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ANOMALY FREE COMPLEX REPRESENTATIONS IN SU(N)

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ABSTRACT

We have found all irreducible, anomaly-free and complex representations of SU(N) up to dimension 4×10^9 and SU(16). None of these representations are asymptotically free. For each SU(N), we have given a complete list of complex reducible representations which satisfy both asymptotic and anomaly freedom. Applications of such solutions are briefly discussed.

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I. INTRODUCTION

To construct a gauge model unifying the strong, electromagnetic and weak interactions, one needs several constraints to be satisfied by the representation. Georgi and Glashow[1], Georgi[2] and Gell-Mann, Ramond and Slansky[3] have suggested the existence of complex representations as a criterion for grand unified theories. Complex representations can be found in gauge groups SU(N), SO(4N+2) and E_6.

Renormalizability of gauge theories necessitates use of anomaly-free representations.[4] Among the groups with complex representations, SO(4N+2) and E_6 are free of anomaly. For SU(N) one usually needs to combine several representations to cancel anomalies with each other. Actually there are two different ways of getting anomaly-free and complex representations in SU(N). One method is to find anomaly-free, irreducible and complex representations (AFICR), and the other is to form anomaly-free combinations with several complex representations.

The highly reducible nature of the fermion representations is cited[5] as one of the least attractive features of the SU(N) models. For this reason, it may be interesting to find AFICR in the SU(N) group. Okubo[6] and Cox[7] have already observed that none is known with dimensionality below $D = 3 \times 10^5$ for SU(N) with $N \leq 6$.

On the other hand, recent developments of grand unified theories[8] and preon dynamics[9] require comprehensive list of the anomaly-free, reducible and complex representations (AFRCR) for model building.

In this paper, AFICR and AFRCR are presented. A thorough search for

AFICR has been carried out with dimensions less than $D = 4 \times 10^9$ in SU(N) for N less than 17. The smallest AFICR occurs in SU(5) with D = 374,556. The next lowest AFICR is in SU(5) with D = 1,357,824. Altogether twenty-eight AFICR are presented in Section II. These representations are only of mathematical curiosity and do not have any practical use due to their awesome dimensionality. In addition, they usually contain color exotics, i.e., those representations other than $\underline{1}$, $\underline{3}$, and $\underline{3}^*$ of the color group SU(3). Furthermore a close examination of the branching rules contained in Section III reveals that the SU(6) representation with D = 374,556 can accommodate only one generation of quarks and leptons, along with many exotic particles.

In Section IV, we obtain for every SU(N) all AFRCR which also satisfy the asymptotic freedom condition. The requirement of asymptotic freedom is needed here to limit the number of dimensions of reducible representations.

II. ANOMALY-FREE IRREDUCIBLE COMPLEX REPRESENTATIONS (AFICR).

Irreducible representations of SU(N) will be specified by a set of integers $(\lambda_1, \lambda_2, \cdots, \lambda_{N-1})$, where λ_i equals the number of columns of the Young tableau with i boxes.[10] This notation agrees with the Cartan labels for the highest weight of an irreducible representation.

Complex representations in SU(N) satisfy $(\lambda_1, \lambda_2, \cdots, \lambda_{N-1}) \neq (\lambda_{N-1}, \cdots, \lambda_1)$. Only very few of them are anomaly-free and irreducible. There are no complex representations in SU(2). For SU(3) and SU(4), no AFICR exist below $D = 4 \times 10^9$. Table I summarizes all AFICR in SU(N) with dimensionality up to $D = 4 \times 10^9$ and $N \leq 16$. Conjugate representations are not repeated in Table I.

The smallest AFICR mentioned before corresponds to (0,5,0,0,4) of SU(6) in the Cartan labels. The next lowest AFICR is (0,7,3,3) of SU(5). For clarity, we show the corresponding Young tableaux in Fig. 1.

III. BRANCHING RULES.

To study the branching rules of the smallest AFICR (0,5,0,0,4) of SU(6), we follow the method of elementary multiplets suggested by Sharp and Patera.[11]

For SU(6) $\rightarrow$ SU(5) $\times$ U(1), there are ten elementary multiplets. We use the notation $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5;\ \alpha_1\alpha_2\alpha_3\alpha_4, Y^a)$, where λ_i and α_i are the Cartan labels for SU(6) and SU(5) respectively, and Y^a is the hypercharge of $U^a(1)$ label.

They are

$$A_1^1 = (10000;\ 0000,\ -5) \tag{1}$$

$$A_2^1 = (10000;\ 1000,\ \ 1) \tag{2}$$

$$A_1^2 = (01000;\ 1000,\ -4) \tag{3}$$

$$A_2^2 = (01000;\ 0100,\ \ 2) \tag{4}$$

$$A_1^3 = (00100;\ 0100,\ -3) \tag{5}$$

$$A_2^3 = (00100;\ 0010,\ \ 3) \tag{6}$$

$$A_1^4 = (00010;\ 0010,\ -2) \tag{7}$$

$$A_2^4 = (00010;\ 0001,\ \ 4) \tag{8}$$

$$A_1^5 = (00001,\ 0001,\ -1) \tag{9}$$

$$A_2^5 = (00001;\ 0000,\ \ 5) \tag{10}$$

Table II shows the branching rules of SU(6) to SU(5) with dimension, anomaly in SU(5), and $U^a(1)$ hypercharge. The anomaly and hypercharge in Table II add up zero as expected.

