Fermi National Accelerator Laboratory

FERMILAB-PUB-81/71-THY

October, 1981

Complete Set of SU(5) Monopole Solutions

I. G. Koh[†]

Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, Illinois 60510

and

Physics Department, Sogang University,
C. P. O. Box 1142, Seoul, Korea

Y. D. Kim, Y. J. Park, C. H. Kim, and Y. S. Kim

Physics Department, Sogang University,
C. P. O. Box 1142, Seoul, Korea

Complete set of SU(5) pointlike monopole solutions are found by using the magnetic symmetry introduced by Cho. New magnetic tensor representations greatly expedite the computations. These magnetic tensors explicitly exhibit the full homotopy class of the mappings classifying the arbitrary allowed magnetic charges.

[†]Senior Fulbright Fellow

PACS Category Numbers: 11.30.Ly, 02.40.Tm, 14.80.Hv

The nontrivial topological structures[1] of the monopoles have been much studied in the gauge field theory. No regular monopole solutions exist for the Abelian gauge group. However for non-Abelian SU(2) gauge group, Wu and Yang have explicitly found regular monopole solutions except the singularity at the origin.[2] The remaining singularity at the origin was removed by 't Hooft to give regular solutions everywhere with finite mass for the monopole.[3] Monopoles in the minimal grand unified gauge group SU(5) get wide attention due to the cosmological implications.[4] To suppress the production rate of monopoles, Guth and Tye,[5] Langacker and Pi[6] have introduced several different scenarios. In these circumstances, it would be very interesting to find a complete set of monopole solution for the grand unified group SU(5). Wilkinson,[7] Bais and Weldon,[8] Wilkinson and Bais,[9] Brandt and Neri,[10] Weinberg,[11] Daniel, Lazarides and Shafi,[12] Dokos and Tomaras[13] have studied monopoles in the non-Abelian gauge group.

Cho[14] has obtained all the monopole solutions for the SU(3) gauge group by examining the topological structure, instead of using the usual methods of solving the equations of motion. The magnetic symmetry, as a set of self-consistent Killing vector fields, plays a crucial role in finding a complete set of monopole solutions. Recently, we have obtained a complete set of monopole solutions in SU(4) gauge group.[15] Cho and we have used the generators in adjoint representations, and the calculations become quite complicated for large SU(N) group. In this paper, we have used 5x5 tensor representation of SU(5) instead of the 24-dimensional representation for the Killing vectors. These tensor representations considerably expedite the computations.

The monopole solutions are classified by the homotopy group[16] and the non-trivial homotopy groups for SU(5) are

$$\pi_2[SU(5) \,/\, U(1) \times U(1) \times U(1) \times U(1)]$$
$$= \pi_1[U(1) \times U(1) \times U(1) \times U(1)] = Z \times Z \times Z \times Z \,, \qquad (1)$$

$$\pi_2[SU(5) \,/\, U(2) \times U(1) \times U(1)]$$
$$= \pi_1[U(2) \times U(1) \times U(1)] = Z \times Z \times Z \,, \qquad (2)$$

$$\pi_2 = [SU(5) \,/\, U(3) \times U(1)]$$
$$= \pi_1[U(3) \times U(1)] = Z \times Z \,, \qquad (3)$$

$$\pi_2 = [SU(5) \,/\, U(4)] = Z \,, \qquad (4)$$

where $U(1) \times U(1) \times U(1) \times U(1)$, $U(2) \times U(1) \times U(1)$, $U(3) \times U(1)$ and $U(4)$ are subgroups associated with λ_3, λ_8, λ_{15} and λ_{24} - like symmetry respectively. Therefore the monopoles must now be classified by four, three, two, and one integers respectively.

The magnetic charges must satisfy the generalized quantization condition[17] given by

$$\exp[4\pi i g\, (\tfrac{1}{2}\lambda_3\, g_1 + \tfrac{1}{2}\lambda_8\, g_2 + \tfrac{1}{2}\lambda_{15}\, g_3 + \tfrac{1}{2}\lambda_{24}\, g_4] = 1 \,, \qquad (5)$$

where λ_3, λ_8, λ_{15}, λ_{24} are the self-commuting generators of the SU(5) group.

