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Infinite Number of Conservation Laws in Two Dimensional Superconformal Models

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Abstract

Superconformal theories in two dimensional space-time are considered. Noether's theorem and the Belinfante improvement procedure are extended to superspace where they are used to construct the supercurrent. With its aid, an infinite number of classical conservation laws are derived. These laws are shown to survive quantization in the supersymmetric, non-linear, O(N) sigma model.

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1. Introduction

Every conformally invariant classical field theory in two dimensional space-time possesses an infinite number of current conservation laws [1]. These laws, if they survive quantization, are sufficient for proving the absence of particle production and the factorization of the S-matrix, which in turn allow the S-matrix to be explicitly calculated [2]. The existence of these quantum conservation laws has been demonstrated in various models [1],[3]. The difficulty encountered in such proofs is that conformal invariance is broken at the quantum level and hence quantum anomalies occur in the conservation laws. One procedure [1] used to show that these anomalies do not spoil the required form of the conservation laws is to simply list the various anomaly operators using the restrictions imposed by the good symmetries of the model and dimensional analysis applied to the anomalous conservation law. It was found that for many models, the anomaly terms could be written as total divergences which still allowed the S-matrix to be determined [1],[3].

This same procedure can be applied to conformal models which are also supersymmetric. However, the infinite set of currents will not be supersymmetry multiplets. The purpose of this paper is to show how to construct an infinite set of superfield currents. The easiest way to keep the supersymmetry manifest is to work in superspace [4]. In addition, we will show how to construct a set of currents which are of lower scaling dimension than the usual infinite set of currents and from which, by spinor differentiation, the usual set can be derived.

To be specific, in section 2 we consider classical field theories in two space-time dimensions and generalize Noether's theorem to superspace [5]. The classical fields, $\phi(x,\theta)$, are functions of points in superspace $z = (x^\mu, \theta_a)$, where $x^\mu = (x^o, x^1)$ are 2-dimensional space-time coordinates and $\theta_a = (\theta_1, \theta_2)$ are Majorana, Grassmann, spinor coordinates [6]. If under the superspace transformations $x'_\mu = x_\mu + \delta x_\mu$, $\theta'_a = \theta_a + \delta\theta_a$, the fields transform intrinsically as

$$\hat{\delta}\phi = \phi'(x,\theta) - \phi(x,\theta), \qquad (1.1)$$

then Noether's theorem takes the form

$$0 = D_a[\frac{\partial \mathcal{L}}{\partial D_a \phi} \hat{\delta}\phi] + \bar{D}_a[\hat{\delta}\phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_a \phi^+}]$$
$$+ \delta x^\mu \partial_\mu \mathcal{L} + \overline{\delta\theta} \frac{\partial \mathcal{L}}{\partial \bar{\theta}} - \delta \mathcal{L} . \qquad (1.2)$$

Here $\delta\mathcal{L} = \mathcal{L}'(x',\theta') - \mathcal{L}(x,\theta)$ is the total variation of the superfield Lagrangian and D_a, $\bar{D}_a = \gamma^o_{ab} D_b$ are the supersymmetry covariant derivatives. The action I, which describes the dynamics of the model, is given in terms of the Lagrangian as $I = \int d^2x d^2\theta \mathcal{L}(x,\theta)$. We further apply Noether's theorem to the superconformal symmetries and show that, as usual, the supercurrent V^μ_a plays a pivotal role in describing the currents associated with these symmetries [7]. The supercurrent contains as component currents the supersymmetry current Q^μ_a and the energy-momentum tensor, $T^{\mu\nu}$ so that

$$V^\nu_a = -\frac{1}{8}[Q^\nu_a - 2i(\gamma_\mu \theta)_a T^{\mu\nu}] . \qquad (1.3)$$

The Belinfante improvement procedure is also generalized so that the energy-momentum tensor in equation (1.3) is symmetric and traceless. Moreover, given the improved V_a^ν, we have

$$T^{\mu\nu} = 2i(\bar{D}\gamma^\nu)_a V_a^\mu \ . \tag{1.4}$$

The space-time translation symmetry of the theory then implies the conservation equation

$$\bar{D}_a V_a^\mu = 0 \ . \tag{1.5}$$

Further, the superconformal symmetries imply the tracelessness of V_a^μ

$$(\gamma_\mu V^\mu)_a = 0 \tag{1.6}$$

and of the improved energy-momentum tensor scalar superfield; $T_\lambda^\lambda = 0$. Together, equations (1.5) and (1.6) imply the space-time divergence conservation equation $\partial_\mu V_a^\mu = 0$.

In section 3, we use the supercurrent to construct the infinite number of classical conserved currents. It is easiest to do this in light-cone coordinates defined via

$$x^\pm = \frac{1}{\sqrt{2}}(x^o \pm x^1) \tag{1.7}$$

which implies

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_o \pm \partial_1) \ . \tag{1.8}$$

Equations (1.4),(1.5), and (1.6) then translate into

$$\begin{aligned} T_{++} &= -i2\sqrt{2}\, D_2 V_{+2} \\ T_{--} &= -i2\sqrt{2}\, D_1 V_{-1} \\ T_{+-} &= 0 = T_{-+} \\ D_2 V_{-1} &= 0 = D_1 V_{+2} \\ V_{+1} &= 0 = V_{-2} \quad . \end{aligned} \tag{1.9}$$

The infinite number of spinor derivative conservation laws thus acquire the form

$$D_2[V_{-1}(T_{--})^n] = 0$$

or

$$D_1[V_{+2}(T_{++})^n] = 0 \tag{1.10}$$

for $n = 0,1,2,\ldots$. By acting upon these with the $\bar{D}D$ derivative, we derive

$$\partial_+(T^n_{--}) = 0 \tag{1.11}$$

for $n = 1,2,\ldots$, which contains the usual form of the conservation laws for two dimensional conformal models [1]. The lower dimensional equation (1.10) is more useful, however, when the theories are quantized. This follows, since for each n, the lower dimension equation has fewer possible anomalies which could possibly spoil the conservation laws. Thus the search for anomalies is greatly simplified.

Finally, in section 4, we apply these techniques to the supersymmetric, non-linear, O(N) sigma model [8]. We consider the quantization of the model via a $\frac{1}{N}$ perturbation expansion [8],[9] and show that the anomaly terms corresponding to quantum corrections for the first two ($n = 0, 1$) currents of equation (1.10) appear as D_1 and D_2 derivatives. This is sufficient to explicitly compute the S-matrix for this model [10].

Various definitions and notation as well as some useful formulae are found in Appendix A. Appendix B contains the definitions for the super conformal algebra in two dimensions and its representation by linear superspace differential operators.

Section 2 - Noether's Theorem in Superspace

In this section, conserved currents associated with symmetries of the superfield Lagrangian will be constructed via Noether's theorem extended to superspace. The classical theories under consideration are made from classical complex scalar superfields $\phi_i(x,\theta)$ where the subscript i denotes that ϕ_i belongs to some representation of an internal Lie group. Here $x^\mu = (x^o, x^1)$ are 2-dimensional space-time coordinates and the $\theta_a = (\theta_1, \theta_2)$ are two component Majorana, Grassmann spinor coordinates. Together, $z = (x^\mu, \theta_a)$ describes a point in two dimensional superspace. The dynamics of the theory are given in terms of the action

$$I = \int d^2xd^2\theta\mathcal{L}(x,\theta), \qquad (2.1)$$

where $\mathcal{L}(x,\theta)$ is the superfield Lagrangian. In general, we ask that the action is an invariant under some group of transformations which include the super and Poincaré symmetries. In classical theories, the fields carry representations of these symmetries at every point in superspace. That is, the classical field ϕ' for the observer S' is related to the field ϕ for the observer S by a symmetry operation G. The intrinsic variation of the field for an infinitesimal G transformation is defined by

$$\hat{\delta}^G\phi \equiv \phi'(z)-\phi(z). \qquad (2.2)$$

These variations in turn carry the representation of the algebra associated with operation G. The total variation of the classical field is given by the value of the field in the transformed frame minus its value in the original frame, both evaluated at the same point which is called z' in S' and z in S. Thus

$$\delta^G\phi \equiv \phi'(z')-\phi(z) \ . \tag{2.3}$$

Since these are infintesimal transformations, the variations are related by

$$\delta^G\phi = \hat{\delta}^G\phi + \delta x^\mu \partial_\mu \phi + \delta\bar{\theta}_a \frac{\partial}{\partial\bar{\theta}_a} \phi \ , \tag{2.4}$$

where $x_\mu' = x_\mu + \delta x_\mu$, $\theta_a' = \theta_a + \delta\theta_a$ relate the superspace points in S' and S. (See appendix A for notation and conventions as well as the definition of operations with Grassmann coordinates.)