To reduce SU(5) further into $SU(3) \times SU(2) \times U^b(1)$, there are ten elementary multiplets (the notation is $(\lambda_1\lambda_2\lambda_3\lambda_4;\ \alpha_1\alpha_2, \alpha, Y^b)$ where λ_i α_i and α are the Cartan labels for SU(5), SU(3) and SU(2) respectively; Y^b is the hypercharge of $U^b(1)$ normalized to have integer value).

$$A_1^1 = (1000;\ 10,\ 0,\ 2) \tag{11}$$

$$A_2^1 = (1000;\ 00,\ 1,\ -3) \tag{12}$$

$$A_1^2 = (0100;\ 01,\ 0,\ 4) \tag{13}$$

$$A_2^2 = (0100;\ 10,\ 1,\ -1) \tag{14}$$

$$A_3^2 = (0100;\ 00,\ 0,\ -6) \tag{15}$$

$$A_1^3 = (0010;\ 00,\ 0,\ 6) \tag{16}$$

$$A_2^3 = (0010;\ 01,\ 1,\ 1) \tag{17}$$

$$A_3^3 = (0010;\ 10,\ 0,\ -4) \tag{18}$$

$$A_2^4 = (0001;\ 00,\ 1,\ 3) \tag{19}$$

$$A_3^4 = (0001;\ 01,\ 0,\ -2) \tag{20}$$

In addition to those listed above, three more composite elementary factors are necessary and they are:

$$A^{13} = (1010;\ 01,\ 0,\ -2) \tag{21}$$

$$A^{14} = (1001;\ 00,\ 0,\ 0) \tag{22}$$

$$A^{24} = (0101;\ 10,\ 0,\ 2)\ . \tag{23}$$

The following pairs of elementary factors are incompatible:[11] A^{13} with A_2^2 or A^{24}; A^{14} with A_2^2 or A_2^3; and A^{24} with A_2^3 or A^{13}.

The electric charge generator[12] can generally be a linear combination of T_3, Y^a and Y^b:

$$Q = T_3 + AY^a + BY^b \tag{24}$$

where A and B are to be determined to give correct charge assignment. An exhaustive search was made for possible values of A and B which give correct charges for 15 chiral fields $(u\ d)_L$, $u^c{}_L$, $d^c{}_L$, $(\nu\ e)_L$, $e^c{}_L$. Correct charge assignment for one generation can be made with nine different choices of A and B in Eq. (24), but there are huge numbers of exotic states.

IV. ANOMALY-FREE REDUCIBLE COMPLEX REPRESENTATIONS (AFRCR).

All of AFICR in Table I are of enormous dimensions and therefore are only of mathematical interest. In physically interesting theories, the dimensions of representations can be limited by the constraint of asymptotic freedom.[13] This condition gives the following group theoretical constraint:[14]

$$\sum_{R_i} T_2(R_i) \le \frac{11}{2} C_2(G) \tag{25}$$

where R_i is the irreducible representation of fermions; C_2 is the quadratic Casimir operator; G is the adjoint representation; and T_2 is defined by

$$T_2(R)\ \dim(G) \equiv C_2(R)\ \dim(R)\ . \tag{26}$$

There are nine irreducible and complex representations of SU(N), $R_1, R_2 \cdots R_9$, which satisfy the asymptotic freedom. They are defined in Table III along with the associated properties of the representation such as the dimension,

T_2, the value of the Casimir operator C_2, the anomaly A and the maximum allowed value of N for asymptotic freedom.

Among simple groups, the only complex irreducible representations, which are both anomaly-free and asymptotic-free, are the following: 16-, 126-´, 144-dimensional representations of SO(10); the lowest dimensional spinorial representations of SO(14) and SO(18); and 27-dimensional representation of E_6. The maximum multiplicities of these representations bounded by the asymptotic freedom are: 22, 1, 1; 8, 2; and 22 respectively. There are no complex irreducible representations which are both anomaly-free and asymptotic-free in SU(N).

Relaxing the condition of irreducibility[15] we have considered reducible complex representations $\sum n_i R_i$, n_i being integers, which are both anomaly-free and asymptotic-free. Anomaly-free complex representations which satisfy asymptotic freedom are greatly constrained, and a complete list of such representations in SU(N) is reported here. We give a separate list of AFRCR with asymptotic freedom that contain tensor representations of rank at most 2 for the obvious reason of simplicity.

Tables IV and V show all anomaly-free and asymptotic-free combinations of the following form:

$$n_9 \,\square\!\square \oplus n_8 \begin{array}{c}\square\\[-6pt]\square\end{array} \oplus \bar{n}_9 \,\bar{\square} = n_7 R_7 \oplus n_8 R_8 \oplus \bar{n}_9 R_9^* \tag{27}$$

where n_9, n_8 and $\bar{n}_9$ are integers, whose magnitudes are constrained by asymptotic freedom as[16]

$$|n_9|(N+2) + |n_8|(N-2) + |\bar{n}_9| \leq 11N. \tag{28}$$

Negative values of n_i are to be interpreted as the appearance of n_i times of the associated complex conjugate representations. Table IV contains all

AFRCR with asymptotic freedom for arbitrarily large values of N, whereas Table V includes only those for finite range of N.

Except for the solutions in Tables IV and V, all other anomaly-free and asymptotic-free representations contain at least one term whose tensor representation has the rank greater than two. Such solutions however exist only for $N \leq 17$, and are listed in Table VI for $3 \leq N \leq 7$, in Table VII for $8 \leq N \leq 10$, and in Table VIII for $11 \leq N \leq 17$. Again the negative n_i's in these Tables represent the occurrence of the associated complex conjugate representation.

It is to be emphasized that all anomaly-free and asymptotic-free complex representations for $N \geq 18$ are only of the type listed in Tables IV and V.

Since there are a number of representations which differ only in the number of occurrence of R_8 and R_9 we group these different possibilities collectively by P. The variable P takes integer values between finite limits as shown in the last column of Tables VI, VII and VIII. The ℓ appearing in the tenth column of these Tables is the maximum magnitude of the multiplicities of the associated representation consistent with the asymptotic freedom.

V. COMMENTS ON RESULTS.

We have found both irreducible and reducible representations which are complex and anomaly-free. Complex irreducible representations can indeed be anomaly-free, although the number of such examples is very limited.

All AFICR with $D \leq 4 \times 10^9$ are listed in Table I up to SU(16). None of these representations satisfy the asymptotic freedom.