To satisfy the above condition, one can take

$$g_1 = g^{-1}[n_1 - \tfrac{1}{2}n_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}n_4] \, , \tag{6}$$

$$g_2 = g^{-1}\, \tfrac{1}{2}\sqrt{3}[n_2 + \tfrac{1}{3}(n_3 + n_4)] \, , \tag{7}$$

$$g_3 = g^{-1}(2/3)^{\frac{1}{2}}[n_3 + \tfrac{1}{4}n_4] \, , \tag{8}$$

$$g_4 = g^{-1}(\sqrt{10}/4)n_4 \, , \tag{9}$$

where n_1, n_2, n_3 and n_4 are arbitrary integers.

The value of the magnetic charges depends on the topological configurations which are expressed in Eqs. (1)-(4).

When n_4 is set to zero, the monopole changes are reduced to the SU(4) case.[15] Here additional zeroes for n_3 would give SU(3) results.[14] Subsequent zeroes for n_2 will finally give SU(2) results.

To find a complete set of the monopole solutions for every set of integers n_1, n_2, n_3, n_4, the Killing symmetry assumptions[14] introduced by Cho are required as

$$D_\mu\, m_1 = 0, \quad D_\mu\, m_2 = 0, \quad D_\mu\, m_3 = 0 \quad \text{and} \quad D_\mu\, m_4 = 0 \, , \tag{10}$$

where $$D_\mu = \partial_\mu - ig\bar{B}_\mu, \quad \text{and} \quad B_\mu = B_\mu^i \frac{\lambda^i}{2} \, . \tag{11}$$

The gauge potential that satisfies the above Killing symmetry assumption should be given by the following form

$$B_\mu = A_\mu^1 m_1 + A_\mu^2 m_2 + A_\mu^3 m_3 + A_\mu^4 m_4$$
$$- ig[m_1, \partial_\mu m_1] - ig[m_2, \partial_\mu m_2] - ig[m_3, \partial_\mu m_3] - ig[m_4, \partial_\mu m_4], \tag{12}$$

where A_μ^1, A_μ^2, A_μ^3 and A_μ^4 are the components not fixed by the condition (10).

The magnetic tensors m_1, m_2, m_3, and m_4 are chosen to exhibit the full homotopy class of the mappings (1)-(3).
These are found by the gauge transformation such that

$$m_1 = U\,\tfrac{1}{2}\lambda_3\,U^\dagger,\quad m_2 = U\,\tfrac{1}{2}\lambda_8\,U^\dagger,\quad m_3 = U\,\tfrac{1}{2}\lambda_{15}\,U^\dagger, \tag{13}$$
$$\text{and}\quad m_4 = U\,\tfrac{1}{2}\lambda_{24}\,U^\dagger\ ,$$

where

$$U = \exp[in_4\,\tfrac{1}{2}\phi(\tfrac{1}{2}\lambda_3 - \sqrt{3}/2\lambda_8 + (3/2)^{\frac{1}{2}}\lambda_{15} - 3/\sqrt{10}\lambda_{24})]\ \ \exp[i\,\tfrac{1}{2}\theta\lambda_{19}]$$
$$\times \exp[in_3\,\tfrac{1}{2}\phi(\tfrac{1}{2}\lambda_3 - \sqrt{3}/2\lambda_8 + (3/2)^{\frac{1}{2}}\lambda_{15})]\ \exp[-i\,\tfrac{1}{2}\theta\lambda_{10}]$$
$$\times \exp[in_2\,\tfrac{1}{2}\phi(\tfrac{1}{2}\lambda_3 - \sqrt{3}/2\lambda_8)]\ \exp[-i\,\tfrac{\theta}{2}\lambda_7]$$
$$\times \exp[-i\,\tfrac{1}{2}\phi\lambda_3(n_1 - \tfrac{1}{2}n_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}n_4)]\ \exp[-i\,\tfrac{1}{2}\theta\lambda_2]\ . \tag{14}$$

Here $\lambda_i(i=1,\ldots, 24)$ are the fundamental representations of the SU(5) generators. Explicit forms of magnetic tensors are shown in Table I.

These m_1, m_2, m_3 and m_4 represent the homotopic mappings (1), (2), (3) and (4) respectively. These magnetic tensors in Table I are regular except at the origin. Since ϕ-part in the magnetic tensors accompanies $\sin\theta$, their derivatives are also regular except at the origin.

