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# On the Quantization of Local Conservation Laws in Two Dimensional Supersymmetric CP(N-1) and Principal Chiral Models

STEVEN GOTTLIEB
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

and

T.E. CLARK and S.T. LOVE
Department of Physics
Purdue University
West Lafayette, IN 47907

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ABSTRACT

We extend our discussion of the infinite series of local conservation laws in superconformal two dimensional models to the CP(N-1), and O(N)×O(N) and SU(N)×SU(N) principal chiral models. The simple argument which was so successful in proving that these laws survive quantization for the bosonic O(N) $\sigma$-model, principal chiral models and supersymmetric O(N) $\sigma$-model does not resolve the issue here. Some new conservation laws are also considered.

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# I. Introduction

This paper is a continuation of our work on local conservation laws and their quantization in superconformal models. In a previous paper[1] we extended Noether's theorem and the Belinfante improvement procedure to superspace in order to construct the supercurrent. Given the supercurrent, it was easy to construct an infinite series of local construction laws. Finally, we were able to prove via a simple argument that the dimension five conservation survives quantization for the supersymmetric non-linear $\sigma$-model just as it does for the ordinary non-linear $\sigma$-model.[2] Here we consider the supersymmetric versions of the other models studied by Goldschmidt and Witten in Ref. 2, namely the CP(N-1) model, and the $O(N)\times O(N)$ and $SU(N)\times SU(N)$ chiral models. We find that the simple argument is sufficient in neither case. It is extremely unlikely, as we will explain below, that the conservation law doesn't hold in the CP(N-1) model. Whether the imposition of the extended supersymmetry of that model is sufficient to prove the conservation law is under investigation. On the other hand, less is known about the chiral models and these results should prompt further study of their dynamics.

Let us briefly review the motivation and background for this work. There are three different lines of development which are relevant. The first is the discovery of theories with a infinite number of conservation laws. The second is the study of supersymmetric two dimensional models, and the third is the construction of exact S-matrices for two dimensional field theories.

The first field began in 1976 when Pohlmeyer[3] showed the close relationship of the O(3) $\sigma$-model to the integrable sine-Gordon theory. An infinite series of local conservation laws was discovered. Shortly thereafter, Lüscher and Pohlmeyer[4] discovered an infinite series of classically conserved non-local charges. Lüscher then showed that the non-local charges are also conserved in the quantum theory.[5] The inverse scattering formalism was applied to a variety of field theories.[6,7]

Ferrara initiated the study of supersymmetric two-dimensional gauge theories in 1975.[8] The supersymmetric $\sigma$-model was formulated by DiVecchia and Ferrara,[9] and by Witten.[10] Soon thereafter Cremmer and Scherk formulated the supersymmetric CP(N-1) model.[11]

The early work on factorized S-matrices was carried out by the Zamolodchikovs and by Berg, Karowski, Thun, Truong and Weisz. S-matrices were found for the sine-Gordon,[12] massive Thirring,[13] O(N) $\sigma$,[14] and chiral invariant SU(N) Thirring[15] models.

At approximately this point, infinite series of conservation laws[16] and exact S-matrices[17] for supersymmetric theories began to make an appearance. The inverse scattering formalism was also applied to supersymmetric models.[18,19] Eichenherr and Forger[20] showed that all classical non-linear $\sigma$-models on symmetric spaces have the dual symmetry of Pohlmeyer.[3] Aspects of the quantum theories were studied by Pisarski.[21] Non-local[22-24] and local[25] conservation laws in the classical supersymmetric $\sigma$-models were constructed.

This brief outline of earlier work should suffice to provide a context for this work. The rest of the paper is organized as follows. Section II is devoted to the CP(N-1) models. The quantization of several conservation laws is considered. Section III deals with the $O(N)\times O(N)$ and $SU(N)\times SU(N)$ models. A new (as far as we know) series of classical conservation laws is derived. For the $SU(N)\times SU(N)$ models the second non-trivial one is shown to survive quantization. Our summary and conclusions are contained in the final section.

## II. The CP(N-1) model

In this section we consider the quantization of the local conservation laws for the supersymmetric CP(N-1) model. The purely bosonic model was constructed by Eichenherr[26] and by Golo and Perelomov.[27] The 1/N expansion was studied by D'Adda, DiVecchia and Lüscher,[28] and by Witten.[29] Such an analysis does not lead one to believe the model is soluble. There are both local[2,30] and non-local[20,31] conservation laws at the classical level; however, recently Abdalla, Abdalla and Gomes[32] have shown that the non-local conservation laws are spoiled by anomalies at the quantum level. An analysis of the local conservation laws also indicates that there may be anomalies there.[2]

The supersymmetric CP(N-1) model was first constructed by Cremmer and Scherk.[33] It was studied in the context of the 1/N expansion by d'Adda, DiVecchia and Lüscher.[34] Our notation is somewhat different from that of the latter group so we will repeat a number of their formulae.

The model is constructed in terms of a complex superfield which transforms as a vector under an SU(N) internal symmetry. Under an Abelian gauge transformation,

$$\phi\rightarrow\phi'=\phi e^{-i\Lambda}, \tag{1}$$

where $\Lambda$ is a real scalar superfield. As usual we wish to define a covariant derivative. Since the theory is supersymmetric, we have spinor derivatives, so we introduce a Majorana spinor superfield $A_a$. The supercovariant derivative acting on $\phi$ and $\phi^*$ is defined by Eqs. (2) and (3).

$$\mathcal{D}_a\phi=(D_a+iA_a)\phi \tag{2}$$

$$\mathcal{D}_a\phi^*=(D_a-iA_a)\phi^* \; . \tag{3}$$

Under a gauge transformation,

$$A \rightarrow A' = A + D\Lambda. \tag{4}$$

The Lagrangian is invariant under supersymmetry, the SU(N) internal symmetry and gauge transformations. A Lagrange multiplier field is included to impose the constraint $\phi^* \cdot \phi = 1$.

$$S = \int d^2x d\bar{\theta} d\theta \, (\overline{\mathcal{D}\phi} \cdot \mathcal{D}\phi + C[\phi^* \cdot \phi - 1]), \tag{5}$$