All complex fermion representations in SU(N) which satisfy the constraints of asymptotic freedom and anomaly cancellation are listed in Tables IV - VIII. The most general solution subject to the anomaly-free condition can be given by the sum of a complex representation C_a listed in Tables IV - VIII and a pseudoreal representation R_a whose general form is

$$R_a = \sum_{i=1}^{9} m_i (R_i \oplus R_i^*) \oplus \sum_j n_j r_j \ . \tag{29}$$

Here $\{R_i\}$ are the nine complex representations defined in Table III and j runs over all pseudoreal irreducible representations r_j. The multiplicity m_i and n_j are non-negative integers; and the condition for the general solution, $C_a + R_a$, to be asymptotically free is simply[16]

$$T_2(C_a) + T_2(R_a) \leq \frac{11}{2} N \ . \tag{30}$$

Our results will be useful in model building within the context of grand unified theories (GUT) with elementary scalar fields, where all gauge interactions are unified into a simple gauge group and the constraints to the model include the two conditions we have imposed. Furthermore, we expect the role of pseudoreal representations to be minimal in view of Georgi's rules[2] for grand unification. Usually additional constraints are needed for GUT to insure that fermions transform as 1,3, and 3^* only under the SU(3) color group.[3]

In dynamical models, a new gauge interaction is introduced which becomes strong at an energy scale much above presently available energies. These kinds of interactions usually have a simple compact group structure of the kinds studied here. Again the fermions must satisfy the conditions of asymptotic freedom and anomaly cancellation with respect to this new gauge interaction to be physically meaningful. Furthermore, the real representation content is relatively unimportant. In these models it is perfectly sensible to regard the ordinary quarks and leptons as bound states of more fundamental objects (preons); and the additional constraint that the representation is totally antisymmetric in SU(N) GUT need not apply.

Thus both schemes of unification requires the same minimal conditions on the fermionic content:[13]

(a) existence of complex representations,

(b) asymptotic freedom,

(c) anomaly cancellation.

We have enumerated all solutions to these conditions in this paper.

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12. For example, see K. Kang and I-G. Koh, "Hypercharge Generators in SU(7) GUT", Phys. Rev. D (in press).

13. See, for example, Ref. 9 in the case of preon dynamics and P. H. Frampton, Phys. Lett. 88B, 299 (1979) in the case of grand unification models.

14. In the special case of equality in Eq. (25) an additional group theoretical constraint (arising in higher orders)

$$-\frac{34}{3}N^2+\frac{10}{3}N\sum_{R_i} T_2(R_i)+2\sum_{R_i} C_2(R_i)\,T_2(R_i)<0$$

must be satisfied for the theory to be asymptotically free. We will however present all solutions of Eq. (25). The two loop beta function (from which the above constraint follows) was first calculated by W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974).

15. Recently A. N. Schellekens has also independently compiled reducible anomaly-free and complex representations with a limit on dimension, instead of the asymptotic freedom requirement imposed here.

16. The same comment as in Ref. 14 applies in the special case of equality in Eq. (28).

TABLE I. Anomaly free irreducible complex representations in SU(N). Weight is given by $(\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$, where λ_i equals the number of Young tableau with i boxes. This notation agrees with the Cartan labels for the highest weight of an irreducible representation.

SU(N)	Weight	Dimension
5	0 7 3 3	1357824
5	1 8 1 5	3048474
5	7 7 15 1	1390411776
6	0 5 0 0 4	374556
6	0 5 3 2 3	192615423
6	0 6 0 3 3	28514304
6	0 10 0 0 8	108645537
6	0 10 0 2 7	1000276992
6	1 5 5 0 5	832637988
6	1 6 1 1 5	128035908
6	2 7 1 0 7	303771468
7	0 0 6 3 0 2	1189284096
7	0 1 6 0 2 2	1540923384
7	0 2 4 2 0 3	1747519488
7	0 3 3 1 1 3	1911816192
7	0 4 2 0 2 3	823350528
7	1 3 4 0 0 5	1941877938
7	1 5 1 0 1 5	1207195704
8	0 0 4 0 0 1 2	37081044
8	0 1 3 0 0 0 3	12360348
8	0 4 1 0 0 2 3	1646701056
8	1 5 0 0 0 1 5	1207195704

TABLE I (Cont'd)

SU(N)	Weight	Dimension
10	0 0 0 3 0 0 0 0 2	19423404
10	0 3 0 1 0 0 0 1 3	3080563200
10	1 3 1 0 0 0 0 0 5	2615590692
12	0 2 0 0 1 0 0 0 0 0 3	266982144
14	0 0 1 0 0 1 0 0 0 0 0 0 2	72813312
16	0 0 1 1 0 0 0 0 0 0 0 0 0 0 3	371804160

TABLE II. Branching of the smallest anomaly free complex representation (4 0 0 5 0) of SU(6) into SU(5)×U^a(1). SU(5) weight is given by $(\lambda_1,\ldots,\lambda_4)$ in Cartan labels.

SU(5) weight	dimension	anomaly in SU(5)	Y^a
0 0 5 0	1176	-1050	-30
1 0 5 0	4410	-3339	-24
2 0 5 0	10780	-4851	-18
3 0 5 0	21560	1232	-12
4 0 5 0	38220	30303	- 6
0 0 4 1	1470	-1323	-24
1 0 4 1	5600	-4320	-18
2 0 4 1	13860	-6633	-12
3 0 4 1	28000	200	- 6
4 0 4 1	50050	35750	0
0 0 3 2	1260	-1287	-18
1 0 3 2	4900	-4445	-12
2 0 3 2	12320	-7832	- 6
3 0 3 2	25200	-4500	0
4 0 3 2	45500	22425	6
0 0 2 3	840	-1032	-12
1 0 2 3	3360	-3816	- 6
2 0 2 3	8624	-7700	0
3 0 2 3	17920	-8448	6
4 0 2 3	32760	5148	12
0 0 1 4	420	-627	-6
1 0 1 4	1750	-2500	0
2 0 1 4	4620	-5643	6
3 0 1 4	9800	-8260	12
4 0 1 4	18200	-4810	18
0 0 0 5	126	-225	0
1 0 0 5	560	-984	6
2 0 0 5	1540	-2453	12
3 0 0 5	3360	-4248	18
4 0 0 5	6370	-4732	24

TABLE III. The nine irreducible and complex representations of SU(N) which satisfy the asymptotic freedom constraint. The first and the second columns define the representations and the corresponding Young tableaux. The dimension of the representation, the second index (T_2), the value of the quadratic Casimir operator (C_2), and the anomaly (A) are given in the next four columns. The final column gives the maximum allowed value for N consistent with asymptotic freedom.

