[4] $[2^2]$	$[2^2]$ $[1^4]$
$[1^2]$	[3,1]	$[2,1^2]$
[3]	[6] [4,2] $[2^3]$	$[3^2]$ $[2^2,1^2]$ $[1^6]$
[2,1]	[5,1] [4,2] [3,2,1]	[3,2,1] $[2^2,1^2]$ $[2,1^4]$
$[1^3]$	$[3^2]$ $[4,1^2]$	$[3,1^3]$ $[2^3]$
[4]	[8] [6,2] [4,4] $[4,2^2]$ $[2^4]$	$[4^2]$ $[3^2,1^2]$ $[2^4]$ $[2^2,1^4]$ $[1^8]$

Table I - cont'd

SU(M)	SU(N)	
$[3,1]$	$[7,1]$	$[4,3,1]$
	$[6,2]$	$[3^2,1^2]$
	$[5,3]$	$[3,2^2,1]$
	$[5,2,1]$	$[3,2,1^3]$
	$[4,3,1]$	$[2^3,1^2]$
	$[4,2^2]$	$[2^2,1^4]$
	$[3,2^2,1]$	$[2,1^6]$
$[2^2]$	$[6,2]$	$[4,2^2]$
	$[5,2,1]$	$[3^2,1^2]$
	$[4^2]$	$[3,2,1^3]$
	$[4,2^2]$	$[2^4]$
	$[3^2,1^2]$	$[2^2,1^4]$
$[2,1^2]$	$[5,3]$	$[4,2,1^2]$
	$[6,1^2]$	$[3^2,2]$
	$[5,2,1]$	$[3,2^2,1]$
	$[4,3,1]$	$[3,2,1^3]$
	$[3^2,2]$	$[2^3,1^2]$
	$[4,2,1^2]$	$[3,1^5]$
$[1^4]$	$[4,3,1]$	$[4,1^4]$
	$[5,1^3]$	$[3,2^2,1]$
$[5]$	$[10]$	$[5^2]$
	$[8,2]$	$[4^2,1^2]$
	$[6,4]$	$[3^2,2^2]$
	$[6,2^2]$	$[3^2,1^4]$
	$[4^2,2]$	$[2^4,1^2]$
	$[4,2^3]$	$[2^2,1^6]$
	$[2^5]$	$[1^{10}]$

Table I - cont'd.

SU(M)	SU(N)	
[4,1]	[9,1]	[5,4,1]
	[8,2]	[4^2,1^2]
	[7,3]	[4,3,2,1]
	[7,2,1]	[4,3,1^3]
	[6,4]	[3^2,2^2]
	[6,3,1]	[3^2,2,1^2]
	[6,2,2]	[3^2,1^4]
	[5,4,1]	[3,2^3,1]
	[5,3,2]	[3,2^2,1^3]
	[5,2^2,1]	[3,2,1^5]
	[4^2,2]	[2^4,1^2]
	[4,3,2,1]	[2^3,1^4]
	[4,2^3]	[2^2,1^6]
	[3,2^3,1]	[2,1^8]
[3,2]	[8,2]	[4^2,1^2]
	[7,3]	[4,3,2,1]
	[6,4]	[4,3,1^3]
	[7,2,1]	[3^2,2^2]
	[6,3,1]	[3^2,2,1^2]
	2 x [6,2^2]	2 x [3^2,1^4]
	[5,4,1]	[5,3,2]
	[5,3,2]	[4,2^2,1^2]
	[5,3,1^2]	[3,2^3,1]
	[5,2^2,1]	[3,2^2,1^3]
	[4^2,2]	[3,2,1^5]
	[4,3,2,1]	[2^4,1^2]
	[4,2^3]	[2^3,1^4]
	[3^2,2,1^2]	[2^2,1^6]
[3,1^2]	[7,3]	[5,3,1^2]
	[8,1^2]	[4^2,2]
	[7,2,1]	[4,3,2,1]

cont'd.

Table I - cont'd.

SU(M)	SU(N)	
	$2 \times [6,3,1]$	$[4,3,1^3]$
	$[5^2]$	$2 \times [3^2,2,1^2]$
	$[5,4,1]$	$[4,2^3]$
	$2 \times [5,3,2]$	$[4,2^2,1^2]$
	$[6,2,1^2]$	$[3^3,1]$
	$[5,3,1^2]$	$[3,2^3,1]$
	$[5,2^2,1]$	$2 \times [3,2^2,1^3]$
	$[4,3^2]$	$[4,2,1^4]$
	$[4^2,1^2]$	$[2^5]$
	$[4,3,2,1]$	$[3,2,1^5]$
	$[3^2,2^2]$	$[2^3,1^4]$
	$[4,2^2,1^2]$	$[3,1^7]$
$[2^2,1]$	$[6,4]$	$[5,2^2,1]$
	$[7,2,1]$	$[4,3^2]$
	$[6,3,1]$	$[4,3,2,1]$
	$[6,2^2]$	$[4,2^2,1^2]$
	$[5,4,1]$	$[4,3,1^3]$
	$[5,3,2]$	$[3^2,2^2]$
	$[6,2,1^2]$	$[3^2,2,1^2]$
	$[5,3,1^2]$	$[3^2,1^4]$
	$[5,2^2,1]$	$[4,2,1^4]$
	$[4^2,2]$	$[3,2^3,1]$
	$[4,3,2,1]$	$[3,2^2,1^3]$
	$[3^3,1]$	$[3,2,1^5]$
	$[4,3,1^3]$	$[2^4,1^2]$
$[2,1^3]$	$[6,3,1]$	$[5,2,1^3]$
	$[5,4,1]$	$[4,3,2,1]$
	$[5,3,2]$	$[4,2^3]$
	$[7,1^3]$	$[4,2^2,1^2]$
	$[6,2,1^2]$	$[4,2,1^4]$
	$[5,3,1^2]$	$[3^3,1]$
	$[4,3^2]$	$[3^2,2,1^2]$
	$[4^2,1^2]$	$[3,2^3,1]$
	$[4,3,2,1]$	$[3,2^2,1^3]$
	$[5,2,1^3]$	$[4,1^6]$

Table I - cont'd.

SU(M)	SU(N)	
$[1^5]$	$[4^2,2]$	$[5,1^5]$
	$[5,3,1^2]$	$[4,2^2,1^2]$
	$[6,1^4]$	$[3^2,2^2]$
$[6]$	$[12]$	$[6^2]$
	$[10,2]$	$[5^2,1^2]$
	$[8,4]$	$[4^2,2^2]$
	$[8,2^2]$	$[4^2,1^4]$
	$[6^2]$	$[3^4]$
	$[6,4,2]$	$[3^2,2^2,1^2]$
	$[6,2^3]$	$[3^2,1^6]$
	$[4^3]$	$[2^6]$
	$[4^2,2^2]$	$[2^4,1^4]$
	$[4,2^4]$	$[2^2,1^8]$
	$[2^6]$	$[1^{12}]$
$[5,1]$	$[11,1]$	$[2,1^{10}]$
	$[10,2]$	$[2^2,1^8]$
	$[9,3]$	$[2^3,1^6]$
	$[9,2,1]$	$[3,2,1^7]$
	$[8,4]$	$[2^4,1^4]$
	$[8,3,1]$	$[3,2^2,1^5]$
	$[8,2^2]$	$[3^2,1^6]$
	$[7,5]$	$[2^5,1^2]$
	$[7,4,1]$	$[3,2^3,1^3]$
	$[7,3,2]$	$[3^2,2,1^4]$
	$[7,2^2,1]$	$[4,3,1^5]$
	$[6,5,1]$	$[3,2^4,1]$
	$2 \times [6,4,2]$	$2 \times [3^2,2^2,1^2]$
	$[6,3,2,1]$	$[4,3,2,1^3]$
	$[6,2^3]$	$[4^2,1^4]$
	$[5,4,3]$	$[3^3,2,1]$
	$[5,4,2,1]$	$[4,3,2^2,1]$
	$[5,3,2^2]$	$[4^2,2,1^2]$

cont'd

Table I - cont'd.