In detail,

$$\overline{\mathcal{D}\phi} \cdot \mathcal{D}\phi = \bar{D}\phi^* \cdot D\phi - i\phi^* \cdot \bar{A}D\phi + i\bar{D}\phi^* A \cdot \phi + \bar{A}A\phi^* \cdot \phi. \tag{6}$$

No dynamics is included for the gauge field so we may use the equations of motion to express A in terms of $\phi$ and $\phi^*$.

$$\begin{aligned} A &= \frac{i}{2} \, \phi^* \cdot \overleftrightarrow{D}\phi \\ &= i \, \phi^* \cdot D\phi, \end{aligned} \tag{7}$$

taking the constraint $\phi^* \cdot \phi = 1$ into account. Finally, the equation of motion for $\phi$ is

$$\bar{\mathcal{D}}\mathcal{D}\phi = -(\bar{\mathcal{D}}\phi^* \cdot \mathcal{D}\phi)\,\phi. \tag{8}$$

Using the results of our previous paper,[1] it is a simple matter to construct the Belinfante improved supercurrent. Working in light cone coordinates, Eq. (3.21) of Ref. 1 states

$$V_{+2} = -i \frac{\partial \mathcal{L}}{\partial D_1 \phi} \cdot \partial_+ \phi + \partial_+ \phi \cdot \frac{\partial \mathcal{L}}{\partial D_2 \phi^*} +$$

$$+ \frac{1}{2\sqrt{2}} D_2 \left[ \frac{\partial \mathcal{L}}{\partial D_1 \phi} \cdot D_2 \phi + \bar{D}_1 \phi^* \cdot \frac{\partial \mathcal{L}}{\partial \bar{D}_2 \phi^+} \right]. \tag{9}$$

Using the Lagrangian implied by Eqs. (5) and (6), we see that the third term, which is the improvement term, vanishes. {Although the supercurrent does not look gauge covariant because of the appearance of $\partial_+\phi$ rather than the gauge covariant derivative defined in Eq. (13), it is easy to see using Eqs. (12) and (13) that changing the ordinary derivative to a covariant derivative doesn't change $V_{+2}$.

$$V_{+2} = \mathcal{D}_2 \phi^* \cdot \nabla_+ \phi + \nabla_+ \phi^* \cdot \mathcal{D}_2 \phi \ . \tag{10}$$

Once again recalling Ref. 1, superconformal invariance implies $V_{-2} = V_{+1} = 0$, so space time translation invariance, $\bar{D}\ V_\mu = 0$ becomes simply $D_1 V_{+2} = D_2 V_{-1} = 0$. The non-vanishing components of the energy-momentum tensor may be expressed as derivatives of the supercurrent, i.e., $T_{++} = -2i\sqrt{2}\ D_2 V_{+2}$, $T_{--} = -2i\sqrt{2}\ D_1 V_{-1}$. So, $D_1\ T_{++} = D_2\ T_{--} = 0$. Thus follows the infinite series of conservation laws whose quantization we wish to study.

$$D_1[V_{+2}(T_{++})^n] = 0. \tag{11}$$

We will study the quantization of Eq. (10) for n=1 by constructing all the possible counter terms which might appear on the r.h.s. and all the possible total derivative operators in terms of which we would like to reexpress the counterterms. If the derivatives span the set of counterterms then we say the conservation law survives quantization because no matter which counterterms appear the quantum equation is in the form of a conservation law.

In order to construct the counterterms we start off by constructing SU(N) singlet operators linear in $\phi$ and $\phi^*$. The only dimensionless operator is the identity since $|\phi|^2=1$. Gauge invariance makes the dimension 1/2 operators vanish.

$$\phi^*\cdot\mathscr{D}_a\phi = \phi^*\cdot(D_a+iA_a)\phi = \phi^*\cdot(D_a-[\phi^*\cdot D_a\phi])\phi = 0, \tag{12}$$

For dimension one operators we would like to construct the ordinary (i.e. non-spinor) covariant derivative. We denote it by $\nabla_\mu$

$$\nabla_\mu\phi = (\partial_\mu-(\phi^*\cdot\partial_\mu\phi))\phi. \tag{13}$$

Obviously, the dimension one operators involving $\nabla$ vanish. Nonvanishing operators may be formed using two spinorial derivatives. They are

$$\bar{\mathscr{D}}\phi^*\cdot\mathscr{D}\phi$$
$$\bar{\mathscr{D}}\phi^*\cdot\gamma^5\mathscr{D}\phi$$
$$\bar{\mathscr{D}}\phi^*\cdot\gamma^\mu\mathscr{D}\phi \quad . \tag{14}$$

The last operator is the conserved bosonic current required by the extended supersymmetry of this model.[34] Next we construct the dimension 3/2 bilinears. The twist zero operators are listed first.(Twist is defined to be dimension minus light cone weight, where light cone weight is +1 for a + component, -1 for a - component, + 1/2 for a spinorial 2 component, and - 1/2 for a spinorial 1 component.)

$$(\phi^*\cdot\nabla_+ \mathscr{D}_2\phi)$$
$$(\nabla_+\mathscr{D}_2\phi^*\cdot\phi)$$
$$(\nabla_+\phi^*\cdot\mathscr{D}_2\phi)$$
$$(\mathscr{D}_2\phi^*\cdot\nabla_+\phi)$$

But it is easy to see that the first two operators may be expressed in terms of the last two. As $(\phi^*\cdot\mathscr{D}_2\,\phi) = 0$,

$$\partial_+(\phi^*\cdot\mathscr{D}_2\phi) = (\nabla_+\phi^*\cdot\mathscr{D}_2\phi)+(\phi^*\cdot\nabla_+\mathscr{D}_2\phi) = 0.$$

For a superconformally invariant theory, the supercurrent, $V_{\mu a}$, obeys $D_1\ V_{+2}=0$. It is easy to show

$$D_1(\nabla_+\phi^*\cdot\mathscr{D}_2\phi) = 0$$

$$D_1(\mathscr{D}_2\phi^*\cdot\nabla_+\phi) = 0 .$$

We expected one conserved current corresponding to the sum of the two above equations from invariance under supersymmetry. The difference is conserved as a consequence of the invariance under extended sypersymmetry.