The magnetic tensors are invariant under the additional gauge transformation by the diagonal generators,

,i.e.,

$$m_1 = U\frac{1}{2}\lambda_3 U^\dagger = U'\frac{1}{2}\lambda_3 U'^\dagger , \tag{15}$$

$$m_2 = U\frac{1}{2}\lambda_8 U^\dagger = U'\frac{1}{2}\lambda_8 U'^\dagger , \tag{16}$$

$$m_3 = U\frac{1}{2}\lambda_{15} U^\dagger = U'\frac{1}{2}\lambda_{15} U'^\dagger , \tag{17}$$

and

$$m_4 = U\frac{1}{2}\lambda_{24} U^\dagger = U'\frac{1}{2}\lambda_{24} U'^\dagger \tag{18}$$

with
$$U' = U \exp[-i\frac{1}{2}(\alpha\lambda_3 + \beta\lambda_8 + \gamma\lambda_{15} + \delta\lambda_{24})] \tag{19}$$

for arbitrary α, β, γ and δ.

Finally the unrestricted A_μ^1, A_μ^2, A_μ^3, A_μ^4 in Eq.(12) are chosen to obtain the desired monopole solutions as

$$A_\mu^1 = -\frac{1}{g}[[(\frac{1}{2}n_2+n_3+n_4) + \frac{1}{4}n_4\cos\theta]\sin^2\theta\partial_\mu\phi + \frac{1}{2}(n_3+n_4)\phi\sin\theta\partial_\mu\theta] ,$$

$$A_\mu^2 = \frac{1}{g}\sqrt{3}/4\ \sin\theta[n_4 \sin\theta\partial_\mu\phi + (2n_3/3)\phi\partial_\mu\theta],$$

$$A_\mu^3 = \frac{1}{g}(2/3)^{\frac{1}{2}}(2n_3 + n_4)\phi\sin\theta\partial_\mu\theta\ , \tag{20}$$

$$A_\mu^4 = 0$$

with

$$\alpha = [-\tfrac{1}{2}(n_3+n_4)\cos\theta + (\tfrac{1}{2}n_2+n_3+n_4)]\phi\ ,$$

$$\beta = -\sqrt{3}[\tfrac{1}{6}n_3\cos\theta + (\tfrac{1}{6}n_3+\tfrac{1}{2}n_4)]\phi\ , \tag{21}$$

$$\gamma = [(2/3)^{\frac{1}{2}}(2n_3+n_4)\cos\theta + \sqrt{6}(\tfrac{1}{6}n_3+\tfrac{1}{4}n_4)]\phi\ ,$$

$$\delta = (-\sqrt{10}/20)n_4\ \phi\ .$$

Earlier results for Eqs.(20) and (21) in the SU(4), SU(3) and SU(2) can be easily recovered by setting n_4, n_3, n_2 to zero subsequently.

Since the magnetic Killing tensors in Table I, their derivatives and the A^1,A_μ^2,A_μ^3 A_μ^4 in Eq. (20) are smooth, the potential (12) is regular everywhere except at the origin. If the gauge potential(12) is expressed in terms of m_1, m_2, m_3 and m_4 and of the pure gauge terms as

$$B_\mu = (g_1 m_1 + g_2 m_2 + g_3 m_3 + g_4 m_4)\cos\theta\partial_\mu\phi - i\ g^{-1}U'\partial_\mu U'^{\dagger}\ , \tag{22}$$

the string singularity in the first term of Eq.(22) is cancelled by the second term. In the λ gauge , the potential (22) reduces to the standard

Dirac potential form with the string singularity as

$$(g_1\frac{\lambda_3}{2} + g_2\frac{\lambda_8}{2} + g_3\frac{\lambda_{15}}{2} + g_4\frac{\lambda_{24}}{2}) \cos\theta\partial_\mu \phi . \tag{23}$$

But it is to be emphasized that the potential (12) with the regular magnetic tensors in Table I and with Eq.(20) is regular everywhere except at the origin.

This potential B_μ describes the desired solutions

$$\begin{aligned} G_{\mu\nu} &= -[g_1 m_1 + g_2 m_2 + g_3 m_3 + g_4 m_4] \\ &\quad \times \sin\theta(\partial_\mu\theta\partial_\nu\phi - \partial_\nu\theta\partial_\mu\phi) , \end{aligned} \tag{24}$$

using $\quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - ig[B_\mu, B_\nu]$.