SU(M)	SU(N)	
	$[5,2^3,1]$	$[5,4,1^3]$
	$[4^2,3,1]$	$[4,3^2,2]$
	$[4^2,2^2]$	$[4^2,2^2]$
	$[4,3,2^2,1]$	$[5,4,2,1]$
	$[4,2^4]$	$[5^2,1^2]$
	$[3,2^4,1]$	$[6,5,1]$
$[4,2]$	$[10,2]$	$[5^2,1^2]$
	$[9,3]$	$[5,4,2,1]$
	$2 \times [8,4]$	$[5,4,1^3]$
	$[9,2,1]$	$2 \times [4^2,2^2]$
	$[8,3,1]$	$[4^2,2,1^2]$
	$2 \times [8,2^2]$	$2 \times [4^2,1^4]$
	$2 \times [7,4,1]$	$[5,3^2,1]$
	$2 \times [7,3,2]$	$[5,3,2,1^2]$
	$[6^2]$	$[4,3^2,1^2]$
	$[6,5,1]$	$2 \times [4,3,2^2,1]$
	$3 \times [6,4,2]$	$2 \times [4,3,2,1^3]$
	$[7,3,1^2]$	$[4,3,1^5]$
	$[7,2^2,1]$	$[3^4]$
	$2 \times [6,3,2,1]$	$[3^3,2,1]$
	$2 \times [6,2^3]$	$3 \times [3^2,2^2,1^2]$
	$[5,4,3]$	$2 \times [3^2,2,1^4]$
	$[5^2,1^2]$	$2 \times [3^2,1^6]$
	$2 \times [5,4,2,1]$	$[6,4,2]$
	$[5,3^2,1]$	$[4,2^4]$
	$[5,3,2^2]$	$[4,2^2,1^4]$
	$[5,3,2,1^2]$	$[3,2^4,1]$
	$[5,2^3,1]$	$2 \times [3,2^3,1^3]$
	$[4^3]$	$[3,2^2,1^5]$
	$2 \times [4^2,2^2]$	$[3,2,1^7]$
	$[4,3^2,1^2]$	$[2^6]$
	$[4,3,2^2,1]$	$2 \times [2^4,1^4]$
	$[4,2^4]$	$[2^3,1^6]$
	$[3^2,2^2,1^2]$	$[2^2,1^8]$

Table I - cont'd.

SU(M)	SU(N)	
$[4,1^2]$	$[9,3]$	$[6,4,1^2]$
	$[10,1^2]$	$[5^2,2]$
	$[9,2,1]$	$[5,4,2,1]$
	$2 \times [8,3,1]$	$[5,4,1^3]$
	$[7,5]$	$2 \times [4^2,2,1^2]$
	$2 \times [7,4,1]$	$[5,3,2^2]$
	$2 \times [7,3,2]$	$[5,3,2,1^2]$
	$[8,2,1^2]$	$[4^2,3,1]$
	$[7,3,1^2]$	$[4,3^2,2]$
	$[7,2^2,1]$	$[4,3^2,1^2]$
	$[6,5,1]$	$2 \times [4,3,2^2,1]$
	$[6,4,2]$	$2 \times [4,3,2,1^3]$
	$2 \times [6,3^2]$	$[5,3,1^4]$
	$2 \times [6,4,1^2]$	$[4,3,1^5]$
	$2 \times [6,3,2,1]$	$[3^3,2,1]$
	$2 \times [5^2,2]$	$[3^2,2^2,1^2]$
	$[5,4,3]$	$2 \times [3^3,1^3]$
	$2 \times [5,4,2,1]$	$2 \times [3^2,2^3]$
	$[5,3^2,1]$	$2 \times [3^2,2,1^4]$
	$2 \times [5,3,2^2]$	$2 \times [4,2^3,1^2]$
	$[6,2^2,1^2]$	$[3,2^4,1]$
	$[5,3,2,1^2]$	$2 \times [3,2^3,1^3]$
	$[5,2^3,1]$	$[4,2^2,1^4]$
	$[4^2,3,1]$	$2 \times [3,2^2,1^5]$
	$[4,3^2,2]$	$[4,2,1^6]$
	$[4^2,2,1^2]$	$[3,2,1^7]$
	$[4,3,2^2,1]$	$[2^5,1^2]$
	$[3^2,2^3]$	$[2^3,1^6]$
	$[4,2^3,1^2]$	$[3,1^9]$

Table I - cont'd.

SU(M)	SU(N)	
$[3^2]$	$[9,3]$	$[5,4,2,1]$
	$[8,3,1]$	$[4^2,2,1^2]$
	$[8,2^2]$	$[4^2,1^4]$
	$[7,5]$	$[6,3^2]$
	$[7,4,1]$	$[5,3,2,1^2]$
	$[7,3,2]$	$[4,3^2,2]$
	$[7,2^2,1]$	$[4,3,2^2,1]$
	$[6,4,2]$	$[4,3,2,1^3]$
	$[6,3^2]$	$[4,3,1^5]$
	$[6,4,1^2]$	$[3^2,2^2,1^2]$
	$[6,3,2,1]$	$[3^3,1^3]$
	$[6,2^3]$	$[3^2,2^3]$
	$[5^2,2]$	$[3^2,2,1^4]$
	$[5,4,2,1]$	$[3^2,1^6]$
	$[5,3,2^2]$	$[4,2^3,1^2]$
	$[4^2,3,1]$	$[3,2^3,1^3]$
	$[5,3,2,1^2]$	$[3,2^2,1^5]$
	$[4,3,2^2,1]$	$[2^5,1^2]$
	$[3^3,1^3]$	$[2^3,1^6]$
$[3,2,1]$	$[4,3,2,1^3]$	$[6,3,2,1]$
	$[3^3,2,1]$	$[5,4,3]$
	$[4,3,2^2,1]$	$[5,4,2,1]$
	$[4,3^2,1^2]$	$[5,3^2,1]$
	$2 \times [5,3,2,1^2]$	$2 \times [5,3,2,1^2]$
	$[5,2^3,1]$	$[5,4,1^3]$
	$[4^2,2^2]$	$[4^2,2^2]$
	$2 \times [5,3,2^2]$	$2 \times [4^2,2,1^2]$
	$[6,2^3]$	$[4^2,1^4]$
	$[4^2,2,1^2]$	$[5,3,2^2]$
	$[4,3^2,2]$	$[4^2,3,1]$
	$[4^2,3,1]$	$[4,3^2,2]$
	$2 \times [5,3^2,1]$	$2 \times [4,3^2,1^2]$
	$3 \times [5,4,2,1]$	$3 \times [4,3,2^2,1]$

cont'd

Table I - cont'd.

SU(M)	SU(N)	
	4 x [6,3,2,1]	4 x $[4,3,2,1^3]$
	$[6,2^2,1^2]$	$[5,3,1^4]$
	2 x $[7,2^2,1]$	2 x $[4,3,1^5]$
	2 x [5,4,3]	2 x $[3^3,2,1]$
	3 x [6,4,2]	3 x $[3^2,2^2,1^2]$
	$[6,3^2]$	$[3^3,1^3]$
	$[5^2,2]$	$[3^2,2^3]$
	3 x [7,3,2]	3 x $[3^2,2,1^4]$
	$[8,2^2]$	$[3^2,1^6]$
	$[5,4,1^3]$	$[5,2^3,1]$
	$[6,3,1^3]$	$[5,2^2,1^3]$
	$[5^2,1^2]$	$[4,2^4]$
	2 x $[6,4,1^2]$	2 x $[4,2^3,1^2]$
	2 x $[7,3,1^2]$	2 x $[4,2^2,1^4]$
	2 x [6,5,1]	2 x $[3,2^4,1]$
	2 x [7,4,1]	2 x $[3,2^3,1^3]$
	2 x [8,3,1]	2 x $[3,2^2,1^5]$
	$[8,2,1^2]$	$[4,2,1^6]$
	[9,2,1]	$[3,2,1^7]$
	[7,5]	$[2^5,1^2]$
	[8.4]	$[2^4,1^4]$
$[3,1^3]$	[8,3,1]	$[6,3,1^3]$
	[7,4,1]	[5,4,2,1]
	[7,3,2]	$[5,3,2^2]$
	$[9,1^3]$	$[5,3,2,1^2]$
	$[8,2,1^2]$	$[5,3,1^4]$
	2 x $[7,3,1^2]$	$[4^2,3,1]$
	[6,5,1]	$[4^2,2,1^2]$
	[6,4,2]	$[4,3^2,2]$
	2 x $[6,3^2]$	2 x $[4,3^2,1^2]$
	2 x $[6,4,1^2]$	2 x $[4,3,2^2,1]$
	2 x [6,3,2,1]	2 x $[4,3,2,1^3]$
	$[5^2,2]$	$[3^3,2,1]$

cont'd.