To be specific, let us consider the symmetries associated with the graded Poincaré group. The generators for such symmetries are denoted by P_μ, the generators for space-time translations, $M_{\mu\nu}$, the generator for Lorentz transformations (just boosts in two dimensions), and Q_a, which is a two component Grassmann, Majorana spinor generating supersymmetry translations. These generators obey the algebra (see Appendix B)

$$[M_{\mu\nu},P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda})$$

$$[M_{\mu\nu},M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma})$$

(= 0 in 2 dimension)

$$\begin{aligned} &[P_\mu,P_\nu] = 0 = [P^\mu,Q_a] \\ &[M^{\mu\nu},Q_a] = -\frac{i}{2}\sigma^{\mu\nu}_{ab}Q_b \\ &\{Q_a,Q_b\} = -2(\gamma^\mu\gamma^o)_{ab}P_\mu \quad . \end{aligned} \tag{2.5}$$

The representation of this algebra on the superfields as given by the intrinsic variations of these fields is;

$$\hat{\delta}^{P}_{\mu}\phi = \partial_{\mu}\phi$$

$$\hat{\delta}^{M}_{\mu\nu}\phi = [x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} + \frac{1}{2}\,\bar{\theta}\sigma_{\mu\nu}\frac{\partial}{\partial\theta}]\phi$$

$$\hat{\delta}^{Q}_{a}\phi = [\frac{\partial}{\partial\bar{\theta}_{a}} + i(\not{\partial}\theta)_{a}]\phi \quad , \tag{2.6}$$

where the linear superspace differential operators $\hat{\delta}^{G}$ obey the same algebra as the corresponding P^{μ}, $M^{\mu\nu}$, Q_{a}.

The Lagrangian can thus be written as a function of ϕ and the supersymmetry covariant derivatives, $D_{a}\phi$ and $\partial_{\mu}\phi$. The spinor covariant derivatives are defined by

$$D_{a} \equiv \frac{\partial}{\partial\bar{\theta}_{a}} - i(\not{\partial}\theta)_{a} \quad . \tag{2.7}$$

Moreover, since $\{D_{a},\bar{D}_{b}\} = 2i\not{\partial}_{ab}$, the Lagrangian need only depend on the spinor derivatives, so that in general,

$$\mathcal{L} = \mathcal{L}(\phi,\phi^{+},D_{a}\phi,\bar{D}_{a}\phi^{+}) \quad . \tag{2.8}$$

The total variation of the Lagrangian is thus given by

$$\delta\mathcal{L} = \hat{\delta}\mathcal{L} + \delta x^{\mu}\partial_{\mu}\mathcal{L} + \overline{\delta\theta}_{a}\frac{\partial\mathcal{L}}{\partial\bar{\theta}_{a}} \quad , \tag{2.9}$$

where for convenience we define

$$\tilde{\delta} = a^{\mu}\tilde{\delta}^{P}_{\mu} + \frac{1}{2}\lambda^{\mu\nu}\tilde{\delta}^{M}_{\mu\nu} + \bar{\xi}_a\tilde{\delta}^{Q}_a \quad . \tag{2.10}$$

Here $\tilde{\delta}^G = \delta^G$ or $\hat{\delta}^G$, with $G\epsilon\{P^\mu, M^{\mu\nu}, Q_a\}$, and a^μ, $\lambda^{\mu\nu} = -\lambda^{\mu\nu}$, ξ_a are the infinitesimal parameters associated with space-time translations, Lorentz transformations, and supersymmetry translations **respectively.** The intrinsic variation of $\mathcal{L}$ is

$$\hat{\delta}\mathcal{L} = \hat{\delta}\phi \frac{\partial\mathcal{L}}{\partial\phi} + \hat{\delta}D_a\phi \frac{\partial\mathcal{L}}{\partial D_a\phi} + \hat{\delta}\phi^+ \frac{\partial\mathcal{L}}{\partial\phi^+} + \hat{\delta}\bar{D}_a\phi^+ \frac{\partial\mathcal{L}}{\partial D_a\phi^+} \quad . \tag{2.11}$$

Since $\hat{\delta}D_a\phi = D_a\phi'(z) - D_a\phi(z) = D_a\hat{\delta}\phi$, we can write Eq.(2.11) as

$$\hat{\delta}\mathcal{L} = E(\phi)\hat{\delta}\phi + \hat{\delta}\phi^+E(\phi^+) + D_a[\frac{\partial\mathcal{L}}{\partial D_a\phi}\hat{\delta}\phi] + \bar{D}_a[\hat{\delta}\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}\phi^+}], \tag{2.12}$$

where E denotes the Euler-Lagrange derivative

$$E(\phi) = \frac{\partial\mathcal{L}}{\partial\phi} - D_a \frac{\partial\mathcal{L}}{\partial D_a\phi}$$

$$E(\phi^+) = \frac{\partial\mathcal{L}}{\partial\phi^+} - \bar{D}_a \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+} \quad . \tag{2.13}$$

Although for classical theories, the Euler-Lagrange derivatives vanish, it is useful to keep track of these terms throughout the various calculations. Thus using the notation

$$W^G \equiv -[E(\phi)\hat{\delta}^G\phi + \hat{\delta}^G\phi^+E(\phi^+)] \tag{2.14}$$

and $W = a^{\mu}W^{P}_{\mu} + \frac{1}{2}\lambda^{\mu\nu}W^{M}_{\mu\nu} + \bar{\xi}_a W^{Q}_{a}$, Equation (2.9) can then be written as

$$W = D_a[\frac{\partial \mathcal{L}}{\partial D_a\phi}\hat{\delta}\phi] + \bar{D}_a[\hat{\delta}\phi^{+}\frac{\partial \mathcal{L}}{\partial \bar{D}_a\phi^{+}}] + \delta x^{\mu}\partial_{\mu}\mathcal{L} + \overline{\delta\theta}\,\frac{\partial \mathcal{L}}{\partial \bar{\theta}} - \delta\mathcal{L}. \tag{2.15}$$

This is the fundamental form of Noether's theorem in superspace. Let's apply this result to the graded Poincaré symmetries.

We will demand that the Lagrangian is invariant under space-time translations, Lorentz transformations and supersymmetry translations so that $\delta\mathcal{L} \equiv 0$. In addition, since the fields in our model will be taken as scalars under these transformations, we have, $\delta\phi = 0$. This yields $\hat{\delta}\phi = -\delta x^{\mu}\partial_{\mu}\phi - \overline{\delta\theta}_a \frac{\partial}{\partial\overline{\theta}_a}\phi$ which implies

$$\begin{aligned} \delta x^{\mu} &= -a^{\mu} - i\bar{\xi}\gamma^{\mu}\theta + \lambda^{\mu\nu}x_{\nu} \\ \delta\theta_a &= -\xi_a + \frac{1}{4}\lambda^{\mu\nu}(\sigma_{\mu\nu}\theta)_a . \end{aligned} \tag{2.16}$$

Thus we find the three fundamental conservation equations

(i) Translation invariance:

$$W^{P}_{\mu} = D_a[\frac{\partial \mathcal{L}}{\partial D_a\phi}\partial_{\mu}\phi] + \bar{D}_a[\partial_{\mu}\phi^{+}\frac{\partial \mathcal{L}}{\partial \bar{D}_a\phi^{+}}] - \partial_{\mu}\mathcal{L} \tag{2.17a}$$

(ii) Supersymmetry:

$$W^{Q}_{a} = D_b[\frac{\partial \mathcal{L}}{\partial D_b\phi}\hat{\delta}^{Q}_{a}\phi] - \bar{D}_b[\hat{\delta}^{Q}_{a}\phi^{+}\frac{\partial \mathcal{L}}{\partial \bar{D}_b\phi^{+}}] - \hat{\delta}^{Q}_{a}\mathcal{L} \tag{2.17b}$$

(iii) Lorentz invariance:

$$W^{M}_{\mu\nu} = D_a[\frac{\partial \mathcal{L}}{\partial D_a\phi}\hat{\delta}^{M}_{\mu\nu}\phi] + \bar{D}_a[\hat{\delta}^{M}_{\mu\nu}\phi^{+}\frac{\partial \mathcal{L}}{\partial \bar{D}_a\phi^{+}}] - \hat{\delta}^{M}_{\mu\nu}\mathcal{L}. \tag{2.17c}$$

We can recast these into the usual form of a conserved space-time divergence equation by noting that

$$\bar{D}D\bar{D}_a = 2i\partial\!\!\!/_{ba}\bar{D}_b$$

and (2.18)

$$\partial_\mu = -\frac{i}{2}\bar{D}\gamma_\mu D \quad .$$

Then the translation invariance equation (2.17a) can be written in the form

$$W^P_\mu = \bar{D}_a V_{\mu a}$$

with

$$V^\mu_a \equiv \gamma^o_{ba}\frac{\partial\mathcal{L}}{\partial D_b\phi}\partial^\mu\phi + \partial^\mu\phi^+\frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+} + \frac{i}{2}(\gamma_\mu D)_a\mathcal{L} \quad . \tag{2.19}$$

Taking $\bar{D}D$ of this equation we find the usual conservation equation for the canonical energy momentum tensor

$$\bar{D}DW^{P\mu} = \partial_\nu T^{\mu\nu} \quad ,$$

$$T^{\mu\nu} \equiv 2i(\bar{D}\gamma^\nu)_a V^\mu_a \quad . \tag{2.20}$$

Notice that since translation is a supersymmetric operation, the translation invariance equations are supercovariant equations. Thus V^μ and $T^{\mu\nu}$ are scalar superfields ($\delta^Q_b V^\mu_a = 0$ and $\delta^Q_b T^{\mu\nu} = 0$). In addition the θ-independent component of $T^{\mu\nu}(x,\theta)$ is the ordinary Noether energy-momentum tensor in terms of the component fields of the model. However, since $T^{\mu\nu}(x,\theta)$ is the Noether tensor, it is not necessarily symmetric.