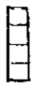
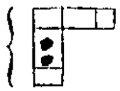
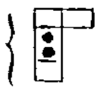
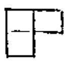
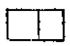
Representation	Young Tableau	Dimension	T_2	C_2	A	N_{max}
R_1		$\frac{N(N-1)(N-2)(N-3)}{24}$	$\frac{(N-2)(N-3)(N-4)}{12}$	$\frac{2}{N}(N^2 - N-4)$	$\frac{(N-8)(N-3)(N-4)}{6}$	12
R_2		$\frac{1}{12} N^2(N^2-1)$	$\frac{N(N^2-4)}{6}$	$\frac{2}{N}(N^2-4)$	$\frac{N(N^2-16)}{3}$	6
R_3		$\frac{1}{6}N(N-1)(N-2)$	$\frac{(N-2)(N-3)}{4}$	$\frac{3(N-3)(N+1)}{2N}$	$\frac{(N-3)(N-6)}{2}$	26
R_4	N-1 	$\frac{1}{2}N(N-1)(N+2)$	$\frac{(N+2)(3N-1)}{4}$	$\frac{(3N-1)(N+1)}{2N}$	$\frac{(N^2+7N-2)}{2}$	5
R_5	N-2 	$\frac{1}{2}N(N+1)(N-2)$	$\frac{(N-2)(3N+1)}{4}$	$\frac{(3N+1)(N-1)}{2N}$	$\frac{(-N^2+7N+2)}{2}$	9
R_6		$\frac{1}{3}N(N^2-1)$	$\frac{N^2-3}{2}$	$\frac{3}{2}(\frac{N^2-3}{N})$	(N^2-9)	11
R_7		$\frac{N(N+1)}{2}$	$\frac{N+2}{2}$	$\frac{(N-1)(N+2)}{N}$	N+4	None
R_8		$\frac{N(N-1)}{2}$	$\frac{N-2}{2}$	$\frac{(N+1)(N-2)}{N}$	N-4	None
R_9		N	$\frac{1}{2}$	$\frac{N^2-1}{2N}$	1	None

TABLE IV. Complex representations of SU(N) of the form of Eq. (26) which satisfy the constraints of anomaly cancellation and asymptotic freedom for arbitrary large N. The representations R_7, R_8 and R_9 are defined in Table III. In a given row, the representation $n_7R_7 + n_8R_8 + \bar{n}_9R_9^*$ is denoted by the integer n_7, n_8 and $\bar{n}_9$. If $n_i < 0$, $n_i R_i$ is to be interpreted as $|n_i| R_i^*$. β gives twice the sum of the $T_2(R)$ for given anomaly free combinations. The last column gives the maximum multiplicity of the representation (denoted by ℓ) consistent with the constraint of asymptotic freedom and its dependence on N. Except for the first row, N is greater than or equal to 5.

n_7	n_8	$\bar{n}_9$	β	ℓ
1	0	N+4	2(N+3)	$\ell=2$ (N=3), $\ell-3$ ($4\leq N\leq 7$), $\ell=4$ ($8\leq N\leq 29$), $\ell=5$ (N>30)
0	1	N−4	2(N−3)	$\ell=13$ (N=5), $\ell=11$ (N=6), $\ell=9$ (N=7), $\ell=8$ ($8\leq N\leq 9$), $\ell=7$ ($10\leq N\leq 14$), $\ell=6$ ($15\leq N\leq 36$) $\ell=5$ ($N\geq 37$)
1	1	2N	4N	$\ell=2$
1	2	3N−4	6N−6	$\ell=2$ ($N\leq 12$), $\ell=1$ ($N\geq 13$)
1	3	4N−8	8N−12	$\ell=1$
1	4	5N−12	10N−18	$\ell=1$
1	−1	8	2N+8	$\ell=3$ ($5\leq N\leq 10$), $\ell=4$ ($11\leq N\leq 39$) $\ell=5$ ($N\geq 40$)
1	−2	−N+12	3N−2+|N−12|	$\ell=2$ (N=5, $N\geq 43$) $\ell=3$ ($6\leq N\leq 42$)
1	−3	−2(N−8)	4N−4+2|N−8|	$\ell=2$ ($N\leq 7$), $\ell=3$ (N=8) $\ell=2$ ($9\leq N\leq 40$), $\ell=1$ ($N\geq 41$)
1	−4	−3N+20	5N−6+|3N−20|	$\ell=2$ ($N\leq 10$), $\ell=1$ ($N\geq 11$)
1	−5	−4(N−6)	6N−8+4|N−6|	$\ell=2$ ($N\leq 7$), $\ell=1$ ($N\geq 8$)
2	1	3N+4	6N+6	$\ell=1$
2	3	5N−4	10N−6	$\ell=1$
2	−1	N+12	4N+14	$\ell=1$ ($N\leq 9$), $\ell=2$ ($N\geq 10$)