Table I - cont'd.

SU(M)	SU(N)	
	$[5,4,3]$	2 x $[3^3,1^3]$
	$[5^2,1^2]$	$[3^2,2^3]$
	2 x $[5,4,2,1]$	$[3^2,2^2,1^2]$
	2 x $[5,3^2,1]$	$[3^2,2,1^4]$
	$[5,3,2^2]$	$[5,2^3,1]$
	$[7,2,1^3]$	2 x $[4,2^3,1^2]$
	$[6,3,1^3]$	$[5,2^2,1^3]$
	$[6,2^2,1^2]$	$[4,2^4]$
	$[5,4,1^3]$	2 x $[4,2^2,1^4]$
	$[5,3,2,1^2]$	$[3,2^4,1]$
	$[4^2,3,1]$	$[3,2^3,1^3]$
	$[4,3^2,2]$	$[3,2^2,1^5]$
	$[4^2,2,1^2]$	$[5,2,1^5]$
	$[4,3,2^2,1]$	$[4,2,1^6]$
	$[5,2^2,1^3]$	$[4,1^8]$
$[2^3]$	$[4^2,2^2]$	$[4^2,2^2]$
	$[6,2^3]$	$[4^2,1^4]$
	$[4,3^2,1^2]$	$[5,3^2,1]$
	$[5,3,2,1^2]$	$[5,3,2,1^2]$
	$[5,3^2,1]$	$[4,3^2,1^2]$
	$[5,4,2,1]$	$[4,3,2^2,1]$
	$[6,3,2,1]$	$[4,3,2,1^3]$
	$[7,2^2,1]$	$[4,3,1^5]$
	$[4^3]$	$[3^4]$
	2 x $[6,4,2]$	2 x $[3^2,2^2,1^2]$
	$[8,2^2]$	$[3^2,1^6]$
	$[4^2,1^4]$	$[6,2^3]$
	$[6,3,1^3]$	$[5,2^2,1^3]$
	$[3^4]$	$[4^3]$
	$[5^2,1^2]$	$[4,2^4]$
	$[7,3,1^2]$	$[4,2^2,1^4]$
	$[7,4,1]$	$[3,2^3,1^3]$
	$[6^2]$	$[2^6]$

Table I - cont'd.

SU(M)	SU(N)	
$[2^2,1^2]$	$[7,4,1]$	$[3,2^3,1^3]$
	$[6,5,1]$	$[3,2^4,1]$
	$[6,4,2]$	$[3^2,2^2,1^2]$
	$[7,3,2]$	$[3^2,2,1^4]$
	$[8,2,1^2]$	$[4,2,1^6]$
	$[7,3,1^2]$	$[4,2^2,1^4]$
	$[7,2^2,1]$	$[4,3,1^5]$
	$[6,3^2]$	$[3^3,1^3]$
	$2 \times [6,4,1^2]$	$2 \times [4,2^3,1^2]$
	$2 \times [6,3,2,1]$	$2 \times [4,3,2,1^3]$
	$[5^2,2]$	$[3^2,2^3]$
	$[5,4,3]$	$[3^3,2,1]$
	$2 \times [5,4,2,1]$	$2 \times [4,3,2^2,1]$
	$[5,3^2,1]$	$[4,3^2,1^2]$
	$[5,3,2^2]$	$[4^2,2,1^2]$
	$[7,2,1^3]$	$[5,2,1^5]$
	$[6,3,1^3]$	$[5,2^2,1^3]$
	$[6,2^2,1^2]$	$[5,3,1^4]$
	$[5,4,1^3]$	$[5,2^3,1]$
	$[5,3,2,1^2]$	$[5,3,2,1^2]$
	$[4^2,3,1]$	$[4,3^2,2]$
	$[4,3^2,2]$	$[4^2,3,1]$
	$[4^2,2,1^2]$	$[5,3,2^2]$
	$[4,3^2,1^2]$	$[5,3^2,1]$
	$[5,3,1^4]$	$[6,2^2,1^2]$
$[2,1^4]$	$[6,4,2]$	$[6,2,1^4]$
	$[7,3,1^2]$	$[5,3,2,1^2]$
	$[6,4,1^2]$	$[5,2^3,1]$
	$[6,3,2,1]$	$[5,2^2,1^3]$
	$[5,4,3]$	$[5,2,1^5]$
	$[5^2,1^2]$	$[4^2,2^2]$
	$[5,4,2,1]$	$[4,3^2,2]$
	$[5,3^2,1]$	$[4,3^2,1^2]$
	$[8,1^4]$	$[4,3,2^2,1]$

cont'd

Table I - cont'd

SU(M)	SU(N)	
	$[7,2,1^3]$	$[4,3,2,1^3]$
	$[6,3,1^3]$	$[4,2^3,1^2]$
	$[5,4,1^3]$	$[4,2^4]$
	$[5,3,2,1^2]$	$[4,2^2,1^4]$
	$[4^2,3,1]$	$[3^3,2,1]$
	$[4^2,2^2]$	$[3^2,2^2,1^2]$
	$[6,2,1^4]$	$[5,1^7]$
$[1^6]$	$[4^3]$	$[6,1^6]$
	$[5,4,2,1]$	$[5,2^2,1^3]$
	$[6,3,1^3]$	$[4,3,2^2,1]$
	$[7,1^5]$	$[3^4]$
$[7]$	$[14]$	$[7^2]$
	$[12,2]$	$[6^2,1^2]$
	$[10,4]$	$[5^2,2^2]$
	$[10,2^2]$	$[5^2,1^4]$
	$[8,6]$	$[4^2,3^2]$
	$[8,4,2]$	$[4^2,2^2,1^2]$
	$[8,2^3]$	$[4^2,1^6]$
	$[6^2,2]$	$[3^4,1^2]$
	$[6,4^2]$	$[3^2,2^4]$
	$[6,4,2^2]$	$[3^2,2^2,1^4]$
	$[6,2^4]$	$[3^2,1^8]$
	$[4^3,2]$	$[2^6,1^2]$
	$[4^2,2^3]$	$[2^4,1^6]$
	$[4,2^5]$	$[2^2,1^{10}]$
	$[2^7]$	$[1^{14}]$
$[2,1^5]$	$[6,4^2]$	$[7,2,1^5]$
	$[7,4,2,1]$	$[6,3,2,1^3]$
	$[6,5,2,1]$	$[6,2^3,1^2]$
	$[6,4,3,1]$	$[6,2^2,1^4]$
	$[6,4,2^2]$	$[6,2,1^6]$
	$[8,3,1^3]$	$[5,4,2^2,1]$
	$[7,4,1^3]$	$[5,3^2,2,1]$
	$[7,3,2,1^2]$	$[5,3^2,1^3]$