Similarly, we find for supersymmetry translations,

$$\bar{D}DW_a^Q = \partial_\mu Q_a^\mu ,$$

$$Q_a^\mu \equiv -2i\,(\gamma^\mu D)_b \left[\frac{\partial\mathcal{L}}{\partial D_b\phi} \quad \hat{\delta}_a^Q \phi\right] -2i(\bar{D}\gamma^\mu)_b \left[\hat{\delta}_a^Q\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right] + 2i\bar{D}_b[(\gamma^\mu\theta)_a D_b\mathcal{L}] \; . \qquad (2.21)$$

Although Q_a^μ and W_a^Q are superfunctions, they are not scalars under supersymmetry transformations. Finally we derive the conservation equation for the angular momentum tensor as

$$\bar{D}DW^{M\mu\nu} = \partial_\rho M^{\mu\nu\rho} \quad ,$$

$$M^{\mu\nu\rho} \equiv x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho} + \frac{1}{2}\,\bar{\theta}\sigma_{\mu\nu}Q^\rho + i\,\frac{\partial\mathcal{L}}{\partial D\phi}\,\gamma^\rho\sigma^{\mu\nu}D\phi - i(\bar{D}\phi^+)\sigma^{\mu\nu}\gamma^\rho\,\frac{\partial\mathcal{L}}{\partial\bar{D}\phi^+} \; .$$

$$(2.22)$$

Again $M^{\mu\nu\rho}$ and $W^{M\mu\nu}$ are superfunctions but are not scalars under supersymmetry.

Next we would like to extend to superspace the Belinfante procedure for symmetrizing the Noether energy-momentum tensor. We can always improve our currents by adding anti-symmetric total divergences. Thus we define

$$T_B^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda G^{\lambda\mu\nu}$$

and $$(2.23)$$

$$Q_{Ba}^\mu \equiv Q_a^\mu + \partial_\lambda Q_a^{\lambda\mu} \quad ,$$

where $G^{\lambda\mu\nu} = -G^{\nu\mu\lambda}$ and $Q_a^{\lambda\mu} = -Q_a^{\mu\lambda}$. Exploiting the antisymmetry of the improvements gives

$$\bar{D}DW^{P\mu} = \partial_\nu T^{\mu\nu} = \partial_\nu T_B^{\mu\nu} \tag{2.24a}$$

$$\bar{D}DW_a^Q = \partial_\mu Q_a^\mu = \partial_\mu Q_{Ba}^\mu \quad . \tag{2.24b}$$

Substituting these Belinfante tensors into $M^{\mu\nu\rho}$, we define

$$\begin{aligned} M_B^{\mu\nu\rho} &= M^{\mu\nu\rho} + \partial_\lambda [x^\mu G^{\lambda\nu\rho} - x^\nu G^{\lambda\mu\rho} + \tfrac{1}{2}\bar{\theta}\sigma^{\mu\nu}Q^{\lambda\rho}] \\ &= x^\mu T_B^{\nu\rho} - x^\nu T_B^{\mu\rho} + \tfrac{1}{2}\bar{\theta}\sigma^{\mu\nu}Q_B^\rho + (G^{\mu\nu\rho} - G^{\nu\mu\rho}) \\ &\quad + i\frac{\partial\mathcal{L}}{\partial D\phi}\gamma^\rho\sigma^{\mu\nu}D\phi - i\bar{D}\phi^+\sigma^{\mu\nu}\gamma^\rho\frac{\partial\mathcal{L}}{\partial\bar{D}\phi^+} , \end{aligned} \tag{2.25}$$

so that

$$\bar{D}DW^{M^{\mu\nu}} = \partial_\rho M^{\mu\nu\rho} = \partial_\rho M_B^{\mu\nu\rho} \quad . \tag{2.26}$$

Moreover, if we take

$$G^{\lambda\mu\nu} = -i(g^{\mu\lambda}\gamma^\nu - g^{\mu\nu}\gamma^\lambda)_{ab}\left[\frac{\partial\mathcal{L}}{\partial D_a\phi}D_b\phi + \bar{D}_a\phi^+\frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right], \tag{2.27}$$

it follows that

$$M_B^{\mu\nu\rho} = x^\mu T_B^{\nu\rho} - x^\nu T_B^{\mu\rho} + \tfrac{1}{2}\bar{\theta}\sigma^{\mu\nu}Q_B^\rho \quad . \tag{2.28}$$

Substituting equation (2.28) into equation (2.26) and using equations (2.24a) and (2.17c), we find

$$T_B^{\mu\nu} - T_B^{\nu\mu} = -2i\bar{\theta}\gamma^\mu D W^{P\nu} + 2i\bar{\theta}\gamma^\nu D W^{P\mu} - (\bar{D}\sigma^{\mu\nu})_a W_a^Q$$
$$= \bar{D}_a\,[E(\phi)(\sigma^{\mu\nu}D\phi)_a + (\sigma^{\mu\nu}D\phi^+)_a E(\phi^+)]\ . \tag{2.29}$$

For classical field theory, the Euler-Lagrange derivative vanish so that the right hand side of the above equation vanishes and $T_B^{\mu\nu}$ is symmetric.

Having improved the energy-momentum tensor, we would similarly like to define the lower dimensional current, $V_{B\ a}^{\ \mu}$, which improves V_a^μ by satisfying

$$W_\mu^P = \bar{D}_a V_{\mu a} = \bar{D}_a V_{B\mu a}$$

and (2.30)

$$T_B^{\mu\nu} = 2i(\bar{D}\gamma^\nu)_a V_{Ba}^\mu\ .$$

Since $\bar{D}\sigma^{\mu\nu}D = 0$, the first of these equations is guaranteed by defining

$$V_{B\ a}^{\ \mu} = V_a^\mu + (\sigma^{\mu\nu}D)_a B_\nu\ . \tag{2.31}$$

Next, fixing B_ν by $\varepsilon^{\mu\lambda}B_\lambda = \frac{1}{8}\,\varepsilon_{\rho\nu}G^{\rho\mu\nu}$, we obtain

$$V_{B\ a}^{\ \mu} = V_a^\mu - \frac{1}{8}(\gamma_5 D)_a \varepsilon_{\nu\rho} G^{\rho\mu\nu} \tag{2.32}$$

which also satisfies the latter of equation (2.30). Using the explicit form for $G^{\rho\mu\nu}$ (cf. equation(2.27)), we secure

$$V_{B\ a}^{\ \mu} = \gamma_{ba}^o \frac{\partial\mathcal{L}}{\partial D_b\phi}\ \partial^\mu\phi + \partial^\mu\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+} + \frac{i}{2}(\gamma^\mu D)_a\mathcal{L}$$
$$-\frac{i}{4}(\gamma_5 D)_a\Big[\frac{\partial\mathcal{L}}{\partial D_b\phi}\ (\gamma_5\gamma^\mu D)_b\phi + (\bar{D}\gamma_5\gamma^\mu)_b\phi^+\ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\Big]. \tag{2.33}$$

Before applying Noether's theorem to some of the other superconformal symmetries, we will apply it to the case of internal symmetries. Let ϕ_i, i = 1,...,k transform according to some k dimensional representation of a Lie group, $\mathcal{G}$ as

$$\delta\phi_i = \hat{\delta}\phi_i = \Lambda^A T^A_{ij}\phi_j \ . \tag{2.34}$$

Here Λ^D, D = 1,..., dim$\mathcal{G}$, parametrizes the group transformations and T^D_{ij} are the anti-hermitian k x k representation matrices. Then assuming $\mathcal{L}$ is a group scalar, we find

$$\Lambda^A W^{\mathcal{G}}_A = \Lambda^A \{D_a [\frac{\partial\mathcal{L}}{\partial D_a\phi_i} T^A_{ij}\phi_j] - \bar{D}_a [\phi^+_j T^A_{ji} \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+_i}]\}. \tag{2.35}$$

Again taking $\bar{D}D$ yields the usual form of the conserved current for internal symmetry transformations

$$\bar{D}DW^{\mathcal{G}}_A = \partial_\mu J^\mu_A \ ,$$

$$J^\mu_A = -2i(\gamma^\mu D)_a [\frac{\partial\mathcal{L}}{\partial D_a\phi_i} T^A_{ij}\phi_j]$$

$$-2i(\bar{D}\gamma^\mu)_a [\phi^+_j T^A_{ji} \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+_i}] \ . \tag{2.36}$$

Finally, let us consider the conservation equations associated with dilatation, special conformal, and conformal supersymmetry transformations. The generators for these symmetry operators are denoted by D,K^μ and the 2 component Majorana Grassmann spinor, S_a, respectively. The algebra of P^μ, $M^{\mu\nu}$, D,K^μ ,Q_a, and S_a is given in Appendix B along with the representation of the generators as linear superspace differential operators.