TABLE IV. (cont'd)

n_7	n_8	$\bar{n}_9$	β	ℓ
2	-3	-N+20	5N-2+|N-20|	ℓ=1 (5≤N≤11, N≥45)
				ℓ=2 (12≤N≤44)
2	-5	-3N+28	7N-6+|3N-28|	ℓ=1
3	1	4N+8	8N+12	ℓ=1
3	2	5N+4	10N+6	ℓ=1 (N≥6)
3	-1	2N+16	6N+20	ℓ=1
3	-2	N+20	6N+22	ℓ=1
3	-4	-N+28	7N-2+|N-28|	ℓ=1 (N≥6)
3	-5	-2N+32	8N-4+2|N-16|	ℓ=1 (N≥6)
4	1	5N+12	10N+18	ℓ=1 (N≥18)
4	-1	3N+20	8N+26	ℓ=1 (N≥9)
4	-3	N+28	8N+30	ℓ=1 (N≥10)
4	-5	-N+36	9N-2+|N-36|	ℓ=1 (N≥12)
5	-1	4N+24	10N+32	ℓ=1 (N≥32)
5	-2	3N+28	10N+34	ℓ=1 (N≥34)
5	-3	2N+32	10N+36	ℓ=1 (N≥36)
5	-4	N+36	10N+38	ℓ=1 (N≥38)

TABLE V. Complex representations of SU(N) of the form $n_7R_7 \oplus n_8R_8 \oplus \bar{n}_9R_9^*$ which satisfy Eq. (25) but only for a finite range of N. The notations for the first three columns are the same as in Table IV. Column 4 gives the range of N for which an asymptotic-free solution exists. The maximum multiplicity of each of these solutions is $\ell = 1$. N is greater than or equal to 5.

n_7	n_8	$\bar{n}_9$	N
1	5	6N-16	$N \leq 24$
1	6	7N-20	$N \leq 10$
1	7	8N-24	$N \leq 7$
1	8	9N-28	$N \leq 6$
1	9	10N-32	$N = 5$
1	-6	-5N+28	$N \leq 38$
1	-7	-6N+32	$N \leq 14$
1	-8	-7N+36	$N \leq 10$
1	-9	-8N+40	$N \leq 8$
1	-10	-9N+44	$N \leq 6$
1	-11	-10N+48	$N \leq 6$
1	-12	-11N+52	$N = 5$
1	-13	-12N+56	$N = 5$
1	-14	-13N+60	$N = 5$
2	5	7N-12	$N \leq 6$
2	-7	-5N+36	$N \leq 15$
2	-9	-7N+44	$N \leq 8$
2	-11	-9N+52	$N \leq 6$
3	-7	-4N+40	$7 \leq N \leq 16$
3	-8	-5N+44	$7 \leq N \leq 10$
4	-7	-3N+44	$13 \leq N \leq 16$
5	-6	-N+44	$42 \leq N \leq 46$

TABLE VI. Complex solutions in SU(N) with $3 \le N \le 7$ which contain at least one irreducible representation of rank greater than 2.

The representation $R_1, R_2, \cdots, R_9$ are defined in Table III. In a given row, the representation $\sum_{i=1}^{9} n_i R_i$ is denoted by the integer $n_1, \cdots n_9$. Again, if $n_i < 0$, $n_i R_i$ is interpreted as $|n_i| R_i^*$. When a number of solutions of similar form exists, they are sometimes denoted collectively by introducing an integer variable P in the solution. In these cases, the values of P which give solutions are given in the last column. The maximum multiplicity of the solution ℓ is given in the 10th column.

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	$\bar{n}_9$	ℓ	P
SU(3)											
	0	0	0	-1	0	0	1	0	7	1	
	0	0	0	-1	0	0	2	0	0	1	
SU(4)											
	0	0	0	0	0	-1	0	0	7	2	
	0	0	0	0	0	-2	1	0	6	1	
	0	0	0	0	0	1	-1	0	1	2	
	0	0	0	0	0	-1	-1	0	15	1	
	0	0	0	0	0	1	-2	0	9	1	
SU(5)											
	0	0	0	0	-1	0	0	P	6-1P	1	$-6 \le P \le 9$
	0	-1	0	0	0	0	0	P	15-1P	1	$-1 \le P \le 2$
	0	-1	0	0	0	0	1	P	6-1P	1	$-1 \le P \le 3$
	0	0	0	0	0	-1	0	P	16-1P	1	$-4 \le P \le 8$
	0	0	0	0	0	-1	1	P	7-1P	1	$-4 \le P \le 8$
	0	0	0	0	1	-1	1	0	1	1	
	0	0	0	0	0	-1	2	P	-2-1P	1	$-5 \le P < 4$
	0	0	0	0	-1	0	1	P	-3-1P	1	$-6 \le P \le 5$
	0	0	0	0	-1	0	-1	P	15-1P	1	$-2 \le P \le 4$
	0	0	0	0	0	-1	-1	0	25	1	
	0	1	0	0	0	0	-2	P	3-1P	1	$0 \le P \le 1$
	0	0	0	0	1	0	-2	P	12-1P	1	$1 \le P \le 2$
	0	0	0	0	0	1	-3	0	11	1	

TABLE VI. - cont'd.

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
SU(6)											
	0	0	0	0	-1	0	0	P	4-2P	1	-4≤P≤5
	0	0	0	0	0	-1	0	P	27-2P	1	-1≤P≤3
	0	0	0	0	0	-1	1	P	17-2P	1	-1≤P≤4
	0	0	0	0	0	-1	2	P	7-2P	1	-1≤P≤4
	0	0	0	0	0	-1	3	P	-3-2P	1	-2≤P≤1
	0	0	0	0	-1	0	1	P	-6-2P	1	-4≤P≤2
	0	0	0	0	-1	0	-1	P	14-2P	1	-1≤P≤3
SU(7)											
	0	0	0	0	-1	0	0	P	1-3P	1	-2≤P≤2
	0	0	-1	0	0	0	0	P	2-3P	1	-8≤P≤-4, 4≤P≤8
										2	p=-2, -3, 3
										3	p=-1, 2
										4	p=1
										6	p=0
	0	0	-1	0	1	0	0	P	1-3P	1	-1≤P≤1
	0	0	-1	0	-1	0	0	P	3-3P	1	-1≤P≤1
	0	0	-2	0	0	0	0	P	4-3P	1	P=-5, -3, 3, 5, 7
										2	P=-1, 1
	0	0	-3	0	0	0	0	P	6-3P	1	P=-5,-4,-2,-1,2,4,5
										2	P=1
	0	0	-4	0	0	0	0	P	8-3P	1	P=-3,-2,-1,1,2,3,5
	0	0	-5	0	0	0	0	P	10-3P	1	P=-1,1,3
	0	0	-6	0	0	0	1	P	1-3P	1	0≤P≤1
	0	0	-6	0	0	0	0	1	9	1	
	0	0	-4	0	0	0	1	P	-3-3P	1	-3<P≤3
	0	0	-5	0	0	0	1	P	-1-3P	1	-2≤P≤2
	0	0	-3	0	0	0	1	P	-5-3P	1	-5≤P≤4
	0	0	-1	0	0	0	1	P	-9-3P	1	-8≤P≤-4, 2≤P≤6
										2	-3≤P≤1
										3	P=6
	0	0	-2	0	0	0	1	P	-7-3P	1	-6≤P≤-2, 1≤P≤5, P=5
										2	P=0, -1
	0	0	1	0	0	0	1	P	-13-3P	1	-8≤2≤-4, 1≤P≤5
										2	3≤P≤0