There are two twist one operators obtained from the above by replacing $\mathscr{D}_2$ with $\mathscr{D}_1$. In Table 1 we list the bilinears needed to construct the counterterms and total divergence operators required below. The final column of the table gives a compact notation for the operators. Ones denoted S(A) are symmetric (antisymmetric) under the interchange of $\phi$ and $\phi^*$. A comma separates the two sets of derivatives which operate on $\phi^*$ and $\phi$.

Just as we did for the $\sigma$-model, we consider the spinorial conservation law related to $\partial_-(T_{++}^2)=0$, i.e. $D_1(V_{+2}T_{++})=0$. We list all the possible counterterms to this statement. In the case of the $\sigma$-model, there were eight possible counterterms. In the CP(N-1) model, there are 24. They are listed in Table 2.

For the supersymmetric $\sigma$-model, it was possible to reexpress the counterterms as derivatives of other operators. For the supersymmetric CP(N-1) models this isn't possible. To show this we list in Table 3 all the dimension

7/2 twist zero or one operators. There are 22.

A linear combination of $S_{+,+}S_{+,2}$ and $S_{+2,2}$ $S_{+,2}$ is $V_{+2}T_{++}$, the operator whose conservation we wish to show. Therefore, there are clearly not enought divergence operators to span the space of counterterm operators. Thus, we are not able to prove, simply on the basis of the supersymmetry that there will be a dimension five conservation law which is true on the quantum level. It is certainly likely that such a law exists for the supersymmetric CP(N-1) model since the model has an exact S-matrix[35] and there is no anomaly in the first non-trivial non-local current.[36] We have not attempted to use the extended sypersymmetry of the model in the above discussion. Whether this additional symmetry is sufficient to show that the conservation law holds at the quantum level is under investigation. On the other hand, it may be possible to shown via direct calculation that not all operators which may mix with $D_1(V_{+2}T_{++})$ under renormalization actually do so. Supersymmetry usually improves the ultraviolet convergence properties of a theory, so this may be likely. Such a situation would be reminicent of the vanishing of the $\beta$ function to three loop order in N=4 supersymmetry.[37]

Related to the issue of extended supersymmetry, we wish to consider the conservation law one gets by substituting the antisymmetric current for the usual supercurrent in the above conservation law. It is easy to show that the counting works out identically.

## III. The O(N)×O(N) and SU(N)×SU(N) Chiral Models

Let us take up the case of the supersymmetric chiral models.[19] The action is given by

$$S=\int d^2x d\bar{\theta} d\theta \; \mathrm{Tr}(\bar{D}g^{-1} Dg). \tag{15}$$

There is also the constraint that g is an element of O(N) or SU(N). The equation of motion is

$$\bar{D}Dg = -\bar{D}g \; Dg^{-1} g. \tag{16}$$

In this model, operators which are singlets under the internal symmetry will be traces. Operators may also be classified according to their property under the discrete symmetries $g \to g^T$, $g \to g^{-1}$. In O(N), this is the same symmetry so an operator will vanish if it transforms oppositely under the two symmetries.

As in the other models we've considered, the dimension 1/2 operators vanish, i.e. $\mathrm{Tr}(g^{-1}D_a g)=\mathrm{Tr}(D_a g^{-1} g)=0$. This immediately implies that for higher dimension operators if all derivative appear on one of g or $g^{-1}$, the operator may be reexpressed in terms of operators with derivatives on both g and $g^{-1}$.

Considering operators of dimension 1, $\mathrm{Tr}(g^{-1}\partial_\mu g)=\mathrm{Tr}(\partial_\mu g^{-1} g)=0$. In fact, the only nonvanishing operator is $\mathrm{Tr}(D_1 g^{-1} D_2 g)$. For instance,

$$\mathrm{Tr}(D_2 g^{-1} D_2 g) = -\mathrm{Tr}(g^{-1} D_2 g\ g^{-1} D_2 g) = -\mathrm{Tr}\ A^2 = 0, \quad (17)$$

where A is a Grassman matrix, so this vanishes. We note that $\mathrm{Tr}(D_1 g^{-1} D_2 g)$ is even under both discrete symmetries.

For dimension 3/2, we may have one ordinary derivative and one spinor derivative, or three spinor derivatives. In the first case we have $\mathrm{Tr}(\partial_\mu g^{-1} D_a g)$. This is even under both discrete symmetries. In the second case, the twist zero and one operators are $\mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_2 g)$ and $\mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_1 g)$, respectively. These operators are odd under the discrete symmetries. As before it is easy to see what the supercurrent must be.

$$V_{+2} = \mathrm{Tr}(\partial_\mu g^{-1} D_2 g), \quad (18)$$

to within a numerical factor. It is easy to verify that $D_1 V_{+2} = 0$. More interesting is the fact that $D_1\ \mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_2 g) = 0$. This is analogous to the CP(N-1) model[34] where there were two conserved dimension 3/2 currents; however, here we can't see how there can be a conserved bosonic current which is a dimension one superfield.

It is also interesting to note at this point that the last conservation law is the first in a series of classical conservation laws. It is easy to see that for all odd m,

$$D_1 \, Tr\left[(g^{-1}D_2 g)^m\right] = 0. \tag{19}$$

Recall that in the non-supersymmetric model[7]

$$\partial_- Tr(g^{-1}\partial_+ g)^n = 0. \tag{20}$$

Eq. (19) is the supersymmetric analog. Table 4 contains many of the traces relevant to the study of our conservation law. The parity of each operator under the two discrete symmetries is listed and operators which vanish in O(N) because of these symmetries are indicated with an x. For dimension 4 we have only listed twist one operators even under both discrete symmetries as these are the ones relevant to the conservation law we consider. The only dimension three operators required are twist zero and even under the symmetries, since there is only one dimension one operator and it is even and twist one.