cont'd

Table I - cont'd

SU(M)	SU(N)	
	$[6,5,1^3]$	$[5,3,2^3]$
	$[6,4,2,1^2]$	$[5,3,2^2,1^2]$
	$[6,3^2,1^2]$	$[5,3,2,1^4]$
	$[5^2,3,1]$	$[5,2^4,1]$
	$[5,4^2,1]$	$[5,2^3,1^3]$
	$[5,4,3,2]$	$[5,2^2,1^5]$
	$[5^2,2,1^2]$	$[4^2,3^2]$
	$[5,4,3,1^2]$	$[4^2,3,2,1]$
	$[5,4,2^2,1]$	$[4^2,2^2,1^2]$
	$[9,1^5]$	$[4,3^3,1]$
	$[8,2,1^4]$	$[4,3^2,2,1^2]$
	$[7,3,1^4]$	$[4,3^2,2^2]$
	$[6,4,1^4]$	$[4,3,2^3,1]$
	$[6,3,2,1^3]$	$[4,3,2^2,1^3]$
	$[4^3,2]$	$[3^4,1^2]$
	$[7,2,1^5]$	$[6,1^8]$
$[1^7]$	$[5,4^2,1]$	$[7,1^7]$
	$[5^2,2^2]$	$[6,2^2,1^4]$
	$[6,4,2,1^2]$	$[5,3,2^2,1^2]$
	$[7,3,1^4]$	$[4^2,2^3]$
	$[8,1^6]$	$[4,3^3,1]$
$[8]$	$[2^8]$	$[8^2]$
	$[4,2^6]$	$[7^2,1^2]$
	$[4^2,2^4]$	$[6^2,2^2]$
	$[1^6,2^5]$	$[6^2,1^4]$
	$[4^3,2^2]$	$[5^2,3^2]$
	$[6,4,2^3]$	$[5^2,2^2,1^2]$
	$[8,2^4]$	$[5^2,1^6]$
	$[4^4]$	$[4^4]$
	$[6,4^2,2]$	$[4^2,3^2,1^2]$
	$[6^2,2^2]$	$[4^2,2^4]$

cont'd

Table I - cont'd

SU(M)	SU(N)	
	$[8,4,2^2]$	$[4^2,2^2,1^4]$
	$[10,2^3]$	$[4^2,1^8]$
	$[6^2,4]$	$[3^4,2^2]$
	$[8,4^2]$	$[3^4,1^4]$
	$[8,6,2]$	$[3^2,2^4,1^2]$
	$[10,4,2]$	$[3^2,2^2,1^6]$
	$[12,2^2]$	$[3^2,1^{10}]$
	$[8^2]$	$[2^8]$
	$[10,6]$	$[2^6,1^4]$
	$[12,4]$	$[2^4,1^8]$
	$[14,2]$	$[2^2,1^{12}]$
	$[16]$	$[1^{16}]$
$[1^8]$	$[5^2,4,2]$	$[8,1^8]$
	$[6,4^2,1^2]$	$[7,2^2,1^5]$
	$[6,5,2^2,1]$	$[6,3,2^2,1^3]$
	$[7,4,2,1^3]$	$[5,4,2^3,1]$
	$[8,3,1^5]$	$[5,3^3,1^2]$
	$[9,1^7]$	$[4^2,3^2,2]$
$[1^9]$	$[5^3,3]$	$[9,1^9]$
	$[6,5,4,2,1]$	$[8,2^2,1^6]$
	$[7,4^2,1^3]$	$[7,3,2^2,1^4]$
	$[6^2,2^3]$	$[6,4,2^3,1^2]$
	$[7,5,2^2,1^2]$	$[6,3^3,1^3]$
	$[8,4,2,1^4]$	$[5^2,2^4]$
	$[9,3,1^6]$	$[5,4,3^2,2,1]$
	$[10,1^8]$	$[4^3,3^2]$

Table I - cont'd.

SU(M)	SU(N)	
$[1^{10}]$	$[5^4]$	$[10,1^{10}]$
	$[6,5^2,3,1]$	$[9,2^2,1^7]$
	$[6^2,4,2^2]$	$[8,3,2^2,1^5]$
	$[7,5,4,2,1^2]$	$[7,4,2^3,1^3]$
	$[8,4^2,1^4]$	$[7,3^3,1^4]$
	$[7,6,2^3,1]$	$[6,4,3^2,2,1^2]$
	$[8,5,2^2,1^3]$	$[6,5,2^4,1]$
	$[9,4,2,1^5]$	$[5^2,3^2,2^2]$
	$[10,3,1^7]$	$[5,4^2,3^2,1]$
	$[11,1^9]$	$[4^5]$
$[1^{11}]$	$[12,1^{10}]$	$[11,1^{11}]$
	$[11,3,1^8]$	$[10,2^2,1^8]$
	$[10,4,2,1^6]$	$[9,3,2^2,1^6]$
	$[9,5,2^2,1^4]$	$[8,4,2^3,1^4]$
	$[9,4^2,1^5]$	$[8,3^3,1^5]$
	$[8,6,2^3,1^2]$	$[7,5,2^4,1^2]$
	$[8,5,4,2,1^3]$	$[7,4,3^2,2,1^3]$
	$[7^2,2^4]$	$[6^2,2^5]$
	$[7,6,4,2^2,1]$	$[6,5,3^2,2^2,1]$
	$[7,5^2,3,1^2]$	$[6,4^2,3^2,1^2]$
	$[6^2,5,3,2]$	$[5^2,4,3^2,2]$
	$[6,5^3,1]$	$[5,4^4,1]$
$[1^{12}]$	$[13,1^{11}]$	$[12,1^{12}]$
	$[12,3,1^9]$	$[11,2^2,1^9]$
	$[11,4,2,1^7]$	$[10,3,2^2,1^7]$
	$[10,5,2^2,1^5]$	$[9,4,2^3,1^5]$
	$[10,4^2,1^6]$	$[9,3^3,1^6]$
	$[9,6,2^3,1^3]$	$[8,5,2^4,1^3]$
	$[9,5,4,2,1^4]$	$[8,4,3^2,2,1^4]$
	$[8,7,2^4,1]$	$[7,6,2^5,1]$
	$[8,6,4,2^2,1^2]$	$[7,5,3^2,2^2,1^2]$

cont'd

Table I - cont'd.

SU(M)	SU(N)	
	$[8,5^2,3,1^3]$	$[7,4^2,3^2,1^3]$
	$[7^2,4,2^3]$	$[6^2,3^2,2^3]$
	$[7,6,5,3,2,1]$	$[6,5,4,3^2,2,1]$
	$[7,5^3,1^2]$	$[6,4^4,1^2]$
	$[6^3,3^2]$	$[5^3,3^3]$
	$[6^2,5^2,2]$	$[5^2,4^3,2]$

TABLE II. The reduction of the tensor products of $[1^3]$ and $[1^4]$ of SU(N) up to five copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(M) into the representation of SU(N) where the defining representation of SU(N) is embedded to the fundamental representation of SU(M) such that M is the dimension of the defining representation of SU(N).

SU(M)	Defining Representations of SU(N)		
$[1]$	$[1^3]$	$[1^4]$	
$[2]$	$[2^3]$ $[2,1^4]$	$[2^4]$ $[2^2,1^4]$	$[1^8]$
$[1^2]$	$[2^2,1^2]$ $[1^6]$	$[2^3,1^2]$	$[2,1^6]$
$[3]$	$[3^3]$ $[3,2^2,1^2]$ $[2^3,1^3]$ $[3,1^6]$	$[3^4]$ $[3^2,2^2,1^2]$ $[3,2^3,1^3]$ $[2^4,1^4]$ $[3^2,1^6]$	$[2^6]$ $[2^3,1^6]$ $[2^2,1^8]$ $[1^{12}]$
$[2,1]$	$[3^2,2,1]$ $[3,2^2,1^2]$ $[2^3,1^3]$ $[3,2,1^4]$ $[2^4,1]$ $[2^2,1^5]$ $[2,1^7]$	$[3^3,2,1]$ $[3^2,2^2,1^2]$ $[3,2^3,1^3]$ $2 \times [2^4,1^4]$ $[3^2,2,1^4]$ $[3,2^4,1]$	$[3,2^2,1^5]$ $[2^5,1^2]$ $[2^3,1^6]$ $[3,2,1^7]$ $[2^2,1^8]$ $[2,1^{10}]$
$[1^3]$	$[3^2,1^3]$ $[3,2^3]$ $[2^3,1^3]$ $[2^2,1^5]$ $[1^9]$	$[3^3,1^3]$ $[3^2,2^3]$ $[3,2^3,1^3]$ $[3,2^2,1^5]$	$[2^5,1^2]$ $[2^3,1^6]$ $[3,1^9]$

Table II - cont'd.