The latter give the intrinsic variation of the classical superfields. As previously, we can associate infinitesimal parameters with these transformations, so that

$$\tilde{\delta} = a^{\mu}\tilde{\delta}^{P}_{\mu} + \frac{1}{2}\lambda^{\mu\nu}\tilde{\delta}^{M}_{\mu\nu} + \varepsilon\tilde{\delta}^{D} + C^{\mu}\hat{\delta}^{K}_{\mu} + \bar{\xi}_{a}\tilde{\delta}^{Q}_{a} + \bar{r}_{a}\tilde{\delta}^{S}_{a}\ , \tag{2.37}$$

where $\tilde{\delta}^{G} = \delta^{G}$ or $\hat{\delta}^{G}$ for $G\varepsilon\{P^{\mu},M^{\mu\nu},D,K^{\mu},Q_{a},S_{a}\}$, and $a^{\mu},\lambda^{\mu\nu},\varepsilon$, C^{μ}, ξ_{a}, r_{a} are the respective infinitesimal parameters. Taking the scaling weight of $\mathcal{L}$ to be one and assuming that the action, I, is superconformally invariant so that $\delta\mathcal{L} = \varepsilon\mathcal{L} + 2\bar{r}\theta\mathcal{L} + 2c^{\mu}x_{\mu}\mathcal{L}$, the fundamental conservation equation becomes

$$\begin{aligned} W &= a^{\mu}W^{P}_{\mu} + \frac{1}{2}\lambda^{\mu\nu}W^{M}_{\mu\nu} + \bar{\xi}_{a}W^{Q}_{a} \\ &\quad + \varepsilon W^{D} + C^{\mu}W^{K}_{\mu} + \bar{r}_{a}W^{S}_{a} \\ &= D_{a}\left[\frac{\partial\mathcal{L}}{\partial D_{a}\phi}\,\hat{\delta}\phi\right] + \bar{D}_{a}\left[\hat{\delta}\phi^{+}\,\frac{\partial\mathcal{L}}{\partial\bar{D}_{a}\phi^{+}}\right] + \delta x^{\mu}\partial_{\mu}\mathcal{L} \\ &\quad + \overline{\delta\theta}\,\frac{\partial\mathcal{L}}{\delta\bar{\theta}} - \varepsilon\mathcal{L} - 2C^{\mu}x_{\mu}\mathcal{L} - 2\bar{r}\theta\mathcal{L}\ . \end{aligned} \tag{2.38}$$

From equations (B.4), we find

$$\delta x^{\mu} = -a^{\mu} + \lambda^{\mu\nu}x_{\nu} - \varepsilon x^{\mu} + [C^{\mu}x^{2} - 2C^{\lambda}x_{\lambda}x^{\mu}] - i\bar{\xi}\gamma^{\mu}\theta - \bar{r}\not{x}\gamma^{\mu}\theta$$

and

$$\delta\theta_{a} = +\frac{1}{4}\lambda^{\mu\nu}(\sigma_{\mu\nu}\theta)_{a} - \frac{1}{2}\varepsilon\theta_{a} - C^{\mu}x^{\nu}(\gamma_{\nu}\gamma_{\mu}\theta)_{a} - \xi_{a} - i(\not{x}r)_{b}\ . \tag{2.39}$$

This yields after substitution in Eq.(2.38),

$$W = D_{a}\left[\frac{\partial\mathcal{L}}{\partial Da\phi}\,\hat{\delta}\phi\right] + \bar{D}_{a}\left[\hat{\delta}\phi^{+}\,\frac{\partial\mathcal{L}}{\partial\bar{D}_{a}\phi^{+}}\right] - \hat{\delta}\mathcal{L}, \tag{2.40}$$

with $d_{\mathcal{L}} = 1$ in $\hat{\delta}\mathcal{L}$.

Once again, if so desired, we could take $\bar{D}D$ of this expression in order to obtain a space-time divergence current conservation equation. Instead, we shall use the above conservation equations to prove that for superconformally invariant theories, the current V^{μ}_{Ba} contains not only the energy-momentum tensor but also the supersymmetry current Q^{μ}_{Ba} and their traces, $T_B{}^{\lambda}{}_{\lambda}$ and $(\gamma_{\mu}Q^{\mu}_{B})_a$. Hence V^{μ}_{Ba} is the supercurrent. That is, it will be proven that

$$\begin{aligned} -8V^{\mu}_{Ba} &= Q^{\mu}_{Ba} - 2i(\gamma_{\nu}\theta)_a T_B{}^{\nu\mu} \\ &\quad + 2i\gamma^{\mu}_{ab}[W^{Q}_{b} - 2i(\gamma^{\lambda}\theta)_b W^{P}_{\lambda}], \end{aligned} \tag{2.41}$$

which implies the space-time divergence conservation equation

$$\begin{aligned} -8\partial_{\mu}V^{\mu}_{Ba} &= \bar{D}DW^{Q}_{a} - 2i(\gamma^{\nu}\theta)_a \bar{D}DW^{P}_{\nu} \\ &\quad + 2i\not{\partial}_{ab}[W^{Q}_{b} - 2i(\gamma^{\lambda}\theta)_b W^{P}_{\lambda}] . \end{aligned} \tag{2.42}$$

We shall also establish the trace condition

$$2i(\gamma_{\mu}V^{\mu}_{B})_a = W^{Q}_{a} - 2i(\gamma^{\mu}\theta)_a W^{P}_{\mu} , \tag{2.43}$$

which immediately yields the energy-momentum tensor trace

$$\begin{aligned} T_B{}^{\lambda}{}_{\lambda} &= 2i\bar{D}\gamma_{\mu}V^{\mu}_{B} \\ &= \bar{D}_a[W^{Q}_{a} - 2i(\gamma^{\lambda}\theta)_a W^{P}_{\lambda}] . \end{aligned} \tag{2.44}$$

Notice that the combination

$$W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu = -[E(\phi)D_a\phi + D_a\phi^+ E(\phi^+)], \qquad (2.45)$$

is manifestly a scalar superfield although the individual components W^Q_a and $-2i(\gamma^\mu\theta)_a W^P$ are not.

In order to prove the above assertions, we make use of the dilatation and conformal supersymmetry fundamental conservation equations:

(i) Dilatations:

$$W^D = D_a[\frac{\partial\mathcal{L}}{\partial D_a\phi}\hat{\delta}^D\phi] + \bar{D}_a[\hat{\delta}^D\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+}] - \hat{\delta}^D\mathcal{L}$$

$$\hat{\delta}^D = d + x^\lambda\partial_\lambda + \frac{1}{2}\bar{\theta}\frac{\partial}{\partial\bar{\theta}} \qquad (2.46a)$$

(ii) Super conformal:

$$W^S_a = D_b[\frac{\partial\mathcal{L}}{\partial D_b\phi}\hat{\delta}^S_a\phi] - \bar{D}_b[\hat{\delta}^S_a\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}] - \hat{\delta}^S_a\mathcal{L}$$

$$\hat{\delta}^S_a = 2d\theta_a - i(\not{x}\hat{\delta}^Q)_a, \qquad (2.46b)$$

where $d = d_\phi$ for fields ϕ and $d = 1$ for the assumed scale invariant Lagrangian. The dilatation conservation equation yields the simple engineering dimensional analysis equation for the scale invariant Lagrangian

$$\mathcal{L} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial D_a\phi}D_a\phi + \frac{1}{2}\bar{D}_a\phi^+\frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+}$$
$$+ d_\phi[E(\phi)\,\phi + \phi^+\,E(\phi^+) + D_a(\frac{\partial\mathcal{L}}{\partial D_a\phi}\phi) + \bar{D}_a(\phi^+\frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+}].$$
$$(2.47)$$

The consequences of the conformal supersymmetry equation requires a bit of analysis. Some simple algebra allows us to rewrite Eq.(2.46b) as

$$W_a^S + i\not{\partial}_{ab} W_b^Q + 2d_\phi [\gamma^o_{ab}(\frac{\partial \mathcal{L}}{\partial D_b \phi}\ \phi)$$

$$-\theta_a D_b(\frac{\partial \mathcal{L}}{\partial D_b \phi}\ \phi) - \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_a \phi^+} - \theta_a \bar{D}_b(\phi^+ \ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+})]$$

$$= -\gamma^\mu_{ac}(\gamma_\mu \theta)_b (\frac{\partial \mathcal{L}}{\partial D_b \phi} D_c \phi)$$

$$+ \gamma^\mu_{ac}(\gamma^o \gamma_\mu \theta)_b (D_c \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+}) - 2\theta_a \mathcal{L} \ . \tag{2.48}$$

Using the dilatation equation (2.47) for $\mathcal{L}$ along with the following 2 useful identities