Now we construct the anomalies which may spoil the classical conservation law $D_1(V_{+2}T_{++})=0$. There are 15 dimension 4 operators listed in Table 4. In addition, we have the following:

$$\mathrm{Tr}\,(D_2g^{-1}\partial_+^2D_2g+D_2g\partial_+^2D_2g^{-1})\,\mathrm{Tr}\,(D_1g^{-1}D_2g)$$

$$\mathrm{Tr}\,(\partial_+g^{-1}\partial_+^2g+\partial_+g\partial_+^2g^{-1})\,\mathrm{Tr}\,(D_1g^{-1}D_2g)$$

$$\mathrm{Tr}\,(\partial_+g^{-1}\partial_+D_2g+\partial_+g\partial_+D_2g^{-1})\,\mathrm{Tr}\,(\partial_+g^{-1}D_1g)$$

$$\mathrm{Tr}\,(\partial_+^2g^{-1}D_2g+\partial_+^2gD_2g^{-1})\,\mathrm{Tr}\,(\partial_+g^{-1}D_1g)$$

$$\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}\partial_+D_2g-gD_2g^{-1}D_2g\partial_+D_2g^{-1})\,\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}D_1g)$$

$$\mathrm{Tr}\,(\partial_+^2g^{-1}D_1g+\partial_+^2gD_1g^{-1})\,\mathrm{Tr}\,(\partial_+g^{-1}D_2g)$$

$$\mathrm{Tr}\,(g^{-1}D_1gD_2g^{-1}\partial_+D_2g-gD_1g^{-1}D_2g\partial_+D_2g^{-1}$$
$$+g^{-1}D_2gD_1g^{-1}\partial_+D_2g-gD_2g^{-1}D_1g\partial_+D_2g^{-1})\,\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}D_2g)$$

$$\mathrm{Tr}\,(D_1g^{-1}D_2gD_2g^{-1}\partial_+g-D_2g^{-1}D_2gD_1g^{-1}\partial_+g)\,\mathrm{Tr}\,(\partial_+g^{-1}D_2g)$$

$$\mathrm{Tr}\,(\partial_+D_2g^{-1}D_2g+\partial_+D_2gD_2g^{-1})\,\mathrm{Tr}\,(\partial_+D_2g^{-1}D_1g+\partial_+D_2gD_1g^{-1})$$

$$\mathrm{Tr}\,(\partial_+g^{-1}\partial_+g)\,\mathrm{Tr}\,(\partial_+D_2g^{-1}D_1g+\partial_+D_2gD_1g^{-1})$$

$$\mathrm{Tr}\,(g^{-1}\partial_+gD_2g^{-1}D_2g)\,\mathrm{Tr}\,(g^{-1}\partial_+gD_1g^{-1}D_2g+g^{-1}\partial_+gD_2g^{-1}D_1g)$$

There are a total of 26 operators which may appear as anomalies. To construct the total divergence operators, we need all dimension 7/2 twist zero or one even operators.

There are 15 dimension 7/2 operators listed in Table 2. The rest are:

$$\mathrm{Tr}(\partial_+ g^{-1}\partial_+ D_2 g + \partial_+ g \partial_+ D_2 g^{-1})\,\mathrm{Tr}(D_1 g^{-1} D_2 g)$$

$$\mathrm{Tr}(\partial_+^2 g^{-1} D_2 g + \partial_+^2 g D_2 g^{-1})\,\mathrm{Tr}(D_1 g^{-1} D_2 g)$$

$$\mathrm{Tr}(\partial_+ D_2 g^{-1} D_2 g + \partial_+ D_2 g D_2 g^{-1})\,\mathrm{Tr}(\partial_+ g^{-1} D_2 g)$$

$$\mathrm{Tr}(\partial_+ D_2 g^{-1} D_2 g + \partial_+ D_2 g^{-1} D_2 g)\,\mathrm{Tr}(\partial_+ g^{-1} D_1 g)$$

$$\mathrm{Tr}(\partial_+ g^{-1}\partial_+ g)\,\mathrm{Tr}(\partial_+ g^{-1} D_2 g)$$

$$\mathrm{Tr}(\partial_+ g^{-1}\partial_+ g)\,\mathrm{Tr}(\partial_+ g^{-1} D_1 g)$$

$$\mathrm{Tr}(g^{-1}\partial_+ g D_2 g^{-1} D_2 g)\,\mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_2 g)$$

$$\mathrm{Tr}(g^{-1}\partial_+ g D_2 g^{-1} D_2 g)\,\mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_1 g)$$

$$\mathrm{Tr}(\partial_+ D_2 g^{-1} D_1 g + \partial_+ D_2 g D_1 g^{-1})\,\mathrm{Tr}(\partial_+ g^{-1} D_2 g)$$

$$\mathrm{Tr}(g^{-1}\partial_+ g D_1 g^{-1} D_2 g + g^{-1}\partial_+ g D_2 g^{-1} D_1 g)\,\mathrm{Tr}(g^{-1} D_2 g D_2 g^{-1} D_2 g)$$

There are a total of 25 divergence operators with the right quantum numbers. But, included in these 25 divergence operators is our original conservation law $D_1(V_{+2}T_{++})$. In addition, classically, $D_1[\mathrm{Tr}(g^{-1}\partial_+ gD_2g^{-1}D_2g)\mathrm{Tr}(g^{-1}D_2gD_2g^{-1}D_2g)]=0$. We have already remarked that the derivative of the second trace vanishes. The first trace is proportional to $D_2\mathrm{Tr}(g^{-1}D_2gD_2g^{-1}D_2g)$ since there is no other operator of dimension two with the right symmetry. So, the derivative of the first trace also vanishes as $\{D_1,D_2\}=0$. Therefore, we have two conservation laws each of which may have 26 operators appearing as anomalies. In general, we may expect to be able to take a linear combination so that only 25 operators appear. The 23 other divergences identified above will not span the space of anomalies. Thus, the simple argument again fails to prove that there exists a quantum conservation law. It is possible, however, that such a law does exist. It would be interesting to carry out a perturbative calculation to see it fewer than 25 of the anomaly operators actually appear in some linear combination of the two classically conserved quantities. One might hope that the supersymmetry would so improve the short distance properties of the theory that fewer than the maximum number of anomalies would actually appear. The quantization of the non-local currents for the supersymmetric chiral models has recently been studied.[40] Zaikov claims there is no anomaly. Local[2] and non-local[38] currents are known to survive quantization in the purely bosonic models. No factorized S-matrix has been constructed. If the local conservation laws of the supersymmetric chiral models suffer from anomalies at the quantum level, it would be an interesting contrast to the CP(N-1) model. There, the bosonic model has an anomaly in the non-local current but the supersymmetric model does not.