TABLE VI. - cont'd

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
(SU(7))											
	0	0	2	0	0	0	1	P	-15-3P	1	$-7\leq P\leq 4$
	0	0	3	0	0	0	1	P	-17-3P	1	$-6\leq P\leq 2$
	0	0	-1	0	0	0	2	P	-20-3P	1	$-8\leq P\leq 3$
	0	0	-2	0	0	0	2	P	-18-3P	1	P=-7,-5,-3,-1,1
	0	0	1	0	0	0	2	P	-24-3P	1	$-9\leq P\leq 3$
	0	0	0	0	1	0	-1	P	10-3P	1	$0\leq P\leq 1$
	0	0	0	0	-1	0	-1	0	12	1	
	0	0	-4	0	0	0	-1	P	19-3P	1	$-1\leq P\leq 4$
	0	0	3	0	0	0	-2	P	16-3P	1	$-1\leq P\leq 5$
	0	0	4	0	0	0	-2	1	11	1	
	0	0	-2	0	0	0	-2	P	26-3P	1	P=-1,1,3,5
	0	0	-3	0	0	0	-2	0	28	1	
	0	0	1	0	0	0	-3	P	31-3P	1	$-1\leq P\leq 4$
	0	0	2	0	0	0	-3	0	29	1	
	0	0	-1	0	0	0	-3	P	35-3P	1	$0\leq P\leq 2$

TABLE VII. Complex solutions in SU(N) with $8 \le N \le 10$ which contains at least oen irreducible representation of rank greater than 2. The notation is the same as that of Table VI.

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
SU(8)										
0	0	-1	0	0	0	0	P	5-4P	1	$-6 \le P \le -3$, $4 \le P \le 7$
									2	P=-2, -1,2,3
									4	P=0,1
0	0	-2	0	0	0	0	P	10-4P	1	P=-3,-1,3,5
									2	P=1
0	0	-3	0	0	0	1	P	3-4P	1	$-3 \le P \le 3$
0	0	-4	0	0	0	1	P	8-4P	1	$-1 \le P \le 2$
0	0	-3	0	0	0	0	P	15-4P	1	$-2 \le P \le 5$
0	0	-4	0	0	0	0	P	20-4P	1	$1 \le P \le 3$
0	0	0	0	-1	0	0	-1	1	1	
0	0	-2	0	0	0	1	P	-2-4P	1	$-5 \le P \le -1$, $1 \le P \le 4$
									2	P=0
0	0	-4	0	0	0	2	-1	0	1	
0	0	-1		0	0	1	P	-7-4P	1	$-7 \le P \le -3$, $2 \le P \le 5$
									2	$-2 \le P \le 1$
0	0	-3	0	0	0	2	P	-9-4P	1	$-3 \le P \le 1$
0	0	1	0	0	0	1	P	-17-4P	1	$-8 \le P \le -2$, $1 \le P \le 4$, p=4
									2	p=-1,0
0	0	-1	0	0	0	2	P	-19-4P	1	$-7 \le P \le 3$
0	0	-2	0	0	0	2	P	-14-4P	1	P=-5,-3,-1,1
0	0	2	0	0	0	1	P	-22-4P	1	$-7 \le P \le 2$
0	0	1	0	0	0	2	P	-29-4P	1	$-8 \le P \le 2$
0	0	-3	0	0	0	-1	P	27-4P	1	$0 \le P \le 3$
0	0	-2	0	0	0	-2	1	30	1	
0	0	1	0	0	0	-3	P	31-4P	1	$-1 \le P \le 6$
0	0	2	0	0	0	-3	P	26-4P	1	$0 \le P \le 1$
0	0	-1	0	0	0	-3	P	41-4P	1	$0 \le P \le 1$

TABLE VII. cont'd

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
SU(9)										
-1	0	0	0	0	0	0	P	5-5P	1	$-4\le P\le -1,\ 2\le P\le 5$
									2	P=0,1
-2	0	0	0	0	0	0	P	10-5P	1	P=-1,1,3
-2	0	1	0	0	0	0	0	1	1	
0	0	-1	0	0	0	0	P	9-5P	1	$-5\le P\le -2,\ 4\le P\le 7$
									2	P=-1,2,3
									3	P=0,1
1	0	-1	0	0	0	0	P	4-5P	1	$-3\le P\le 3$
-1	0	-1	0	0	0	0	P	14-5P	1	$-2\le P\le 4$
-1	0	-1	0	0	0	1	P	1-5P	1	$-2\le P\le 2$
0	0	-2	0	0	0	1	P	5-5P	1	$-3\le P\le 4$
0	0	-2	0	0	0	0	P	18-5P	1	P=-3,-1,1,3,5
1	0	-2	0	0	0	0	P	13-5P	1	$0\le P\le 2$
1	0	-2	0	0	0	1	0	0	1	
-1	0	-2	0	0	0	1	0	10	1	
0	0	-3	0	0	0	1	P	14-5P	1	$0\le P\le 3$
0	0	-3	0	0	0	2	P	1-5P	1	$-1\le P\le 1$
0	0	-3	0	0	0	0	P	27-5P	1	P=1,2,4
-2	0	0	0	0	0	1	P	-3-5P	1	$-1\le P\le 1$
0	0	-1	0	0	0	1	P	-4-5P	1	$-5\le P\le -2,\ 2\le P\le 5$
									2	P=-1,0,1
-1	0	0	0	0	0	1	P	-8-5P	1	$-5\le P\le 3$
1	0	-1	0	0	0	1	P	-9-5P	1	$-3\le P\le 1$
0	0	-2	0	0	0	2	P	-8-5P	1	P=-3,-1,1
1	0	0	0	0	0	1	P	-18-5P	1	$-5\le P\le 2$
-1	0	1	0	0	0	1	P	-17-5P	1	$-4\le P\le 1$
0	0	-1	0	0	0	2	P	-17-5P	1	$-6\le P\le 3$
0	0	1	0	0	0	1	P	-22-5P	1	$-7\le P\le 3$