1) $$\gamma_{\mu ab}\gamma^\mu_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd} - \gamma_{5ab}\gamma_{5cd} \tag{2.49a}$$

$$= \delta_{ad}\delta_{bc} - \gamma_{5ad}\gamma_{5cb}$$

2) $$-\gamma^\mu_{ac}(\gamma_\mu \theta)_b (\frac{\partial \mathcal{L}}{\partial D_b \phi} D_c \phi) + \gamma^\mu_{ac}(\gamma^o \gamma_\mu \theta)_b (D_c \phi^+ \ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+})$$

$$+ \theta_a \frac{\partial \mathcal{L}}{\partial D_b \phi} D_b \phi - \theta_a \bar{D}_b \phi^+ \ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+}$$

$$= (\gamma_5 \theta)_a [\frac{\partial \mathcal{L}}{\partial D_b \phi}(\gamma_5 D)_b \phi + (\bar{D}\gamma_5)_b \phi^+ \ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+}] \ , \tag{2.49b}$$

the conformal supersymmetry equation (2.48) can be recast as

$$W_a^S + i\not{\partial}_{ab} W_b^Q + 2d_\phi [\gamma^o_{ab}(\frac{\partial \mathcal{L}}{\partial D_b \phi}\ \phi) - \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_a \phi}$$

$$+ \theta_a (E(\phi)\ \phi + \phi^+\ E(\phi^+))]$$

$$= (\gamma_5 \theta)_a [\frac{\partial \mathcal{L}}{\partial D_b \phi}(\gamma_5 D)_b \phi + (\bar{D}\gamma_5)_b \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+}] \ . \tag{2.50}$$

Moreover, using the expressions for $\hat{\delta}^Q_a$ and $\hat{\delta}^S_a$ from Appendix B, it follows by definition that

$$W^S_a + i \not{x}_{ab} W^Q_b = -2d_\phi \theta_a (E(\phi)\ \phi + \phi^+\ E(\phi^+)) \ . \tag{2.51}$$

Hence, we finally obtain

$$(\gamma_5\theta)_a [\frac{\partial \mathcal{L}}{\partial D_b \phi} (\gamma_5 D)_b \phi + (\bar{D}\gamma_5)_b \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+}]$$

$$= 2d_\phi [\gamma^o_{ab} (\frac{\partial \mathcal{L}}{\partial D_b \phi} \phi) - \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_a \phi^+}] \ . \tag{2.52}$$

Since we are interested in superconformally invariant theories, we set $d_\phi = 0$, so that the right hand side vanishes. Further, the terms in the bracket on the left hand side of (2.52) are scalar superfields. Since if $\theta_a F = 0$ and F is a scalar superfield then F = 0, we finally arrive at the simple form of the conformal supersymmetry conservation equation

$$\frac{\partial \mathcal{L}}{\partial D_b \phi} (\gamma_5 D)_b \phi + (\bar{D}\gamma_5)_b\ \phi^+ \frac{\partial \mathcal{L}}{\partial \bar{D}_b \phi^+} = 0 \tag{2.53}$$

We are now in a position to establish that V^μ_{Ba} is indeed the supercurrent. Using Eq.(2.30), the scalar superfield of Eq.(2.45) can be recast as

$$W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu = W^Q_a + 2i\bar{D}_b [(\gamma_\mu\theta)_a V^\mu_{B_b}] + 2i(\gamma_\mu V^\mu_B)_a . \tag{2.54}$$

Substituting Eq.(2.17b) for W^Q_a on the right hand side and Eq.(2.33) for V^μ_{Ba} in the second term on the right hand side, we find

$$W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu = D_a\mathcal{L} + \frac{1}{2} D_a\left(\frac{\partial\mathcal{L}}{\partial D_b\phi} D_b\phi\right)$$
$$+ \frac{1}{2}(\gamma_5 D)_a\left[\frac{\partial\mathcal{L}}{\partial D_b\phi}(\gamma_5 D)_b\phi\right] - \bar{D}_b\left(D_a\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right)$$
$$- D_b\left(\bar{D}_b\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+}\right) + \frac{1}{2} D_a\left(\bar{D}_b\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right)$$
$$+ \frac{1}{2}(\gamma_5 D)_a\left[\bar{D}_b\phi^+ (\gamma_5)_{bc} \frac{\partial\mathcal{L}}{\partial\bar{D}_c\phi^+}\right] + 2i(\gamma_\mu V^\mu_B)_a .$$
(2.55)

We now apply the dilatation equation (2.47) for $D_a\mathcal{L}$ to obtain

$$W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu = 2i(\gamma_\mu V^\mu_B)_a$$
$$+ \frac{1}{2}(\gamma_5 D)_a\left[\frac{\partial\mathcal{L}}{\partial D_b\phi}(\gamma_5 D)_b\phi + (\bar{D}\gamma_5)_b\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right]$$
$$+ d_\phi D_a\left[D_b\left(\frac{\partial\mathcal{L}}{\partial D_b\phi}\phi\right) + \bar{D}_b\left(\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right)\right.$$
$$\left. + E(\phi)\,\phi + \phi^+ E(\phi^+)\right],$$
(2.56)

where use of the identity

$$D_a\left(\bar{D}_b\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right) - \bar{D}_b\left(D_a\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_b\phi^+}\right) - D_b\left(\bar{D}_b\phi^+ \frac{\partial\mathcal{L}}{\partial\bar{D}_a\phi^+}\right) = 0$$
(2.57)

has been made. Next we can use the conformal supersymmetry equation (2.53) to secure equation (2.43), $2i(\gamma_\mu V^\mu_B)_a = W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu$.

Finally we can use $W^{P\mu} = \bar{D}_a V^\mu_{Ba}$ in conjunction with the above to exhibit the space-time divergence conservation equation for V^μ_{Ba}. We begin by noting that

$$\bar{D}DW^Q_a - 2i(\gamma^\mu\theta)_a\bar{D}DW^P_\mu = \partial_\nu[Q^\nu_{Ba} - 2i(\gamma_\mu\theta)_a T^{\mu\nu}_B] \tag{2.58}$$

is a scalar superfield equation. Thus

$$\begin{aligned}\bar{D}D[W^Q_a - 2i(\gamma^\mu\theta)_a W^P_\mu] &= \bar{D}D[2i(\gamma_\mu V^\mu_B)_a] \\ &= \partial_\nu[Q^\nu_{Ba} - 2i(\gamma_\mu\theta)_a T^{\mu\nu}_B] - 4i(\gamma_\mu D)_a\bar{D}_b V^\mu_{Bb} \ .\end{aligned} \tag{2.59}$$

Using $D_c\bar{D}_b = -\frac{1}{2}\delta_{bc}\bar{D}D + i\partial\!\!\!/_{cb}$ and a little algebra we find

$$\begin{aligned}&\partial_\nu[8V^\nu_{Ba} + Q^\nu_{Ba} - 2i(\gamma_\mu\theta)_a T^{\mu\nu}_B] \\ &\quad = \partial_\nu[-2i\gamma^\nu_{ab}(W^Q_b - 2i(\gamma^\mu\theta)_b W^P_\mu)] \ .\end{aligned} \tag{2.60}$$

Up to this point we have not completely specified Q^ν_{Ba}; however, we have specified V^ν_{Ba} and $T^{\mu\nu}_B$. When Eq. (2.60) is integrated it is necessary to introduce a function whose divergence is zero. This function is of the form of an improvement to Q^ν_a and hence can be absorbed into the definition of Q^ν_{Ba}. Thus we obtain our final result (cf. Eq. (2.41)).

$$8V^{\nu}_{Ba} = -Q^{\nu}_{Ba} + 2i(\gamma_{\mu}\theta)_a T^{\mu\nu}_B$$
$$-2i\gamma^{\nu}_{ab}(W^Q_b - 2i(\gamma^{\mu}\theta)_b W^P_{\mu}) \ . \tag{2.61}$$

In summary, in this section we have shown how to construct the fundamental current conservation equations using Noether's theorem in superspace. We have shown that the space-time translation equation can be written as

$$\bar{D}_a V^{\mu}_{Ba} = 0, \tag{2.62}$$

where the symmetric, traceless, and conserved energy momentum tensor is given by

$$T^{\mu\nu}_B = 2i(\bar{D}\gamma^{\nu})_a V^{\mu}_{Ba} \ . \tag{2.63}$$

Further, for superconformal theories, V^{μ}_{Ba} is the conserved supercurrent, $\partial_{\mu} V^{\mu}_{Ba} = 0$. It contains the conserved supersymmetry current Q^{μ}_{Ba} as well as the conserved energy-momentum tensor and is given by

$$V^{\nu}_{Ba} = -\frac{1}{8}[Q^{\nu}_{Ba} - 2i(\gamma_{\mu}\theta)_a T^{\mu\nu}_B]. \tag{2.64}$$

Moreover, V^{ν}_{Ba} is traceless in the sense that the Belinfante tensors are already the improved tensors. Thus,

$$(\gamma_{\mu} V^{\mu}_B)_a = 0$$

which implies $$(\gamma_{\mu} Q^{\mu}_B)_a = 0 \tag{2.65}$$

and $$T^{\lambda}_{B\lambda} = 0.$$

Section 3 - Infinite number of conserved classical currents

In two dimensions it is extremely useful to change from Minkowski coordinates to light cone coordinates. This transformation is defined for the coordinates by

$$x^{\pm} = \frac{1}{\sqrt{2}} (x^o \pm x^1) \tag{3.1}$$

with

$$x_{\pm} = x^{\mp}$$

Then for derivatives we find

$$\partial_{\pm} \equiv \frac{\partial}{\partial x^{\pm}} = \frac{1}{\sqrt{2}} (\partial_o \pm \partial_1)$$

and

$$\partial^{\pm} = \partial_{\mp} \quad . \tag{3.2}$$

Under the Lorentz boost of velocity $\vec{v}$, $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$, with

$$\Lambda^o{}_o = \Lambda^1{}_1 = \cosh\Lambda = \frac{1}{\sqrt{1 - v^2}}$$

$$\Lambda^o{}_1 = \Lambda^1{}_o = -\sinh\Lambda = \frac{-v}{\sqrt{1 - v^2}} , \tag{3.3}$$

every light cone vector, $A_{\pm}$, transforms as

$$A'_{\pm}(x') = e^{\pm\Lambda} A_{\pm} \quad . \tag{3.4}$$

In a similar manner, we can define tensor components in light cone coordinates. For example, the energy-momentum tensor has light cone coordinates

$$T_{++} = \frac{1}{2}(T_{00} + T_{11} + T_{01} + T_{10})$$
$$T_{--} = \frac{1}{2}(T_{00} + T_{11} - T_{01} - T_{10})$$
$$T_{+-} = \frac{1}{2}(T_{00} - T_{11} - T_{01} + T_{10})$$
$$T_{-+} = \frac{1}{2}(T_{00} - T_{11} + T_{01} - T_{10}) \quad . \tag{3.5}$$