We have also examined several other conservation laws to see if they might survive quantization. Classically, there are three dimension 7/2 operators odd under both discrete symmetries whose derivatives vanish.

$$D_1[\mathrm{Tr}(g^{-1}D_2gD_2g^{-1}D_2g)\,\mathrm{Tr}(\partial_+D_2g^{-1}D_2g+D_2g^{-1}\partial_+D_2g-2i\sqrt{2}\partial_+g^{-1}\partial_+g)]=0$$

$$D_1[\mathrm{Tr}(\partial_+g^{-1}D_2g)\,\mathrm{Tr}(g^{-1}\partial_+gD_2g^{-1}D_2g)]=0$$

$$D_1\mathrm{Tr}(g^{-1}D_2gD_2g^{-1}D_2gD_2g^{-1}D_2gD_2g^{-1}D_2g)=0$$

There are 28 possible anomalies, but only 22 divergences, so the simple argument cannot prove that this law survives quantization. The product of the two twist zero dimension 3/2 operators has a vanishing derivative classically; however, at the quantum level, 16 operators may mix and there are only 10 divergences, hence quantum anomalies are possible.

Quite interesting is the quantization of Eq. (19) for $n=5$. The trace is odd under $g\to g^{-1}$, but even under $g\to g^T$ so it vanishes in O(N), but not in SU(N). Here we find that a conservation law does survive quantization. There are four possible anomalies.

$$\mathrm{Tr}\,(g^{-1}D_1gD_2g^{-1}\partial_+^2g-gD_1g^{-1}D_2g\partial_+^2g^{-1})+g\to g^T$$

$$\mathrm{Tr}\,(g^{-1}D_1g\partial_+g^{-1}\partial_+D_2g-gD_1g^{-1}\partial_+g\partial_+D_2g^{-1})+g\to g^T$$

$$\mathrm{Tr}\,(g^{-1}D_1gD_2g^{-1}D_2gD_2g^{-1}\partial_+g)+\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}D_2gD_1g^{-1}\partial_+g)$$

$$\mathrm{Tr}\,(g^{-1}D_2gD_1g^{-1}D_2gD_2g^{-1}\partial_+g)+\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}D_1gD_2g^{-1}\partial_+g)$$

There are four divergence operators.

$$D_1\mathrm{Tr}\,(g^{-1}\partial_+g\partial_+g^{-1}D_2g)$$

$$D_2\mathrm{Tr}\,(g^{-1}\partial_+g\partial_+g^{-1}D_1g)$$

$$D_2\mathrm{Tr}\,(g^{-1}D_2gD_2g^{-1}D_2gD_2g^{-1}D_1g)$$

$$D_2\mathrm{Tr}\,(g^{-1}D_1gD_2g^{-1}\partial_+D_2g-gD_1g^{-1}D_2g\partial_+D_2g^{-1}$$
$$-g^{-1}D_2gD_1g^{-1}\partial_+D_2g+gD_2g^{-1}D_1g\partial_+D_2g^{-1})$$

This conservation law is quite analogous to one considered by Goldschmidt and Witten.[2] They considered the quantization of Eq. (20) for n=3. In that case also, the trace vanishes in O(N) but not in SU(N) since it is odd under $g \to g^{-1}$. In both cases it is the first conservation law in the series which has dimension higher than energy-momentum tensor conservation.

Although the result of Ref. 2, that the classical conservation law is valid quantum mechanically, is correct, the argument is incorrect. The authors state that there are two counterterms and two divergences, however, both are double counted. The two divergences are

$$B_1 = \partial_t \mathrm{Tr}(g_{tt} g_s^{-1} - g_{tt}^{-1} g_s)$$

$$B_2 = \partial_t \mathrm{Tr}(g^{-1} g_t g_t^{-1} g_s) \ .$$

Using $g_s^{-1} = -g^{-1} g_s g^{-1}$ and $g_{tt} g^{-1} = -2 g_t g_t^{-1} - g g_{tt}^{-1}$, we see $\mathrm{Tr}(g_{tt} g_s^{-1} - g_{tt}^{-1} g_s) = 2\mathrm{Tr}(g^{-1} g_t g_t^{-1} g_s)$. Thus, $B_1 = 2\, B_2$. The two counterterms, $A_1 = \mathrm{Tr}(g_{ttt}^{-1} g_s - g_{ttt} g_s^{-1})$ and $A_2 = \mathrm{Tr}(g^{-1} g_s g_t^{-1} g_{tt} - g g_s^{-1} g_t g_{tt}^{-1} + g^{-1} g_t g_s^{-1} g_{tt} - g g_t^{-1} g_s g_{tt}^{-1})$, may similarly be shown to be related by eliminating $g_{ttt}$ from $A_1$ and making manifest the antisymmetry under $g \to g^{-1}$ and the symmetry under $g \to g^T$.

## IV. Summary and Conclusions

Two dimensional integrable field theories are of interest to high energy physicists not only because they provide interesting examples of field theories, but because it is believed that quantum chromodynamics may in some sense be an integrable system. Certainly, it is not yet known whether this is the case. However, if the classical theory is shown to be integrable, precisely the type of question addressed here will be of utmost relevance. That is, given that a system is classically integrable is it still integrable at the quantum level? So far we only have an understanding of the answer to this question in specific examples, rather than a general approach. Let's review all that is known for the three types of models discussed here: non-linear $\sigma$-model, CP(N-1) model, principal chiral model.

For the non-linear $\sigma$-model, the purely bosonic model has a factorized S-matrix,[14] conserved non-local currents[5] and conserved local currents.[2] In the supersymmetric case, there is a factorized S-matrix,[17] and the non-local[38] and local currents[1,39] are known to be conserved.

For the CP(N-1) model, the purely bosonic model has no factorized S-matrix,[2] an anomaly in the non-local current,[32] and possibly an anomaly in the local current.[2] For the supersymmetric case, there is a factorized S-matrix[35] and the anomaly disappears from the non-local conservation law.[36] However, for the local conservation law, we have not been able to prove that there is no anomaly.