TABLE VII. cont'd

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
(SU(9))										
-1	0	0	0	0	0	2	P	-21-5P	1	$-5\le P\le 1$
0	0	-1	0	0	0	3	P	-30-5P	1	$-6\le P\le 1$
0	0	1	0	0	0	2	P	-35-5P	1	$-7\le P\le 1$
-1	0	-1	0	0	0	-1	P	27-5P	1	$0\le P\le 2$
0	0	-2	0	0	0	-1	P	31-5P	1	$-1\le P\le 6$
1	0	1	0	0	0	-2	P	12-5P	1	$0\le P\le 2$
-1	0	0	0	0	0	-2	P	31-5P	1	$0\le P\le 5$
0	0	2	0	0	0	-3	P	21-5P	1	$0\le P\le 1$
SU(10)										
-1	0	0	0	0	0	0	P	14-6P	1	$-2\le P\le 4$
-1	0	0	0	0	0	1	P	0-6P	1	$-3\le P\le 3$
-1	0	1	0	0	0	0	P	0-6P	1	$-1\le P\le 1$
0	0	-1	0	0	0	0	P	14-6P	1	$-4\le P\le -1$, $3\le P\le 6$
									2	P=0,1,2
0	0	-1	0	0	0	1	P	0-6P	1	$-5\le P\le -2$, $2\le P\le 5$
									2	p=-1,0,1
-1	0	-1	0	0	0	1	0	14	1	
-1	0	-1	0	0	0	2	0	0	1	
0	0	-2	0	0	0	1	P	14-6P	1	$-2\le P\le 4$
0	0	-2	0	0	0	0	P	28-6P	1	P=-1,1,3,5
0	0	-2	0	0	0	2	P	0-6P	1	P=-1,1
-1	0	0	0	0	0	2	P	-14-6P	1	$-3\le P\le 1$
0	0	-1	0	0	0	2	P	-14-6P	1	$-5\le P\le 3$
0	0	1	0	0	0	1	P	-28-6P	1	$-7\le P\le 3$

TABLE VII. cont'd

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
(SU(10))										
1	0	0	0	0	0	1	P	-28-6P	1	$-5\leq P\leq 1$
0	0	-1	0	0	0	3	P	-28-6P	1	$-5\leq P\leq 1$
0	0	1	0	0	0	2	P	-42-6P	1	$-7\leq P\leq 1$
-1	0	1	0	0	0	-1	0	14	1	
1	0	-1	0	0	0	-1	0	14	1	
0	0	-2	0	0	0	-1	0	42	1	
0	0	2	0	0	0	-3	P	14-6P	1	$0\leq P\leq 2$

TABLE VIII. Complex solutions in SU(N) with $11 \leq N \leq 17$ which contain at least one irreducible representation of rank greater than 2. No solutions of this type exist for $N \geq 18$. The notation is the same as that of Table VI.

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
SU(11)											
	-1	0	0	0	0	0	0	P	28-7P	1	$0\leq P\leq 4$
	-1	0	0	0	0	0	1	P	13-7P	1	$0\leq P\leq 2$
	0	0	-1	0	0	0	0	P	20-7P	1	$-4\leq P\leq -1,\ 3\leq P\leq 6$
										2	$0\leq P\leq 2$
	0	0	-1	0	0	0	1	P	5-7P	1	$-4\leq P\leq -1,\ 2\leq P\leq 4$
										2	P=0,1
	0	0	-2	0	0	0	1	P	25-7P	1	$0\leq P\leq 3$
	0	0	-2	0	0	0	0	P	40-7P	1	P=1,3
	0	0	-2	0	0	0	2	1	3	1	
	0	0	-1	0	0	0	2	P	-10-7P	1	$-4\leq P\leq 3$
	0	0	-1	0	0	0	3	P	-25-7P	1	$-4\leq P\leq 1$
	0	0	1	0	0	0	1	P	-35-7P	1	$-6\leq P\leq 2$
	1	0	0	0	0	0	-2	0	2	1	
	0	0	-1	0	0	0	-2	P	50-7P	1	$0\leq P\leq 4$
	0	0	2	0	0	0	-3	0	5	1	
SU(12)											
	0	0	-1	0	0	0	0	P	27-8P	1	$-3\leq P\leq 6$
	0	0	-1	0	0	0	1	P	11-8P	1	$-3\leq P\leq 4$
	0	0	-1	0	0	0	2	P	-5-8P	1	$-3\leq P\leq 3$
	0	0	-1	0	0	0	3	P	-21-8P	1	$-3\leq P\leq 1$
	0	0	1	0	0	0	1	P	-43-8P	1	$-6\leq P\leq 1$
	0	0	-1	0	0	0	-2	0	59	1	
SU(13)											
	0	0	-1	0	0	0	0	P	35-9P	1	$-2\leq P\leq 6$
	0	0	-1	0	0	0	1	P	18-9P	1	$-2\leq P\leq 4$
	0	0	-1	0	0	0	2	P	1-9P	1	$-2\leq P\leq 2$
	0	0	-1	0	0	0	3	P	-16-9P	1	$-2\leq P\leq 1$
	0	0	1	0	0	0	1	P	-52-9P	1	$-6\leq P\leq 1$

TABLE VIII. - cont'd.