SU(M)	SU(N)		
[4]	$[4^3]$	$[4^4]$	2 x $[3^2,2^2,1^6]$
	$[4,3^2,1^2]$	$[4^2,3^2,1^2]$	$[4,2^6]$
	$[3^3,1^3]$	$[4,3^3,1^3]$	$[3,2^5,1^3]$
	$[4,2^4]$	2 x $[3^4,1^4]$	$[3,2^4,1^5]$
	$[4,2^2,1^4]$	$[4^2,2^4]$	$[3,2^3,1^7]$
	$[3^2,2^2,1^2]$	$[4^2,2^2,1^4]$	$[2^8]$
	$[3,2^3,1^3]$	$[4,3^2,2^2,,1^2]$	2 x $[2^6,1^4]$
	$[3,2^2,1^5]$	$[4,3,2^3,1^3]$	2 x $[2^4,1^8]$
	$[2^6]$	$[4,3,2^2,1^5]$	$[4^2,1^8]$
	$[4,1^8]$	$[3^4,2^2]$	$[3^2,2,1^8]$
		$[3^3,2^3\ 1]$	$[3^2,1^{10}]$
		$[3^3,2,1^5]$	$[2^3,1^{10}]$
		2 x $[3^2,2^4,1^2]$	$[2^2,1^{12}]$
		$[3^2,2^3,1^4]$	$[1^{16}]$
[3,1]	$[4^2,3,1]$	$[4^3,3,1]$	$[4,2^5,1^2]$
	$[4,3^2,1^2]$	$[4^2,3^2,1^2]$	$[4,2^4,1^4]$
	2 x $[3^3,1^3]$	2 x $[4,3^3,1^3]$	$[4,2^3,1^6]$
	$[4,3,2^2,1]$	2 x $[3^4,1^4]$	$[3^2,2^5]$
	$[4,3,2,1^3]$	$[4^2,3,2^2,1]$	2 x $[3,2^6,1]$
	$[3^3,2,1]$	$[4^2,3,2,1^3]$	3 x $[3,2^5,1^3]$
	$[3^2,2^2,1^2]$	$[4^2,2^3,1^2]$	3 x $[3,2^4,1^5]$
	2 x $[3^2,2,1^4]$	$[4^2,2^2,1^4]$	3 x $[3,2^3,1^7]$
	$[4,2^3,1^2]$	$[4,3^3,2,1]$	2 x $[2^7,1^2]$
	$[4,2^2,1^4]$	$[4,3^2,2^3]$	2 x $[2^6,1^4]$
	$[3^2,2^3]$	$[4,3^2,2^2,1^2]$	3 x $[2^5,1^6]$
	2 x $[3,2^4,1]$	2 x $[4,3^2,2,1^4]$	2 x $[2^4,1^8]$
	2 x $[3,2^3,1^3]$	2 x $[4,3,2^4,1]$	$[4^2,2,1^6]$
	2 x $[3,2^2,1^5]$	2 x $[4,3,2^3,1^3]$	$[4,3,2,1^7]$
	$[2^5,1^2]$	2 x $[4,3,2^2,1^5]$	$[4,3,1^9]$
	$[2^4,1^4]$	$[3^4,2^2]$	2 x $[3^2,2,1^8]$

cont'd.

Table II - cont'd.

SU(M)	SU(N)		
	$[2^3,1^6]$		
	$[4,2,1^6]$	$2 \times [3^4,2,1^2]$	$[3^2,1^{10}]$
	$[3,2,1^7]$	$2 \times [3^3,2^3,1]$	$[3^3,1^7]$
	$[3,1^9]$	$3 \times [3^3,2^2,1^3]$	$2 \times [3,2^2,1^9]$
		$3 \times [3^3,2,1^5]$	$2 \times [2^3,1^{10}]$
		$3 \times [3^2,2^4,1^2]$	$[3,2,1^{11}]$
		$4 \times [3^2,2^3,1^4]$	$[2^2,1^{12}]$
		$4 \times [3^2,2^2,1^6]$	$[2,1^{14}]$
$[2^2]$	$[4^2,2^2]$	$[4^3,2^2]$	$3 \times [3^2,2^2,1^6]$
	$[4,3^2,1^2]$	$[4^2,3^2,1^2]$	$[4,2^6]$
	$[4,3,2,1^3]$	$[4^2,3,2,1^3]$	$2 \times [4,2^4,1^4]$
	$[3^3,2,1]$	$[4,3^3,2,1]$	$[3,2^6,1]$
	$2 \times [3^2,2^2,1^2]$	$2 \times [4,3^2,2^2,1^2]$	$2 \times [3,2^5,1^3]$
	$[3^2,2,1^4]$	$[4,3^2,2,1^4]$	$2 \times [3,2^4,1^5]$
	$[4,2^4]$	$[3^4,2,1^2]$	$2 \times [3,2^3,1^7]$
	$[4,2^2,1^4]$	$2 \times [3^4,1^4]$	$2 \times [2^8]$
	$[3,2^4,1]$	$2 \times [3^3,2^2,1^3]$	$2 \times [2^6,1^4]$
	$[3,2^3,1^3]$	$2 \times [3^3,2,1^5]$	$[2^5,1^6]$
	$[3,2^2,1^5]$	$[4^2,2^4]$	$2 \times [2^4,1^8]$
	$[2^6]$	$[4^2,2^2,1^4]$	$[4,3^2,1^6]$
	$2 \times [2^4,1^4]$	$[4,3,2^4,1]$	$[4,3,2,1^7]$
	$[3^2,1^6]$	$[4,3,2^3,1^3]$	$[3^2,2,1^8]$
	$[3,2,1^7]$	$[4,3,2^2,1^5]$	$[4,2^2,1^8]$
	$[2^2,1^8]$	$[3^4,2^2]$	$[3,2^2,1^9]$
		$[3^3,2^3,1]$	$[3^2,1^{10}]$
		$3 \times [3^2,2^4,1^2]$	$[3,2,1^{11}]$
		$2 \times [3^2,2^3,1^4]$	$[2^2,1^{12}]$
$[2,1^2]$	$[4^2,2,1^2]$	$[4^3,2,1^2]$	$2 \times [4,2^3,1^6]$
	$[4,3^2,2]$	$[4^2,3^2,2]$	$2 \times [3^2,2^5]$
	$[4,3,2^2,1]$	$[4^2,3,2^2,1]$	$2 \times [3,2^6,1]$

cont'd.

Table II - cont'd.

SU(M)	SU(N)		
	$[4,3,2,1^3]$	$[4^2,3,2,1^3]$	$3 \times [3,2^5,1^3]$
	$[3^3,2,1]$	$[4,3^3,2,1]$	$4 \times [3,2^4,1^5]$
	$[3^3,1^3]$	$[4,3^3,1^3]$	$3 \times [3,2^3,1^7]$
	$2 \times [3^2,2^2,1^2]$	$2 \times [4,3^2,2^2,1^2]$	$2 \times [2^7,1^2]$
	$2 \times [3^2,2,1^4]$	$2 \times [4,3^2,2,1^4]$	$[2^6,1^4]$
	$[4,2^3,1^2]$	$2 \times [3^4,2,1^2]$	$2 \times [2^5,1^6]$
	$[3^2,2^3]$	$[3^4,1^4]$	$[2^4,1^8]$
	$[3,2^4,1]$	$4 \times [3^3,2^2,1^3]$	$[4^2,3,1^5]$
	$2 \times [3,2^3,1^3]$	$3 \times [3^3,2,1^5]$	$[4,3^2,1^6]$
	$2 \times [3,2^2,1^5]$	$[4^2,2^3,1^2]$	$[4,3,2,1^7]$
	$2 \times [2^5,1^2]$	$[4,3^2,2^3]$	$2 \times [3^3,1^7]$
	$[2^4,1^4]$	$[4,3,2^4,1]$	$2 \times [3^2,2,1^8]$
	$2 \times [2^3,1^6]$	$2 \times [4,3,2^3,1^3]$	$[3^5,1]$
	$[4,3,1^5]$	$2 \times [4,3,2^2,1^5]$	$[4,2^2,1^8]$
	$[3^2,1^6]$	$2 \times [3^3,2^3,1]$	$2 \times [3,2^2,1^9]$
	$[3,2,1^7]$	$2 \times [3^2,2^4,1^2]$	$[2^3,1^{10}]$
	$[2^2,1^8]$	$4 \times [3^2,2^3,1^4]$	$[4,2,1^{10}]$
	$[2,1^{10}]$	$2 \times [3^2,2^2,1^6]$	$[3,2,1^{11}]$
		$2 \times [4,2^5,1^2]$	$[3,1^{13}]$
		$[4,2^4,1^4]$	
$[1^4]$	$[4^2,1^4]$	$[4^3,1^4]$	$[4,2^6]$
	$[4,3,2^2,1]$	$[4^2,3,2^2,1]$	$[4,2^4,1^4]$
	$[3^2,2^2,1^2]$	$[4,3^2,2^2,1^2]$	$[4,2^3,1^6]$
	$[3^2,2,1^4]$	$[4,3^2,2,1^4]$	$[3^2,2^4,1^2]$
	$[3^2,1^6]$	$[4,3^2,1^6]$	$[3^2,2^2,1^6]$
	$[3^4]$	$[3^3,2^3,1]$	$[3,2^6,1]$
	$[3,2^3,1^3]$	$[3^3,2^2,1^3]$	$[3,2^5,1^3]$
	$[2^6]$	$[3^3,2,1^5]$	$[3,2^4,1^5]$
	$[2^4,1^4]$	$[3^3,1^7]$	$[3,2^3,1^7]$
	$[2^3,1^6]$	$[4,3^4]$	$[4,2^2,1^8]$
	$[2^2,1^8]$	$[4,3,2^3,1^3]$	$[3,2^2,1^9]$
	$[1^{12}]$	$[3^4,2,1^2]$	$[2^6,1^4]$
		$[3^2,2^3,1^4]$	$[4,1^{12}]$