Then under Lorentz transformations

$$T_{++} \to e^{+2\Lambda}\, T_{++}$$
$$T_{--} \to e^{-2\Lambda} T_{--}$$
$$T_{+-} \to T_{+-}$$
$$T_{-+} \to T_{-+} \quad . \tag{3.6}$$

As was shown by Goldschmidt and Witten,[1] it is almost a trivial matter to now construct an infinite number of conserved currents for conformally invariant models. We can also apply their technique in our superspace model since using light cone coordinates is a super-covariant operation. In Section 2, we showed that the scalar superfield Belinfante improved energy-momentum tensor has the attributes

1) Symmetry: $T^{\mu\nu} = T^{\nu\mu}$
2) Tracelessness: $T^{\lambda}_{\lambda} = 0$
3) Conservation: $\partial_\mu T^{\mu\nu} = 0$. (3.7)

Notice that here and for the remainder of the paper, we use only the Belinfante improved objects defined in Section 2. Thus, for notational simplicity we drop the subscript B. In light-cone coordinates, these equations imply

$$\begin{aligned} &1)\ \text{Symmetry:} && T_{+-} = T_{-+} \\ &2)\ \text{Tracelessness:} && T_{+-} = 0 = T_{-+} \\ &3)\ \text{Conservation:} && \partial_+ T_{--} = 0 = \partial_- T_{++} \ , \end{aligned} \tag{3.8}$$

where now

$$T_{++} = T_{00} + T_{01}$$

$$T_{--} = T_{00} - T_{01} \quad .$$

Hence an infinite number of conservation laws are obtained by raising these components to the power $n = 1,2,3,\ldots$ as

$$\partial_+ (T_{--})^n = 0$$

or

$$\partial_- (T_{++})^n = 0 \quad . \tag{3.9}$$

At this point, we would like to digress somewhat in order to recall what happens to these infinite number of conservation laws when we quantize a model. In general, the superconformal symmetry becomes broken due to the necessity of introducing a non-invariant mass scale in order to consistently renormalize the theory. Hence anomalous terms arise in the equations of motion needed to derive equations (3.9). These anomaly terms are not entirely arbitrary. They must obey the unbroken symmetries, discrete and continuous, of the model. In particular, they must be Poincaré and supersymmetry covariant. Finally the additional and most severe constraint they must obey is that of power counting consistency. The anomaly terms must be of the same power counting dimension as the conservation equation to which they belong. Thus we must list all independent (modulo equations of motion) operators of a fixed dimension (basis operators) as possible anomaly terms; for example, $\partial_+(T_{--})^n$ is of dimension 2n + 1 and so we need basis operators of dimension 2n + 1. If we can derive such a space-time divergence equation from a lower dimension spinor derivative equation by simply taking additional derivatives we will be able to restrict the possible anomaly structure further. To be specific, derivatives (spinor or space-time) do not alter the structure of renormalized relations between composite operators. Hence, if we can derive a lower dimensional relation without the presence of anomalies, then, by taking derivatives, we will have a higher dimensional relation of the same form; no new anomalous terms will be introduced. Since equations (3.9) are sufficient in order to derive the S-matrix of the quantum model, we will show that by using the supercurrent, V^μ_a, we can derive these equations from lower dimension equations by merely taking spinor derivatives. The simplest example of this procedure is the fundamental conservation equation (2.61) for space-time translations

$$\bar{D}_a V^\mu_a = 0 \ . \qquad (3.10)$$

Applying $\bar{D}D$ to this equation we found

$$0 = \bar{D}D\bar{D}_a V^\mu_a = 2i\bar{D}\not{\partial}V^\mu = \partial_\nu T^{\mu\nu} \quad . \tag{3.11}$$

Thus from the dimension 2 conservation equation (3.10), the conservation equation (3.11) of dimension 3 can be derived.

First, we would like to write the results of Section 2 in light cone coordinates. Under Lorentz transformations (3.3), a spinor $\psi_a(x)$, a = 1,2, transforms as

$$\begin{aligned} \psi_1'(x') &= e^{-\frac{1}{2}\Lambda}\,\psi_1(x) \\ \psi_2'(x') &= e^{+\frac{1}{2}\Lambda}\,\psi_2(x) \quad . \end{aligned} \tag{3.12}$$

Thus the one component of a spinor transforms like the square root of the vector A_- and the two component like the square root of the vector A_+. Similarly

$$\begin{aligned} D_1\phi &\to e^{-\frac{1}{2}\Lambda}\,D_1\phi \\ D_2\phi &\to e^{+\frac{1}{2}\Lambda}\,D_2\phi \ . \end{aligned} \tag{3.13}$$

For spinors, we will exhibit the 1 and 2 component subscripts explicitly. Finally, we define light cone Dirac matrices as

$$\gamma^{\pm} = \frac{1}{\sqrt{2}}\,(\gamma^o \pm \gamma^1) = \gamma_{\mp} \quad . \tag{3.14}$$

Some useful relations between the derivatives now become

$$
\begin{aligned}
D_1 D_1 &= -i\sqrt{2}\,\partial_- \\
D_2 D_2 &= -i\sqrt{2}\,\partial_+ \\
\bar{D}D &= 2iD_1 D_2 = -2iD_2 D_1 \\
\bar{D}_1 &= -iD_2, \qquad \bar{D}_2 = iD_1 \quad .
\end{aligned}
\tag{3.15}
$$

Thus defining

$$
V_a^{\pm} = \frac{1}{\sqrt{2}} (V_a^o \pm V_a^1) = V_{\mp\, a} \ , \tag{3.16}
$$

we can write equation (2.63) in light-cone coordinates as

$$
\begin{aligned}
T_{++} &= -i2\ \sqrt{2}\ D_2 V_{+2} \\
T_{--} &= -i2\ \sqrt{2}\ D_1 V_{-1} \\
T_{+-} &= -i2\ \sqrt{2}\ D_1 V_{+1} \\
T_{-+} &= -i2\ \sqrt{2}\ D_2 V_{-2} \quad .
\end{aligned}
\tag{3.17}
$$

The space-time translation equation (2.62) now becomes

$$
D_2 V_{\pm\, 1} = D_1 V_{\pm\, 2} \quad . \tag{3.18}
$$

In addition, for the superconformally invariant theories we can write equation (2.65) as

$$
\begin{aligned}
V_{-2} &= 0 \\
V_{+1} &= 0 \quad .
\end{aligned}
\tag{3.19}
$$

Combined with equation (3.18), this yields

$$T_{++} = -i2\sqrt{2}\, D_2 V_{+2}$$

$$T_{--} = -i2\sqrt{2}\, D_1 V_{-1}$$

$$T_{+-} = 0 = T_{-+}$$

$$D_2 V_{-1} = 0 = D_1 V_{+2} \quad , \tag{3.20}$$

where the non-vanishing light-cone components of the supercurrent are (cf. Equations (2.33) and (3.16))

$$V_{-1} = i\,\frac{\partial \mathcal{L}}{\partial D_2\phi}\,\partial_-\phi + \partial_-\phi^+\,\frac{\partial\mathcal{L}}{\partial\bar{D}_1\phi^+} - \frac{1}{2\sqrt{2}}\,D_1\left[\frac{\partial\mathcal{L}}{\partial D_2\phi}\,D_1\phi + \bar{D}_2\phi^+\,\frac{\partial\mathcal{L}}{\partial\bar{D}_1\phi^+}\right]$$

and

$$V_{+2} = -i\,\frac{\partial\mathcal{L}}{\partial D_1\phi}\,\partial_+\phi + \partial_+\phi^+\,\frac{\partial\mathcal{L}}{\partial\bar{D}_2\phi^+} + \frac{1}{2\sqrt{2}}\,D_2\left[\frac{\partial\mathcal{L}}{\partial D_1\phi}\,D_2\phi + \bar{D}_1\phi^+\,\frac{\partial\mathcal{L}}{\partial\bar{D}_2\phi^+}\right] \ . \tag{3.21}$$

From these equations, the lower dimension infinite number of spinor derivative conserved currents can be defined for $n = 0,1,2,\ldots$ as

$$D_2[V_{-1}(T_{--})^n] = 0$$

or

$$D_1[V_{+2}(T_{++})^n] = 0 \quad . \tag{3.22}$$

These infinite number of conservation laws imply the higher dimension laws (3.9) by taking $\bar{D}D$ derivatives. For n = 0 we have $D_2V_{-1} = 0$. Acting on this with D_1D_2 gives

$$0 = D_1\partial_+V_{-1} = \partial_+D_1V_{-1} = \partial_+T_{--} , \qquad (3.23)$$

which gives the energy-momentum conservation law in light-cone coordinates (Equation (3.8)). To obtain the first higher conservation law, we start with Equation (3.22) for n = 1; $D_2(V_{-1}T_{--}) = 0$. Taking D_1D_2 of this equation and using the n = 0 conservation law, Equation (3.23) yields $\partial_+(T_{--})^2 = 0$ which is the first higher conservation law. The remaining laws can be proven in a similar manner by induction.