The magnetic charges g_m^1, g_m^2, g_m^3, g_m^4 of the solutions can then be defined as

$$\begin{aligned} g_m^1 &= 2\int dS^{\mu\nu}\mathrm{Tr}[m_1 G_{\mu\nu}] = 4\pi g^{-1}[n_1 - \tfrac{1}{2}n_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}n_4] , \\ g_m^2 &= 2\int dS^{\mu\nu}\mathrm{Tr}[m_2 G_{\mu\nu}] = 4\pi g^{-1}\tfrac{\sqrt{3}}{2}[n_2 + \tfrac{1}{3}n_3 + \tfrac{1}{3}n_4] , \\ g_m^3 &= 2\int dS^{\mu\nu}\mathrm{Tr}[m_3 G_{\mu\nu}] = 4\pi g^{-1}(2/3)^{\frac{1}{2}}[n_3 + \tfrac{1}{4}n_4] , \\ g_m^4 &= 2\int dS^{\mu\nu}\mathrm{Tr}[m_4 G_{\mu\nu}] = 4\pi g^{-1}\tfrac{\sqrt{10}}{4}n_4 . \end{aligned} \tag{25}$$

The potential (12) with magnetic tensors in Table I and with Eq.(20) indeed describes all the homotopically inequivalent point-like SU(5) monopoles.

At this point, it is appropriate to comment why we take such a long compututation to find the gauge potential (22). To obtain the field tensor (24) for monopole, one would simply write down Eq. (23) with help of the generalized quantization condition (5). But there is no easy practical way to find the regular gauge potential (12) or (22) from the singular one (23). But the new method of magnetic tnesor described in this paper is quite straightforward to obtain the regular gauge potential for the monopoles.

ACKNOWLEDGEMENT

We are grateful for Prof. Y. M. Cho's critical comments and encouragements. One of us(I.G.K.) would like to thank members of the Fermilab Theory Group, in particular, Prof. W. Bardeen, Dr. C. K. Zachos, Dr. J. F. Schonfeld for the helpful discussion and also for the warm hospitality. We appreciate the help of W. S. l'Yi, H. S. Lee in computation. These calculations are also checked independently by symbolic computation program REDUCE[18], for this program we thank Prof. A. C. Hearn. This research was supported in part by Korea Science and Engineering Foundation and Korea Traders Foundation.

[1] J. Arafune, P. G. O. Freund, and C. G. Goebel, J. Math. Phys. 16, 433 (1975); S. Coleman, in Proceedings of the International School of Subnuclear Physics, "Ettore Majorana," Erice, Italy, 1975 (Academic, New York, to be published).

[2] T.T. Wu and C.N. Yang, Phys. Rev. D 12, 3845 (1975).

[3] G. 't Hooft, Nucl. Phys. B 79, 276 (1974).

[4] J.P. Preskill, Phys. Rev. Lett. 43, 1365 (1979).

[5] A. Guth and S.-H. Tye, Phys. Rev. Lett. 44, 631 (1980); M.B. Einhorn, D.L. Stein, and D. Toussaint, Univ. of Michigan preprint, UM HE 80-1.

[6] P. Langacker, and S. Pi, Phys. Rev. Lett. 45, 1(1980).

[7] D. Wilkinson, Nucl. Phys. B 125, 423 (1977).

[8] F.A. Bais and H.A. Weldon, Phys. Rev. Lett. 41, 601 (1978).

[9] D. Wilkinson and F.A. Bais, Phys. Rev. D 19, 2410 (1979).

[10] R.A. Brandt and F. Neri, New York University Report No. NYU/TRI/80(1980).

[11] E.J. Weinberg, Nucl. Phys. B 167, 500 (1980).

[12] M. Daniel; G. Lazarides, and Q. Shafi, Nucl. Phys. B 170, 156 (1980).

[13] C.P. Dokos and T.N. Tomaras, Phys. Rev. D 21, 2940 (1980).

[14] Y.M. Cho, Phys. Rev. Lett. 44, 1115 (1980); Y.M. Cho, Phys. Rev. D 21, 1080 (1980).

[15] Y.D. Kim, I.G. Koh, Y.J. Park, to be published in Phys. Rev. D.

[16] A. Sinha, Phys. Rev. D 14, 2016 (1976); A. Actor, Rev. Mod. Phys. 51, 461 (1979); E. Corrigan, D. Olive, D.B. Fairlie, and J. Nuyts, Nucl. Phys. B 106, 475 (1976).

[17] Y.S. Tyupkin, V.A. Fateev, and A.S. Shvarts, Pis'ma Zh. Eksp. Teor. Fiz. 21, 91 (1975) [JETP Lett. 21, 42 (1975)]; M. I. Mastyrsky and A.M. Perelomov, Pis'ma Zh. Eksp. Teor. Fiz. 21, 94 (1975) JETP Lett. 21, 43 (1975) ; F. Englert and P. Windey, Phys. Rev. D 14, 2728 (1976); P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. B 125, 1 (1977).