Table II - cont'd.

SU(M)	SU(N)
[5]	$[5^3]$
	$[5,4^2,1^2]$
	$[4^3,1^3]$
	$[5,3^2,2^2]$
	$[5,3^2,1^4]$
	$[4^2,3,2,1^2]$
	$[4,3^2,2^2,1]$
	$[4,3^2,2,1^3]$
	$[4,3^2,1^5]$
	$[3^3,2^3]$
	$[3^3,2,1^4]$
	$[5,2^4,1^2]$
	$[4,3,2^3,1^2]$
	$[4,2^4,1^3]$
	$[5,2^2,1^6]$
	$[4,3,2^2,1^4]$
	$[4,2^3,1^5]$
	$[4,2^2,1^7]$
	$[3^4,1^3]$
	$[3^2,2^4,1]$
	$[3^2,2^2,1^5]$
	$[3,2^5,1^2]$
	$[5,1^{10}]$
$[1^5]$	$[5^2,1^5]$
	$[5,4,2^2,1^2]$
	$[4^2,2^3,1]$
	$[4^2,2^2,1^3]$
	$[4^2,2,1^5]$
	$[4^2,1^7]$
	$[5,3^2,2^2]$
	$[4^2,3^2,1]$
	$[4,3^3,1^2]$

cont'd.

Table II - cont'd.

SU(M)	SU(N)
$[4,3^2,2,1^3]$	
$[4,3,2^3,1^2]$	
$[4,3,2^2,1^4]$	
$[3^4,1^3]$	
$[3^3,2^3]$	
$[3^3,2,1^4]$	
$[3^2,2^4,1]$	
$[3^2,2^3,1^3]$	
$2 \times [3^2,2^2,1^5]$	
$[3^2,2,1^7]$	
$[3^2,1^9]$	
$[3,2^5,1^2]$	
$[3,2^3,1^6]$	
$[2^6,1^3]$	
$[2^5,1^5]$	
$[2^4,1^7]$	
$[2^3,1^9]$	
$[2^2,1^{11}]$	
$[1^{15}]$	

Table III. The reduction of the tensor products of $[2,1]$, $[2,2]$, and $[2,1^2]$ of SU(N) up to three copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(M) into the representations of SU(N) where the defining representations $[2,1]$, $[2,2]$ and $[2,1^2]$ are embedded to the fundamental representation of SU(M) respectively. Here M is the dimension of the defining representation of SU(N).

SU(M)	Defining Representations of SU(N)			
$[1]$	$[2,1]$	$[2,2]$	$[2,1^2]$	
$[2]$	$[4,2]$	$[4^2]$	$[4,2^2]$	$[3,2,1^3]$
	$[3,2,1]$	$[4,2^2]$	$[4,1^4]$	$[2^4]$
	$[3,1^3]$	$[3^2,1^2]$	$[3^2,1^2]$	$[2^2,1^4]$
	$[2^3]$	$[2^4]$	$[3,2^2,1]$	
$[1^2]$	$[4,1^2]$	$[4,3,1]$	$[4,2,1^2]$	$[3,2,1^3]$
	$[3^2]$	$[3,2^2,1]$	$[3^2,2]$	$[3,1^5]$
	$[3,2,1]$		$[3,2^2,1]$	$[2^3,1^2]$
	$[2^2,1^2]$			
$[3]$	$[6,3]$	$[6^2]$	$[6,3^2]$	$[5,2,1^5]$
	$[5,3,1]$	$[6,4,2]$	$[6,2^2,1^2]$	$2 \times [4,3,1^5]$
	$[4^2,1]$	$[5^2,1^2]$	$[5,4,2,1]$	$[4,2^2,1^4]$
	$[5,2^2]$	$[5,4,2,1]$	$[5,3^2,1]$	$[4,2,1^6]$
	$[5,2,1^2]$	$2 \times [4^2,2^2]$	$2 \times [5,3,2,1^2]$	$[3^3,2,1]$
	$[4,3,2]$	$[6,2^3]$	$[4^2,3,1]$	$2 \times [3^3,1^3]$
	$[4,3,1^2]$	$[5,3^2,1]$	$2 \times [4^2,2,1^2]$	$[4^2,1^4]$
	$2 \times [4,2^2,1]$	$[4^3]$	$[5,3,2^2]$	$2 \times [3^2,2^3]$
	$[3^3]$	$[5,3,2,1^2]$	$[5,2^3,1]$	$2 \times [3^2,2^2,1^2]$
	$[3^2,2,1]$	$[4,3^2,1^2]$	$2 \times [5,2^2,1^3]$	$2 \times [3^2,2,1^4]$
	$[4,2,1^3]$	$[4,3,2^2,1]$	$2 \times [4,3^2,2]$	$[3^4]$
	$[3^2,1^3]$	$[4,2^4]$	$[4,3^2,1^2]$	$[3,2^4,1]$
	$[3,2^3]$	$[3^4]$	$4 \times [4,3,2^2,1]$	$2 \times [3,2^3,1^3]$
	$[3,2^2,1^2]$	$[4^2,1^4]$	$3 \times [4,3,2,1^3]$	$[3^2,1^6]$
	$[4,1^5]$	$[3^2,2^2,1^2]$	$3 \times [4,2^3,1^2]$	$[3,2^2,1^5]$
		$[2^6]$	$[6,1^6]$	$[2^5,1^2]$
			$[5,3,1^4]$	$[2^3,1^6]$

Table III - cont'd.