For the purely bosonic principal chiral models, O(N)×O(N) and SU(N)×SU(N), it is not known whether or not there is a factorized S-matrix. Goldschmidt and Witten have shown that the local conservation law survives quantization. Zaikov[38] purports to have shown that the non-local conservation laws survive quantization. The supersymmetric version of these models is in much the same state[40] except that our argument does not indicate whether or not the local conservation laws are still conserved when the theory is quantized. There are some interesting open questions which remain to be resolved. Is it possible that the CP(N-1) model is an example of a model where supersymmetry removes the anomaly, but in the O(N)xO(N) and SU(N)xSU(N) models the bosonic models have no anomaly whereas the supersymmetric models have one? We have certainly seen that the simple arguments for quantized local conservation laws in the bosonic models are not as useful for the supersymmetric models. Further study of the CP(N-1) model taking into account the extended supersymmetry, and of the supersymmetric principal chiral models should indicate whether or not this type of analysis is misleading.

TABLE 1.

| Dimension | Twist | | |
|---|---|---|---|
| 1 | 0 | $(\mathcal{D}_2\phi^*\cdot\mathcal{D}_2\phi)$ | $A_{2,2}$ |
| | 1 | $(\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\mathcal{D}_2\phi)$ | $S_{2,1}$ |
| | 1 | $(\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\mathcal{D}_2\phi)$ | $A_{2,1}$ |
| 3/2 | 0 | $(\nabla_+\phi^*\cdot\mathcal{D}_2\phi)+(\mathcal{D}_2\phi^*\cdot\nabla_+\phi)$ | $S_{+,2}$ |
| | 0 | $(\nabla_+\phi^*\cdot\mathcal{D}_2\phi)-(\mathcal{D}_2\phi^*\cdot\nabla_+\phi)$ | $A_{+,2}$ |
| | 1 | $(\nabla_+\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\phi)$ | $S_{+,1}$ |
| | 1 | $(\nabla_+\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\phi)$ | $A_{+,1}$ |
| 2 | 0 | $(\nabla_+\phi^*\cdot\nabla_+\phi)$ | $S_{+,+}$ |
| | 0 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_2\phi)-(\mathcal{D}_2\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $S_{+2,2}$ |
| | 0 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_2\phi)+(\mathcal{D}_2\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $A_{+2,2}$ |
| | 1 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $S_{+2,1}$ |
| | 1 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $A_{+2,1}$ |
| 5/2 | 0 | $(\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_2\phi)+(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\phi)$ | $S_{++,2}$ |
| | 0 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\phi)+(\nabla_+\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $S_{+2,+}$ |
| | 0 | $(\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_2\phi)-(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\phi)$ | $A_{++,2}$ |
| | 0 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\phi)-(\nabla_+\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $A_{+2,+}$ |
| | 1 | $(\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\phi)$ | $S_{++,1}$ |
| | 1 | $(\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\phi)$ | $A_{++,1}$ |
| 3 | 0 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_2\phi)-(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $S_{++2,2}$ |
| | 0 | $(\nabla_+\nabla_+\phi^*\cdot\nabla_+\phi)+(\nabla_+\phi^*\cdot\nabla_+\nabla_+\phi)$ | $S_{++,+}$ |
| | 0 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_2\phi)+(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $A_{++2,2}$ |

TABLE 1 (cont.)

| | | | |
|---|---|---|---|
| | 0 | $(\nabla_+\nabla_+\phi^*\cdot\nabla_+\phi)-(\nabla_+\phi\cdot\nabla_+\nabla_+\phi)$ | $A_{++,+}$ |
| | 0 | $(\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\mathcal{D}_2\phi)$ | $A_{+2,+2}$ |
| | 1 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $S_{++2,1}$ |
| | 1 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $A_{++2,1}$ |
| 7/2 | 0 | $(\nabla_+\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_2\phi)+(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\nabla_+\phi)$ | $S_{+++,2}$ |
| | 0 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\phi)+(\nabla_+\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $S_{++2,+}$ |
| | 0 | $(\nabla_+\nabla_+\phi^*\cdot\nabla_+\mathcal{D}_2\phi)+(\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\phi)$ | $S_{++,+2}$ |
| | 0 | $(\nabla_+\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_2\phi)-(\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\nabla_+\phi)$ | $A_{+++,2}$ |
| | 0 | $(\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\phi)-(\nabla_+\phi^*\cdot\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $A_{++2,+}$ |
| | 0 | $(\nabla_+\nabla_+\phi^*\cdot\nabla_+\mathcal{D}_2\phi)-(\nabla_+\mathcal{D}_2\phi^*\cdot\nabla_+\nabla_+\phi)$ | $A_{++,+2}$ |
| | 1 | $(\nabla_+\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\nabla_+\phi)$ | $S_{+++,1}$ |
| | 1 | $(\nabla_+\nabla_+\nabla_+\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\nabla_+\phi)$ | $A_{+++,1}$ |
| 4 | 1 | $(\nabla_+\nabla_+\nabla_+\mathcal{D}_2\phi^*\cdot\mathcal{D}_1\phi)+(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $S_{+++2,1}$ |
| | 1 | $(\nabla_+\nabla_+\nabla_+\ {}_2\phi^*\cdot\mathcal{D}_1\phi)-(\mathcal{D}_1\phi^*\cdot\nabla_+\nabla_+\nabla_+\mathcal{D}_2\phi)$ | $A_{+++2,1}$ |

TABLE 2

| Class | Operators |
| --- | --- |
| (4) | $S_{+++2,1}$ |
| (3)(1) | $S_{++2,2}\ S_{2,1}$; $S_{++,+}\ S_{2,1}$; $A_{++2,2}\ A_{2,1}$; |
| | $A_{++,+}\ A_{2,1}$; $A_{+2,+2}\ A_{2,1}$; $A_{++2,1}\ A_{2,2}$ |
| (5/2)(3/2) | $S_{++,2}\ S_{+,1}$; $S_{+2,+}\ S_{+,1}$; $A_{++,2}\ A_{+,1}$; |
| | $A_{+2,+}\ A_{+,1}$; $S_{++,1}\ S_{+,2}$; $A_{++,1}\ A_{+,2}$ |
| $(2)^2$ | $S_{+,+}\ S_{+2,1}$; $S_{+2,2}\ S_{+2,1}$; $A_{+2,2}\ A_{+2,1}$ |
| $(2)(1)^2$ | $S_{+,+}\ A_{2,2}\ A_{2,1}$; $S_{+2,2}\ A_{2,2}\ A_{2,1}$; |
| | $A_{+2,2}\ A_{2,2}\ S_{2,1}$; $S_{+2,1}(A_{2,2})^2$; |
| $(3/2)^2(1)$ | $S_{+,2}\ A_{+,2}\ A_{2,1}$; $S_{+,2}\ A_{+,1}\ A_{2,2}$; $A_{+,2}\ S_{+,1}\ A_{2,2}$ |
| $(1)^4$ | $(A_{2,2})^3\ A_{2,1}$ |