In summary, we have constructed for any superconformal model an infinite number of scalar superfield currents which obey a spinor derivative conservation equation:

$$0 = D_2[V_{-1}(T_{--})^n] , \quad n = 0,1,2,\ldots \quad . \qquad (3.24)$$

From this equation, the infinite number of higher dimension conservation equations follow by taking the $\bar{D}D$ derivative;

$$0 = \partial_+(T^n_{--}), \quad n = 1,2,3,\ldots \quad . \qquad (3.25)$$

Section 4: Classical and Quantum Mechanical Currents in the Non-Linear Supersymmetric Sigma Model

The classical non-linear supersymmetric sigma model is described by a real scalar superfield $\phi_i(x,\theta)$, $i = 1,\ldots,N$, which carries the fundamental representation of the O(N) group. The scalar superfield Lagrangian for the model is given by

$$\mathcal{L} = \frac{1}{2}\bar{D}_a\phi\cdot D_a\phi \tag{4.1}$$

with the constraint

$$\phi\cdot\phi = R^2 , \tag{4.2}$$

where R = constant. We have introduced a dot product which dictates a sum over the (suppressed) O(N) indices. The constraint can be handled most easily by the method of Lagrange multipliers. Since equation (4.2) is a super-covariant equation we introduce a scalar superfield Lagrange multiplier, $\lambda(x,\theta)$, so that the effective Lagrangian describing the super non-linear sigma model is given by

$$\mathcal{L} = \frac{1}{2}\bar{D}_a\phi\cdot D_a\phi - \lambda(\phi^2 - R^2) \tag{4.3}$$

We can next use the Euler-Lagrange equations to eliminate the multiplier field from the ϕ equations of motion thus obtaining

$$D_1D_2\phi_i = -\frac{1}{R^2}\phi_i(D_1\phi\cdot D_2\phi)$$

$$\phi^2 = R^2 . \tag{4,4}$$

In order to construct the infinite number of currents, we simply apply equation (3.21) to our Lagrangian again using the Euler-Lagrange equations to eliminate the multiplier field.

We also note that for this model there are no Belinfante improvement terms. Thus we find

$$V_{-1} = D_1\phi\cdot\partial_-\phi$$

$$V_{+2} = D_2\phi\cdot\partial_+\phi$$

$$V_{-2} = 0 = V_{+1} \quad . \tag{4.5}$$

The light-cone coordinate energy-momentum tensor is then given via equation (3.20) as

$$T_{++} = -2\sqrt{2}\ i\ D_2[D_2\phi\cdot\partial_+\phi]$$

$$T_{--} = -2\sqrt{2}\ i\ D_1[D_1\phi\cdot\partial_-\phi]$$

$$T_{+-} = 0 = T_{-+} \quad . \tag{4.6}$$

It can be checked explicitly by using equation (4.4) that equation (3.20), $D_2V_{-1} = 0 = D_1V_{+2}$, is satisfied, hence implying the validity of equation (3.8), $\partial_+T_{--} = 0 = \partial_-T_{++}$. Finally, equation (3.22) yields the infinite number of conservation laws

$$D_2[V_{-1}(T_{--})^n] = 0 \tag{4.7}$$

or

$$D_1[V_{+2}(T_{++})^n] = 0 \quad , \quad n = 0,1,2,\ldots \quad .$$

Thus we have explicitly constructed the infinite number of classical conservation laws for the super, non-linear sigma model. We next examine the modifications to these conservation laws due to quantization. To do this, we keep in mind that quantization is accomplished via a $\frac{1}{N}$ perturbation expansion. In general, the superconformal invariance of the model is broken and anomalous terms will appear on the right-hand side of equation (4.7) so that

$$D_2[V_{-1}(T_{--})^n] = \sum_i r_i^{(n)} A_i^{(n)} \ . \tag{4.8}$$

Here, the constants $r_i^{(n)}$ are a power series in $\frac{1}{N}$ and the $A_i^{(n)}$ are composite field operators which are formally made from monomials in the field operators ϕ and their derivatives. (The composite fields are defined from the field monomials by means of Zimmermann's normal product algorithm [11] applied to the $\frac{1}{N}$ Feynman rules). The sum over composite operators is restricted so that the $A_i^{(n)}$ have the same discrete symmetries as the left-hand side of equation (4.8) as well as the same Lorentz transformation properties. In addition, the $A_i^{(n)}$ must be O(N) singlets and most importantly must have the same scaling weight (power counting dimension) as the left-hand side.

If all anomalies for these conservation laws can be written as total D_1 or D_2 derivatives, then the S-matrix for this model is exactly calculable.[10] That is, if equation (4.8) has the form

$$D_2[V_{-1}(T_{--})^n] = D_2 B_1^{(n)} + D_1 B_2^{(n)} \ , \tag{4.9}$$

where the $B_a^{(n)}$ are anomaly operators with appropriate symmetry and dimension, then acting with $\bar{D}D$ on this equation, we find

$$\partial_+[T_{--}^{n+1}] = -2\sqrt{2}i\ \partial_+[D_1 B_1^{(n)} + V_{-1} D_1 (T_{--})^n]$$
$$+ 2\sqrt{2}\ i\ \partial_- D_2 B_2^{(n)} \quad . \qquad (4.10)$$

This form of the infinite number of conservation laws at the quantum level is sufficient to determine the S-matrix.[10]

With these various restrictions in mind, we now show that equation (4.9) is true for the first two currents n = 0 and n = 1. Use will be made of the field equations and constraints so that only linearly independent terms are counted. Thus for n = 0, we find only two dimension 2 anomaly terms so that

$$D_2 V_{-1} = r_1^{(o)} D_2 (D_1 \phi \cdot \partial_- \phi) + r_2^{(o)} D_1 (D_2 \phi \cdot \partial_- \phi), \qquad (4.11)$$

which is already in the form of equation (4.9). We can trivially rewrite this in the form

$$\bar{D}_a J_{a-}^{(o)} = 0 \ , \qquad (4.12)$$

with

$$J_{1-}^{(o)} = i(1 - r_1^{(o)}) V_{-1}$$
$$J_{2-}^{(o)} = i r_2^{(o)} D_2 \phi \cdot \partial_- \phi \quad . \qquad (4.13)$$

For the n = 1 case, we must list all dimension four operators which transform like $(\partial_-)^3$. There are eight possible anomaly terms for the n = 1 version of equation (4.8),

$$D_2(V_{-1}T_{--}) = \sum_i r_i^{(1)} A_i^{(1)} \quad . \tag{4.14}$$

Listing these operators gives the set

$$\begin{aligned}
A_1^{(1)} &= D_2\phi\cdot\partial_-\partial_-\partial_- D_1\phi \\
A_2^{(1)} &= (D_1\phi\cdot D_2\phi)(D_1\phi\cdot\partial_-\partial_- D_1\phi) \\
A_3^{(1)} &= (D_1\phi\cdot D_2\phi)(\partial_-\phi\cdot\partial_-\partial_-\phi) \\
A_4^{(1)} &= (D_1\phi\cdot\partial_-\phi)(D_2\phi\cdot\partial_-\partial_-\phi) \\
A_5^{(1)} &= (D_2\phi\cdot\partial_-\phi)(D_1\phi\cdot\partial_-\partial_-\phi) \\
A_6^{(1)} &= (D_2\phi\cdot\partial_-\phi)(\partial_- D_1\phi\cdot\partial_-\phi) \\
A_7^{(1)} &= (D_1\phi\cdot\partial_- D_1\phi)(D_2\phi\cdot\partial_- D_1\phi) \\
A_8^{(1)} &= (\partial_-\phi\cdot\partial_-\phi)(D_2\phi\cdot\partial_- D_1\phi) \quad .
\end{aligned} \tag{4.15}$$

After application of the field equations and constraints, it is possible to write these 8 anomaly operators as linear combinations of spinor derivatives of the following 8 independent operators $B_i^{(1)}$, i = 1,...,8 as

$$
\begin{aligned}
B_1^{(1)} &= D_2\phi\cdot\partial_-\partial_-\partial_-\phi \\
B_2^{(1)} &= (D_1\phi\cdot D_2\phi)(D_1\phi\cdot\partial_-\partial_-\phi) \\
B_3^{(1)} &= (D_1\phi\cdot D_2\phi)(\partial_- D_1\phi\cdot\partial_-\phi) \\
B_4^{(1)} &= (D_1\phi\cdot\partial_-\phi)(D_2\phi\cdot\partial_- D_1\phi) \\
B_5^{(1)} &= (D_2\phi\cdot\partial_-\phi)(D_1\phi\cdot\partial_- D_1\phi) \\
B_6^{(1)} &= (D_2\phi\cdot\partial_-\phi)(\partial_-\phi\cdot\partial_-\phi) \\
B_7^{(1)} &= D_1\phi\cdot\partial_-\partial_-\partial_-\phi \\
B_8^{(1)} &= \partial_-\partial_- D_1\phi\cdot\partial_-\phi
\end{aligned}
\tag{4.16}
$$

That is

$$
D_2[V_{-1}T_{--}] = \sum_{i=1}^{6} \beta_i D_1 B_i^{(1)} + \sum_{i=7}^{8} \beta_i D_2 B_i^{(1)} , \tag{4.17}
$$

where the β_i are linear combinations of the r_i. Thus we have proven the first higher quantum conservation law

$$
\bar{D}_a J_a^{(1)} = 0 \tag{4.18}
$$

with

$$
\begin{aligned}
J_1^{(1)} &= V_{-1}T_{--} - \beta_7 B_7^{(1)} - \beta_8 B_8^{(1)} \\
J_2^{(1)} &= \sum_{i=1}^{6} \beta_i B_i^{(1)} .
\end{aligned}
\tag{4.19}
$$

This quantum mechanical law was assumed and used[10] to construct the exact S-matrix for the non-linear, supersymmetric sigma model.