SU(M)	SU(N)			
$[2,1]$	$[6,2,1]$	$[6,5,1]$	$[6,3,2,1]$	$[5,1^7]$
	$[5,4]$	$[6,4,2]$	$[6,2^2,1^2]$	4 x $[4,2^2,1^4]$
	2 x $[5,3,1]$	$[5^2,1^2]$	$[5,4,3]$	2 x $[4,2,1^6]$
	$[4^2,1]$	2 x $[5,4,2,1]$	$[5,4,2,1]$	$[5,4,1^3]$
	$[5,2^2]$	2 x $[4^2,2^2]$	2 x $[5,3^2,1]$	2 x $[4^2,2^2]$
	2 x $[5,2,1^2]$	$[6,3,2,1]$	4 x $[5,3,2,1^2]$	4 x $[3^3,2,1]$
	3 x $[4,3,2]$	$[5,4,3]$	2 x $[4^2,3,1]$	2 x $[3^3,1^3]$
	3 x $[4,3,1^2]$	$[5,3^2,1]$	3 x $[4^2,2,1^2]$	$[4^2,1^4]$
	3 x $[4,2^2,1]$	$[5,3,2^2]$	2 x $[5,3,2^2]$	2 x $[3^2,2^3]$
	$[5,1^4]$	$[4^2,3,1]$	3 x $[5,2^3,1]$	5 x $[3^2,2^2,1^2]$
	2 x $[4,2,1^3]$	$[4,3^2,2]$	3 x $[5,2^2,1^3]$	4 x $[3^2,2,1^4]$
	3 x $[3^2,2,1]$	$[5,4,1^3]$	3 x $[4,3^2,2]$	2 x $[4,2^4]$
	$[3^2,1^3]$	$[5,3,2,1^2]$	4 x $[4,3^2,1^2]$	3 x $[3,2^4,1]$
	$[3,2^3]$	$[4^2,2,1^2]$	6 x $[4,3,2^2,1]$	3 x $[3,2^3,1^3]$
	2 x $[3,2^2,1^2]$	2 x $[4,3,2^2,1]$	7 x $[4,3,2,1^3]$	$[3^2,1^6]$
	$[3,2,1^4]$	$[5,2^3,1]$	5 x $[4,2^3,1^2]$	2 x $[3,2^2,1^5]$
	$[2^4,1]$	$[4,3^2,1^2]$	$[6,2,1^4]$	$[2^5,1^2]$
		$[4,2^4]$	2 x $[5,3,1^4]$	$[3,2,1^7]$
		$[3^3,2,1]$	2 x $[5,2,1^5]$	$[2^4,1^4]$
		$[4,3,2,1^3]$	3 x $[4,3,1^5]$	
		$[3^2,2^2,1^2]$		
		$[3,2^4,1]$		
$[1^3]$	$[6,1^3]$	$[6,3^2]$	$[6,3,1^3]$	2 x $[4,2^4]$
	$[5,3,1]$	$[6,4,1^2]$	$[6,2^3]$	2 x $[4,2^3,1^2]$
	$[5,2^2]$	$[5^2,2]$	$[5,4,2,1]$	3 x $[4,2^2,1^4]$
	$[5,2,1^2]$	$[5,4,2,1]$	$[5,3,2^2]$	$[4^3]$
	$[4,3,2]$	$[5,3,2^2]$	2 x $[5,3,2,1^2]$	$[4^2,3,1]$
	2 x $[4,3,1^2]$	$[4^2,3,1]$	$[5,3,1^4]$	$[3^3,2,1]$
	$[4,2^2,1]$	$[4,3^2,2]$	$[4^2,2^2]$	2 x $[3^3,1^3]$
	$[4,2,1^3]$	$[5,3,2,1^2]$	$[4^2,2,1^2]$	3 x $[3^2,2^2,1^2]$
	$[4^2,1]$	$[4^2,2,1^2]$	$[4^2,1^4]$	$[3^4]$
	$[3^2,2,1]$	$[4,3,2^2,1]$	$[5,3^2,1]$	$[3^2,2^3]$
	$[3^3]$	$[3^2,2^3]$	$[5,2^3,1]$	$[3^2,2,1^4]$

cont'd.

Table III. - cont'd.

SU(M)	SU(N)			
	$[3^2,1^3]$	$[4,2^3,1^2]$	$2 \times [5,2^2,1^3]$	$[3,2^4,1]$
	$[3,2^3]$	$[3^3,1^3]$	$[4,3^2,2]$	$2 \times [3,2^3,1^3]$
	$[3,2^2,1^2]$		$3 \times [4,3^2,1^2]$	$[4,2,1^6]$
	$[2^3,1^3]$		$3 \times [4,3,2^2,1]$	$[3^2,1^6]$
			$3 \times [4,3,2,1^3]$	$[3,2^2,1^5]$
			$[5,2,1^5]$	$[4,1^8]$
			$[4,3,1^5]$	$[2^6]$

TABLE IV. The reduction of the tensor products of the spinor representation (0,0,0,0,1) of D_5 up to six copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(16) into the representations of D_5 where the defining representation (0,0,0,0,1) of D_5 is embedded to the fundamental representation [1] of SU(16). Square bracket is used to denote the Young diagrams of SU(16) in the first column. Representations of D_5 are given in Dynkin notation in parentheses as defined in Fig. 1.

SU(16)	Defining representation of D_5	
[1]	(0,0,0,0,1)	
[2]	(0,0,0,0,2)	(1,0,0,0,0)
$[1^2]$	(0,0,1,0,0)	
[3]	(0,0,0,0,3)	(1,0,0,0,1)
[2,1]	(0,0,1,0,1)	(0,0,0,1,0)
	(1,0,0,0,1)	
$[1^3]$	(0,1,0,1,0)	
[4]	(0,0,0,0,4)	(1,0,0,0,2)
	(2,0,0,0,0)	
[3,1]	(0,0,1,0,2)	(1,0,1,0,0)
	(0,1,0,0,0)	(1,0,0,0,2)
	(0,0,0,1,1)	
$[2^2]$	(0,0,2,0,0)	(0,0,0,1,1)
	(0,0,0,0,0)	(1,0,0,0,2)
	(2,0,0,0,0)	
$[2,1^2]$	(0,1,0,1,1)	(0,0,0,1,1)
	(0,1,0,0,0)	(1,0,1,0,0)
$[1^4]$	(1,0,0,2,0)	
	(0,2,0,0,0)	

Table IV - cont'd.

SU(16)	D_5	
[5]	(0,0,0,0,5)	(2,0,0,0,1)
	(1,0,0,0,3)	
[4,1]	(0,0,1,0,3)	(1,0,0,1,0)
	(1,0,0,0,3)	(0,1,0,0,1)
	(1,0,1,0,1)	(2,0,0,0,1)
	(0,0,0,1,2)	
[3,2]	(0,0,2,0,1)	(2,0,0,0,1)
	(1,0,1,0,1)	(0,0,0,0,1)
	(1,0,0,0,3)	(1,0,0,1,0)
	(0,0,0,1,2)	(0,1,0,0,1)
	(0,0,1,1,0)	
$[3,1^2]$	(1,0,0,1,0)	(0,1,0,1,2)
	2 x (0,1,0,0,1)	(1,1,0,1,0)
	(0,0,0,1,2)	(0,0,1,1,0)
	(1,0,1,0,1)	(0,0,0,0,1)
$[2^2,1]$	(0,1,1,1,0)	(2,0,0,0,1)
	(1,0,1,0,1)	(0,0,0,0,1)
	(0,0,0,1,2)	(1,0,0,1,0)
	(0,0,1,1,0)	(0,1,0,0,1)
$[2,1^3]$	(1,0,0,2,1)	(0,1,0,0,1)
	(0,2,0,0,1)	(0,0,1,1,0)
	(1,0,0,1,0)	(1,1,0,1,0)
$[1^5]$	(0,0,0,3,0)	(1,1,0,1,0)

Table IV - cont'd.

SU(16)	D_5	
[6]	(0,0,0,0,6)	(2,0,0,0,2)
	(1,0,0,0,4)	(3,0,0,0,0)
[5,1]	(0,0,1,0,4)	(2,0,0,0,2)
	(1,0,0,0,4)	(2,0,1,0,0)
	(1,0,1,0,2)	(1,0,0,1,1)
	(0,1,0,0,2)	(1,1,0,0,0)
	(0,0,0,1,3)	
$[2,1^4]$	(1,1,0,1,1)	(0,1,1,0,0)
	(0,0,0,3,1)	(0,0,0,2,0)
	(2,0,0,2,0)	(1,1,0,0,0)
	(1,2,0,0,0)	(1,0,0,1,1)
	(0,1,0,2,0)	

TABLE V. The reduction of the tensor products of (0,0,0,0,2) and (1,0,0,0,1) of D_5, and (0,0,0,0,0,0,0,0,1) of D_9 up to two copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(M) into D_5 and D_9 where the defining representation (0,0,0,0,2) and (1,0,0,0,1) of D_5 and (0,0,0,0,0,0,0,0,1) of D_9 are embedded to the fundamental representation [1] of SU(M). Here M is the dimension of the defining representation. Square brackets and parentheses are used for the Young diagrams and Dynkin diagrams respectively.