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	ℓ	P
SU(14)											
	0	0	-1	0	0	0	0	P	44-10P	1	$-2\leq P\leq 6$
	0	0	-1	0	0	0	1	P	26-10P	1	$-2\leq P\leq 4$
	0	0	-1	0	0	0	2	P	8-10P	1	$-2\leq P\leq 2$
	0	0	-1	0	0	0	3	P	-10-10P	1	$-2\leq P\leq 1$
	0	0	-1	0	0	0	-1	P	62-10P	1	$0\leq P\leq 5$
SU(15)											
	0	0	-1	0	0	0	0	P	54-11P	1	$-1\leq P\leq 5$
	0	0	-1	0	0	0	1	P	35-11P	1	$-1\leq P\leq 4$
	0	0	-1	0	0	0	2	P	16-11P	1	$-1\leq P\leq 2$
	0	0	-1	0	0	0	3	P	-3-11P	1	$-1\leq P\leq 1$
SU(16)											
	0	0	-1	0	0	0	0	P	65-12P	1	$0\leq P\leq 5$
	0	0	-1	0	0	0	1	P	45-12P	1	$0\leq P\leq 4$
	0	0	-1	0	0	0	2	P	25-12P	1	$0\leq P\leq 2$
	0	0	-1	0	0	0	3	P	5-12P	1	$-1\leq P\leq 1$
SU(17)											
	0	0	-1	0	0	0	0	P	77-13P	1	$0\leq P\leq 2$
	0	0	-1	0	0	0	1	P	56-13P	1	$0\leq P\leq 3$
	0	0	-1	0	0	0	2	P	35-13P	1	$0\leq P\leq 2$
	0	0	-1	0	0	0	3	P	14-13P	1	$0\leq P\leq 1$

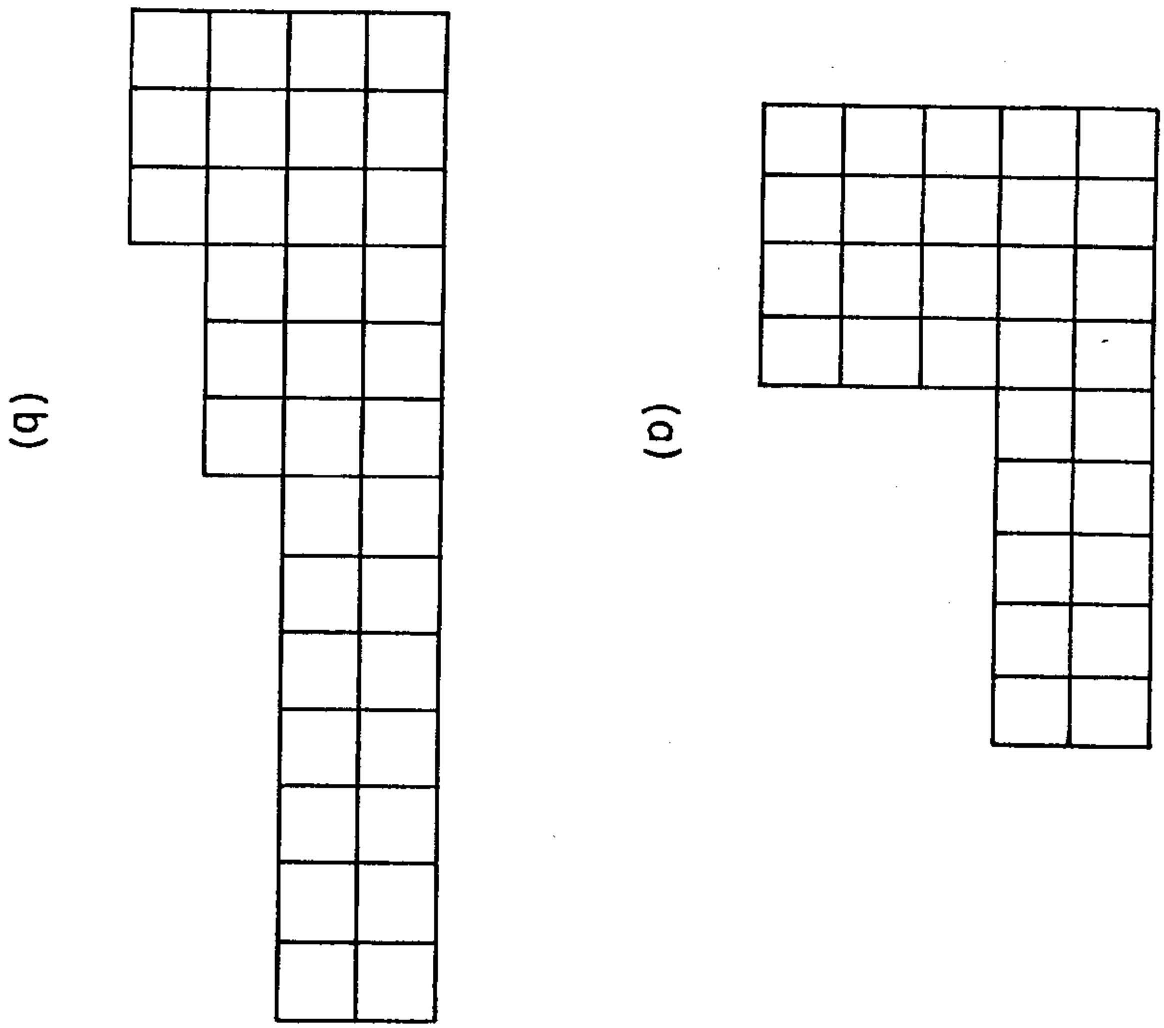


Fig. 1 (a) (0,5,0,0,4) of SU(6) with D = 374,556;
(b) (0,7,3,3) of SU(5) with D = 1,357,824.

These are the two lowest dimensional AFICR