TABLE 3

| Class | $\tau=0$ | $\tau=1$ |
|---|---|---|
| (7/2) | $S_{+++,2}$; $S_{++2,+}$; $S_{++,+2}$ | $S_{+++,1}$ |
| (5/2)(1) | $A_{++,2}\ A_{2,2}$; $A_{+2,+}A_{2,2}$ | $S_{++,2}\ S_{2,1}$; $S_{+2}{}_{,;}\ S_{2,1}$;<br>$A_{++,2}\ A_{2,1}$; $A_{+2,+}\ A_{2,1}$;<br>$A_{++,1}\ A_{2,2}$ |
| (2)(3/2) | $S_{+,+}\ S_{+,2}$; $S_{+2,2}\ S_{+,2}$;<br>$A_{+2,2}\ A_{+,2}$ | $S_{+,+}\ S_{+,1}$; $S_{+2,2}\ S_{+,1}$;<br>$A_{+2,2}A_{+,1}$; $S_{+2,1}\ S_{+,2}$;<br>$A_{+2,1}\ A_{+,2}$ |
| $(3/2)(1)^2$ | $S_{+,2}\ (A_{2,2})^2$ | $S_{+,2}\ A_{2,2}\ A_{2,1}$; $A_{+,2}A_{2,2}S_{2,1}$;<br>$S_{+,1}(A_{2,2})^2$ |

TABLE 4

| Dimension | Twist | | $Pg^{-1}$ | $Pg^{T}$ | O(N) |
|---|---|---|---|---|---|
| 1 | 1 | $Tr(D_1g^{-1}D_2g)$ | + | + | |
| 3/2 | 0,1 | $Tr(\partial_+g^{-1}D_ag)$ | + | + | |
| | 0,1 | $Tr(g^{-1}D_2gD_2g^{-1}D_ag)$ | - | - | |
| 2 | 0,1 | $Tr(\partial_+D_2g^{-1}D_ag+\partial_+D_2gD_ag^{-1})$ | + | + | |
| | 0 | $Tr(\partial_+g^{-1}\partial_+g)$ | + | + | |
| | 0 | $Tr(g^{-1}\partial_+gD_2g^{-1}D_2g)$ | - | - | |
| | 1 | $Tr(g^{-1}\partial_+gD_1g^{-1}D_2g-g^{-1}\partial_+gD_2g^{-1}D_1g)$ | - | + | x |
| | 1 | $Tr(g^{-1}\partial_+gD_1g^{-1}D_2g+g^{-1}\partial_+gD_2g^{-1}D_1g)$ | - | - | |
| | 1 | $Tr(D_2g^{-1}D_2gD_2g^{-1}D_1g)$ | + | - | x |
| 5/2 | 0 | $Tr(\partial_+g^{-1}\partial_+D_2g+\partial_+g\partial_+D_2g^{-1})$ | + | + | |
| | 0,1 | $Tr(\partial_+^2g^{-1}D_ag+\partial_+^2gD_ag^{-1})$ | + | + | |
| | 0,1 | $Tr(g^{-1}\partial_+g\partial_+g^{-1}D_ag)$ | - | + | x |
| | 0 | $Tr(g^{-1}D_2gD_2g^{-1}\partial_+D_2g-gD_2g^{-1}D_2g\partial_+D_2g^{-1})$ | - | - | |
| | 0 | $Tr(D_2g^{-1}D_2gD_2g^{-1}\partial_+g)$ | + | - | x |
| | 0,1 | $Tr(g^{-1}D_2gD_2g^{-1}D_2gD_2g^{-1}D_ag)$ | - | + | x |

TABLE 4 (cont.)

| | | | | | |
|---|---|---|---|---|---|
| | 1 | $\mathrm{Tr}(g^{-1}D_1gD_2g^{-1}\partial_+D_2g-gD_1g^{-1}D_2g\partial_+D_2g^{-1})$ $+g^{-1}D_2gD_1g^{-1}\partial_+D_2g-gD_2g^{-1}D_1g\partial_+D_2g^{-1})$ | - | - | |
| | 1 | $\mathrm{Tr}(g^{-1}D_1gD_2g^{-1}\partial_+D_2g-gD_1g^{-1}D_2g\partial_+D_2g^{-1}$ $-g^{-1}D_2gD_1g^{-1}\partial_+D_2g+gD_2g^{-1}D_1g\partial_+D_2g^{-1})$ | - | + | x |
| | 1 | $\mathrm{Tr}(D_2g^{-1}D_1gD_2g^{-1}\partial_+g)$ | + | - | x |
| | 1 | $\mathrm{Tr}(D_1g^{-1}D_2gD_2g^{-1}\partial_+g-D_2g^{-1}D_2gD_1g^{-1}\partial_+g)$ | + | + | |
| | 1 | $\mathrm{Tr}(D_1g^{-1}D_2gD_2g^{-1}\partial_+g+D_2g^{-1}D_2gD_1g^{-1}\partial_+g)$ | + | - | x |
| 3 | 0 | $\mathrm{Tr}(D_2g^{-1}\partial_+^2D_2g+D_2g\partial_+^2D_2g^{-1})$ | + | + | |
| | 0 | $\mathrm{Tr}(\partial_+g^{-1}\partial_+^2g+\partial_+g\partial_+^2g^{-1})$ | + | + | |
| 7/2 | 0 | $\mathrm{Tr}(\partial_+^3g^{-1}D_2g+\partial_+^3gD_2g^{-1})$ | + | + | |
| | 0 | $\mathrm{Tr}(\partial_+g^{-1}\partial_+^2D_2g+\partial_+g\partial_+^2D_2g^{-1})$ | + | + | |
| | 0 | $\mathrm{Tr}(\partial_+D_2g^{-1}\partial_+^2g+\partial_+D_2g\partial_+^2g^{-1})$ | + | + | |
| | 0 | $\mathrm{Tr}(\partial_+D_2g^{-1}\partial_+gD_2g^{-1}D_2g-\partial_+g^{-1}\partial_+D_2gD_2g^{-1}D_2g)+(g\to g^{-1})$ | + | + | |
| | 0 | $\mathrm{Tr}(\partial_+g^{-1}\partial_+g\partial_+g^{-1}D_2g)$ | + | + | |
| | 0 | $\mathrm{Tr}(D_2g^{-1}D_2gD_2g^{-1}D_2gD_2g^{-1}\partial_+g)$ | + | + | |