SU(M)	Defining Representations		
[1]	(0,0,0,0,2) of D_5	(1,0,0,0,1) of D_5	(0,0,0,0,0,0,0,0,1) of D_9
[2]	(0,0,0,0,4) (2,0,0,0,0) (1,0,0,0,2) (0,0,2,0,0)	(2,0,0,0,2) (0,1,1,0,0) (1,0,0,1,1) (1,1,0,0,0) (3,0,0,0,0) 2 x(0,0,0,0,2) (1,0,0,0,0)	(0,0,0,0,0,0,0,0,2) (0,0,0,0,1,0,0,0,0) (1,0,0,0,0,0,0,0,0)
$[1^2]$	(1,0,1,0,0) (0,0,1,0,2)	(2,0,1,0,0) (1,0,0,1,1) (1,1,0,0,0) 2 x(0,0,1,0,0) (0,1,0,0,2)	(0,0,0,0,0,0,1,0,0) (0,0,1,0,0,0,0,0,0)

TABLE VI. The reduction of the tensor products of the spinor representation (0,0,0,0,0,0,1) of D_7 up to four copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(64) into the representation of D_7 where the defining representation (0,0,0,0,0,0,1) of D_7 is embedded to the fundamental representaiton [1] of SU(64). Square bracket is used to denote the Young diagrams of SU(64) in the first column. Representations of D_7 are given in Dynkin notation in parentheses as defined in Fig. 1.

SU(64)	Defining representation of D_7	
[1]	(0,0,0,0,0,0,1)	
[2]	(0,0,0,0,0,0,2)	(0,0,1,0,0,0,0)
$[1^2]$	(1,0,0,0,0,0,0)	(0,0,0,0,1,0,0)
[3]	(0,0,0,0,0,0,3)	(0,0,1,0,0,0,1)
	(1,0,0,0,0,0,1)	
[2,1]	(0,0,0,0,1,0,1) ,	(1,0,0,0,0,0,1)
	(0,0,1,0,0,0,1)	(0,0,0,0,0,1,0)
	(0,1,0,0,0,1,0)	
$[1^3]$	(0,0,0,0,0,1,0)	(0,0,0,1,0,1,0)
	(1,0,0,0,0,0,1)	
[4]	(0,0,0,0,0,0,4)	(2,0,0,0,0,0,0)
	(0,0,1,0,0,0,2)	(0,0,0,1,0,0,0)
	(0,0,2,0,0,0,0)	(1,0,0,0,1,0,0)
	(1,0,0,0,0,0,2)	
[3,1]	2 x (1,0,0,0,0,0,2)	(0,0,1,0,1,0,0)
	(0,0,0,0,1,0,2)	(0,0,1,0,0,0,2)
	(0,1,0,1,0,0,0)	(0,1,0,0,0,0,0)
	(1,0,0,0,1,0,0)	2 x (0,0,0,0,0,1,1)
	2 x (1,0,1,0,0,0,0)	(0,0,0,1,0,0,0)
		(0,1,0,0,0,1,1)

Table VI - cont'd.

SU(64)	D_7	
$[2^2]$	(0,0,1,0,0,0,2)	(0,0,0,0,0,0,0)
	(0,0,0,0,2,0,0)	2 x (1,0,0,0,1,0,0)
	(0,0,2,0,0,0,0)	(0,0,0,0,0,1,1)
	(0,1,0,0,0,1,1)	(2,0,0,0,0,0,0)
	(0,2,0,0,0,0,0)	(0,0,0,1,0,0,0)
$[2,1^2]$	(0,0,0,1,0,1,1)	2 x (0,1,0,0,0,0,0)
	2 x (0,0,0,1,0,0,0)	(1,0,1,0,0,0,0)
	(0,0,1,0,1,0,0)	(1,0,0,0,1,0,0)
	2 x (0,0,0,0,0,1,1)	(0,1,0,0,0,1,1)
	2 x (1,0,0,0,0,0,2)	(0,1,0,1,0,0,0)
$[1^4]$	(0,0,1,0,0,2,0)	(0,0,0,1,0,0,0)
	(0,0,0,2,0,0,0)	(0,0,0,0,0,1,1)
	(0,0,0,0,0,0,0)	(1,0,0,0,1,0,0)
	(2,0,0,0,0,0,0)	

TABLE VII. The reduction of the tensor products of the (0,0,0,0,1,0) of E_6 up to five copies with the given symmetry property of the Young diagram in the first column. The table can also be used to obtain the branching of SU(27) into the representation of E_6 where the defining representation (0,0,0,0,1,0) of E_6 is embedded to the fundamental representation [1] of SU(27). Square bracket is used to denote the Young diagram of SU(27) and parentheses are used to denote the Dynkin diagrams of E_6 as defined in Fig. 1.

SU(27) [1]	Defining representation of E_6 (0,0,0,0,1,0)	
[2]	(0,0,0,0,2,0)	(1,0,0,0,0,0)
[1^2]	(0,0,0,1,0,0)	
[3]	(0,0,0,0,3,0)	(0,0,0,0,0,0)
	(1,0,0,0,1,0)	
[2,1]	(0,0,0,1,1,0)	(0,0,0,0,0,1)
	(1,0,0,0,1,0)	
[1^3]	(0,0,1,0,0,0)	
[4]	(0,0,0,0,4,0)	(0,0,0,0,1,0)
	(1,0,0,0,2,0)	(2,0,0,0,0,0)
[3,1]	(0,0,0,1,2,0)	(0,0,0,0,1,0)
	(1,0,0,0,2,0)	(0,1,0,0,0,0)
	(1,0,0,1,0,0)	(0,0,0,0,1,1)
[2^2]	(0,0,0,2,0,0)	(2,0,0,0,0,0)
	(1,0,0,0,2,0)	(0,0,0,0,1,1)
	(0,0,0,0,1,0)	

Table VII - cont'd.

SU(27)	E_6	
$[2,1^2]$	(0,1,0,0,0,0)	(0,0,1,0,1,0)
	(0,0,0,0,1,1)	(1,0,0,1,0,0)
$[1^4]$	(0,1,0,0,0,1)	
[5]	(0,0,0,0,5,0)	(0,0,0,0,2,0)
	(1,0,0,0,3,0)	(1,0,0,0,0,0)
	(2,0,0,0,1,0)	
[4,1]	(0,0,0,1,3,0)	(0,1,0,0,1,0)
	(1,0,0,1,1,0)	(1,0,0,0,0,1)
	(1,0,0,0,3,0)	(1,0,0,0,0,0)
	(0,0,0,0,2,1)	(0,0,0,1,0,0)
	(2,0,0,0,1,0)	(0,0,0,0,2,0)
[3,2]	(0,0,0,0,2,0)	(2,0,0,0,1,0)
	(0,0,0,2,1,0)	(0,0,0,0,2,1)
	(1,0,0,0,3,0)	(0,0,0,0,2,0)
	(1,0,0,1,1,0)	(0,1,0,0,1,0)
	(0,0,0,1,0,1)	(1,0,0,0,0,1)
	(0,0,0,1,0,0)	(1,0,0,0,0,0)
$[3,1^2]$	(0,0,1,0,2,0)	
	(1,0,0,1,1,0)	(1,0,0,0,0,1)
	(0,1,0,0,1,0)	(0,1,0,0,1,0)
	(0,0,0,0,2,1)	(0,0,0,1,0,1)
	2 x (0,0,0,1,0,0)	(1,0,1,0,0,0)
$[2^2,1]$	(2,0,0,0,1,0)	(0,1,0,0,1,0)
	(0,0,0,1,0,0)	(0,0,0,0,2,0)
	(1,0,0,0,0,1)	(0,0,1,1,0,0)
	(1,0,0,0,0,0)	(0,0,0,1,0,1)
	(0,0,0,0,2,1)	(1,0,0,1,1,0)

Table VII. - cont'd.

SU(27)	E_6	
$[2,1^3]$	(0,1,0,0,1,1)	(0,0,0,1,0,1)
	(1,0,0,0,0,1)	(1,0,1,0,0,0)
	(0,1,0,0,1,0)	
$[1^5]$	(1,0,0,0,0,2)	(0,2,0,0,0,0)