[18] A.C. Hearn, REDUCE 2, User's Manual, University of Utah (1973).

TABLE 1. Explicit forms of the magnetic tensors with $a = \sin\theta$, $b = \cos\theta$, $c = \sin\frac{\theta}{2}$ and $d = \cos\frac{\theta}{2}$.

$$
m_1 = \frac{1}{2}\begin{pmatrix}
\frac{1}{2}b(1+b) & \frac{1}{2}(1+b)ad\exp[-i(n_1-n_2-n_3-n_4)\phi] & \frac{1}{2}a^2\exp[-in_1\phi] & \frac{1}{2}abc\exp[i(n_3+n_4)\phi] & \frac{1}{2}a^2d\exp[-i(n_1-n_2-n_3)\phi] \\
\frac{1}{2}(1+b)ad\exp[i(n_1-n_2-n_3-n_4)\phi] & -\frac{1}{4}b(1+b)^2 & -\frac{1}{2}abd\exp[-i(n_2+n_3+n_4)\phi] & \frac{1}{2}a^2d\exp[i(n_1-n_2)\phi] & -\frac{1}{4}ab(1+b)\exp[-in_4\phi] \\
\frac{1}{2}a^2\exp[in_1\phi] & -\frac{1}{2}abd\exp[i(n_2+n_3+n_4)\phi] & -\frac{1}{2}b(1-b) & -\frac{1}{2}(1-b)ac\exp[i(n_1+n_3+n_4)\phi] & -\frac{1}{2}abc\exp[i(n_2+n_3)\phi] \\
\frac{1}{2}abc\exp[-i(n_3+n_4)\phi] & \frac{1}{2}a^2d\exp[-i(n_1-n_2)\phi] & \frac{1}{2}(1-b)ac\exp[-i(n_1+n_3+n_4)\phi] & \frac{1}{2}b(1-b) & \frac{1}{2}a^2c\exp[-i(n_1-n_2+n_4)\phi] \\
\frac{1}{2}a^2d\exp[i(n_1-n_2-n_3)\phi] & -\frac{1}{4}ab(1+b)\exp[in_4\phi] & -\frac{1}{2}abc\exp[-i(n_2+n_3)\phi] & \frac{1}{2}a^2c\exp[i(n_1-n_2+n_4)\phi] & -\frac{1}{4}(1-b^2)b
\end{pmatrix}
$$

(a)

$$
a_2 = \frac{1}{2\sqrt{3}}\begin{bmatrix}
\frac{1}{2}(1+b) & 0 & 0 & \frac{1}{2}a\exp[i(n_3+n_4)\phi] & 0 \\
0 & \frac{1}{4}(3b-1)(b+1) & \frac{3}{2}da\exp[-i(n_2+n_3+n_4)\phi] & 0 & \frac{1}{4}a(3b-1)\exp[-in_4\phi] \\
0 & \frac{3}{2}\,[\text{illegible}]\exp[i(n_2+n_3+n_4)\phi] & \frac{1}{2}(1+3b) & 0 & \frac{3}{2}ac\exp[i(n_2+n_3)\phi] \\
\frac{1}{2}a\exp[-i(n_3+n_4)\phi] & 0 & 0 & \frac{1}{2}(1-b) & 0 \\
0 & \frac{1}{4}a(3b-1)\exp[in_4\phi] & \frac{3}{2}ac\exp[-i(n_2+n_3)\phi] & 0 & -\frac{1}{4}(3b-1)(b-1)
\end{bmatrix}
$$

(b)

$$
m_3 = \frac{1}{2\sqrt{6}}
\begin{bmatrix}
2b-1 & 0 & 0 & 2a\exp[i(n_3+n_4)\phi] & 0 \\
0 & \frac{1}{2}(1+b) & 0 & 0 & \frac{1}{2}a\exp[-in_4\phi] \\
0 & 0 & 1 & 0 & 0 \\
2a\exp[-i(n_3+n_4)\phi] & 0 & 0 & -2b-1 & 0 \\
0 & \frac{1}{2}a\exp[in_4\phi] & 0 & 0 & \frac{1}{2}(1-b)
\end{bmatrix}
$$

(c)

$$a_4 = \frac{1}{2\sqrt{10}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2}+\frac{5}{2}b & 0 & 0 & \frac{5}{2}a\exp[-in_4\phi] \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{5}{2}a\exp[in_4\phi] & 0 & 0 & -\frac{3}{2}-\frac{5}{2}b \end{pmatrix}$$

(d)