TABLE 4 (cont.)

| | | | | |
|---|---|---|---|---|
| | 1 | $\mathrm{Tr}(\partial_+^3 g^{-1} D_1 g + \partial_+^3 g D_1 g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+^2 g^{-1} D_1 g D_2 g^{-1} D_2 g - \partial_+^2 g^{-1} D_2 g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} \partial_+ g \partial_+ D_2 g^{-1} D_1 g - \partial_+ D_2 g^{-1} \partial_+ g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+ g^{-1} \partial_+ D_2 g D_2 g^{-1} D_1 g - D_2 g^{-1} \partial_+ D_2 g \partial_+ g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+ D_2 g^{-1} D_2 g \partial_+ g^{-1} D_1 g - \partial_+ g^{-1} D_2 g \partial_+ D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+ g^{-1} \partial_+ g \partial_+ g^{-1} D_1 g)$ | + | + |
| | 1 | $\mathrm{Tr}(D_1 g^{-1} D_2 g D_2 g^{-1} D_2 g D_2 g^{-1} \partial_+ g + D_2 g^{-1} D_2 g D_2 g^{-1} D_2 g D_1 g^{-1} \partial_+ g)$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} D_1 g D_2 g^{-1} D_2 g D_2 g^{-1} \partial_+ g + D_2 g^{-1} D_2 g D_2 g^{-1} D_1 g D_2 g^{-1} \partial_+ g)$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} D_2 g D_1 g^{-1} D_2 g D_2 g^{-1} \partial_+ g)$ | + | + |
| 4 | 1 | $\mathrm{Tr}(D_1 g^{-1} \partial_+^3 D_2 g + D_1 g \partial_+^3 D_2 g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} D_2 g \partial_+^2 D_2 g^{-1} D_1 g - \partial_+^2 D_2 g^{-1} D_2 g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} \partial_+ g \partial_+^2 g^{-1} D_1 g + \partial_+^2 g^{-1} \partial_+ g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} \partial_+^2 g \partial_+ g^{-1} D_1 g + \partial_+ g^{-1} \partial_+^2 g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+ g^{-1} D_2 g \partial_+^2 g^{-1} D_1 g + \partial_+^2 g^{-1} D_2 g \partial_+ g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(D_2 g^{-1} \partial_+ D_2 g \partial_+ D_2 g^{-1} D_1 g - \partial_+ D_2 g^{-1} \partial_+ D_2 g D_2 g^{-1} D_1 g) + (g \to g^{-1})$ | + | + |
| | 1 | $\mathrm{Tr}(\partial_+ g^{-1} \partial_+ D_2 g \partial_+ g^{-1} D_1 g + \partial_+ g \partial_+ D_2 g^{-1} \partial_+ g D_1 g^{-1})$ | + | + |

TABLE 4 (cont.)

| | | | |
|---|---|---|---|
| 1 | $\mathrm{Tr}(\partial_+ D_2 g^{-1}\partial_+ g\partial_+ g^{-1}D_1 g+\partial_+ g^{-1}\partial_+ g\partial_+ D_2 g^{-1}D_1 g)+(g\to g^{-1})$ | + | + |
| 1 | $\mathrm{Tr}(D_2 g^{-1}D_2 g\partial_+ D_2 g^{-1}D_2 g D_2 g^{-1}D_1 g)+(g\to g^{-1})$ | + | + |
| 1 | $\mathrm{Tr}(D_2 g^{-1}\partial_+ D_2 g D_2 g^{-1}D_2 g D_2 g^{-1}D_1 g+D_2 g^{-1}D_2 g D_2 g^{-1}\partial_+ D_2 g D_2 g^{-1}D_1 g)$ $+(g\to g^{-1})$ | + | + |
| 1 | $\mathrm{Tr}(\partial_+ D_2 g^{-1}D_2 g D_2 g^{-1}D_2 g D_2 g^{-1}D_1 g+D_2 g^{-1}D_2 g D_2 g^{-1}D_2 g\partial_+ D_2 g^{-1}D_1 g)$ $+(g\to g^{-1})$ | + | + |
| 1 | $\mathrm{Tr}(\partial_+ g^{-1}\partial_+ g D_2 g^{-1}D_2 g D_2 g^{-1}D_1 g-D_2 g^{-1}D_2 g D_2 g^{-1}\partial_+ g\partial_+ g^{-1}D_1 g)$ | + | + |
| 1 | $\mathrm{Tr}(\partial_+ g^{-1}D_2 g\partial_+ g^{-1}D_2 g D_2 g^{-1}D_1 g-D_2 g^{-1}D_2 g\partial_+ g^{-1}D_2 g\partial_+ g^{-1}D_1 g)$ | + | + |
| 1 | $\mathrm{Tr}(\partial_+ g^{-1}D_2 g D_2 g^{-1}\partial_+ g D_2 g^{-1}D_1 g-D_2 g^{-1}\partial_+ g D_2 g^{-1}D_2 g\partial_+ g^{-1}D_1 g)$ | + | + |
| 1 | $\mathrm{Tr}(D_2 g^{-1}\partial_+ g\partial_+ g^{-1}D_2 g D_2 g^{-1}D_1 g-D_2 g^{-1}D_2 g\partial_+ g^{-1}\partial_+ g D_2 g^{-1}D_1 g)$ | + | + |

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