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Appendix A

Notation, conventions, and useful formulae

The metric tensor $g_{\mu\nu}$ is defined so that its only non-vanishing elements are $g_{oo} = 1 = -g_{11}$. The Levi-Civita tensor $\varepsilon^{\mu\nu}$ is defined by $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, with $\varepsilon^{01} = -\varepsilon_{o1} = +1$ Thus

$$\varepsilon_{\mu\nu}\varepsilon^{\lambda\rho} = \delta^{\rho}_{\mu}\delta^{\lambda}_{\nu} - \delta^{\lambda}_{\mu}\delta^{\rho}_{\nu} \quad . \tag{A.1}$$

The two component Dirac matrices γ^{μ} are defined in terms of the Pauli matrices σ^{i};

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} , \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \tag{A.2}$$

by

$$\gamma^{o} = \sigma^{2}; \quad \gamma^{1} = i\sigma^{1}; \; \gamma_5 = \gamma^{o}\gamma^{1} = \sigma^{3} , \tag{A.3}$$

so that $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$. In addition, their commutator is defined by the matrix $\sigma^{\mu\nu}_{ab}$ as

$$\frac{1}{2}[\gamma^{\mu},\gamma^{\nu}] \equiv \sigma^{\mu\nu} = \varepsilon^{\mu\nu}\gamma_5 \quad . \tag{A.4}$$

Charge conjugation is realized through the γ^{o} matrix since

$$\gamma^{o}\gamma^{T}_{\mu}\gamma^{o} = - \gamma_{\mu}$$
and
$$\gamma^{oT} = -\gamma^{o}, \tag{A.5}$$

where the superscript T signifies transposition. Some useful Dirac matrix identities are listed below:

1) $(\gamma^{\mu}\gamma^{\nu})_{ab} = g^{\mu\nu}\,\delta_{ab} + \sigma^{\mu\nu}_{ab}$

2) $\gamma^{\mu}_{ab}\,\gamma_{\mu cd} = (\gamma_5\gamma^{\mu})_{ab}\,(\gamma_5\gamma_{\mu})_{cd}$ (A.6)

$= 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd} - \gamma_{5\,ab}\,\gamma_{5\,cd}$

3) $\gamma_5\gamma^{\mu} = \varepsilon^{\mu\nu}\gamma_{\nu}$

4) $\sigma^{\mu\nu}\gamma^{\lambda} = \gamma^{\mu}g^{\nu\lambda} - \gamma^{\nu}g^{\mu\lambda}$

The two component, complex (Dirac) Grassmann spinors ψ_a, $a = 1,2$ have an inner product defined by

$$\bar{\psi}_a \chi_a \equiv \psi_a^+ \gamma_{ab}^o \chi_b \tag{A.7}$$

The adjoint spinor, $\bar{\psi}_a$, is defined as

$$\bar{\psi}_a = (\psi^+ \gamma^o)_a = \psi_b^* \gamma_{ba}^o \tag{A.8}$$

and the charge conjugate spinor ψ^C is given by

$$\psi_a^c \equiv (\gamma^o \, \bar{\psi}^T) = -\psi_a^* \quad . \tag{A.9}$$

A Majorana spinor is defined as self charge conjugate. Hence if

$$\psi^c = \psi = -\psi^+ \tag{A.10}$$

then ψ is a Majorana spinor (note $\bar{\psi} = \gamma^o \psi$ in this case). The Grassmann parameters θ_a of superspace are 2 component Majorana spinors. (Note $\theta_a \theta_b = \frac{1}{2} \gamma_{ab}^o \, \bar{\theta}\theta$) .

Differentiation with respect to the anti-commuting parameters θ_a is defined by

$$\phi(\theta + \delta\theta) = \phi(\theta) + \overline{\delta\theta} \frac{\partial}{\partial\bar{\theta}} \phi(\theta) \ ,$$

so that

$$\frac{\partial\theta_a}{\partial\bar{\theta}_b} = \gamma_{ab}^o$$

and

$$\frac{\partial\bar{\theta}_a}{\partial\bar{\theta}_b} = \delta_{ab} \quad . \tag{A.11}$$

Integration is defined so that it is translationally invariant

$$\int d\theta_a \, f(\theta) = \int d\theta_a \, f(\theta + \xi_a) .$$

Thus

$$\int d\theta_a = 0 \tag{A.12}$$

and

$$\int d\theta_a \theta_b = \frac{\partial}{\partial\bar{\theta}_a} \theta_b = -\gamma_{ab}^o \ .$$

A scalar integration is made by use of the inner product

$$\int d^2\theta \equiv \int d\bar{\theta}_a d\theta_a \equiv \frac{\partial}{\partial\bar{\theta}_a} \gamma^o_{ab} \frac{\partial}{\partial\bar{\theta}_b} \quad . \tag{A.13}$$

Spinor covariant derivatives with respect to supersymmetry transformations can be defined as

$$\begin{aligned} D_a &\equiv [\frac{\partial}{\partial\bar{\theta}_a} - i(\gamma^\mu\theta)_a\partial_\mu] \\ \bar{D}_a &\equiv \gamma^o_{ab}D_b \quad . \end{aligned} \tag{A.14}$$

These anti-commute to give

$$\{D_a, \bar{D}_b\} = 2i\not{\partial}_{ab} , \tag{A.15}$$

where

$$\not{\partial}_{ab} = \gamma^\mu_{ab}\partial_\mu \quad .$$

In addition they obey the following useful identities:

$$\begin{aligned} &1) \quad \bar{D}_a D_b = \frac{1}{2}\delta_{ab}\bar{D}D + i\not{\partial}_{ba} \\ &2) \quad \bar{D}DD_a = -2i(\not{\partial}D)_a \\ &3) \quad \bar{D}\gamma_5 D = 0 = \bar{D}\sigma^{\mu\nu} D \\ &4) \quad \bar{D}\gamma^\mu D = 2i\partial^\mu \\ &5) \quad \bar{D}\gamma_5\gamma^\mu D = 2i\,\varepsilon^{\mu\nu}\partial_\nu \\ &6) \quad \bar{\theta}\gamma^\mu D = -\bar{D}\gamma^\mu\theta \quad . \end{aligned} \tag{A.16}$$

Thus supersymmetry invariant integration reduces to

$$\int d^2x\, d^2\theta = -\int d^2x\bar{D}D \quad . \tag{A17}$$

Functional differentiation for superfields can be defined as

$$\frac{\delta\phi(x_1, \theta_1)}{\delta\phi(x_2, \theta_2)} = \delta(1,2) , \tag{A.18}$$

where

$$\delta\,(1,2) \equiv \frac{1}{4}\,\bar{\theta}_{12}\theta_{12}\;\delta^2(x_1 - x_2) \quad ,$$

with

$$\theta_{12} = \theta_1 - \theta_2 \tag{A.19}$$

and

$$\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 \quad .$$

Then

$$\int d^2x_2\; d^2\theta_2\; \delta(1,2)\; \phi(x_2,\theta_2) = \phi(x_1,\theta_1) \ . \tag{A.20}$$

Parity transformations on superfields are given by

$$\begin{aligned} P^{-1}\,\phi(x,\theta)P &= \eta\phi(x_p,\,\theta_p) \\ P^{-1}\,\phi_a(x,\theta)P &= -\eta\; i\gamma^o_{ab}\;\phi_b(x_p,\,\theta_p) \ , \end{aligned} \tag{A.21}$$

where η is the intrinsic parity and the parity transformed coordinates are

$$\begin{aligned} x^\mu_p &= (x^o,\, -x^1) \ , \\ \theta_{pa} &= +\, i\gamma^o_{ab}\;\theta_b \quad , \\ \text{and} \qquad \bar{\theta}_{pa} &= -i\;\bar{\theta}_b\;\gamma^o_{ba} \quad . \end{aligned} \tag{A.22}$$

Appendix B

The Superconformal algebra

We consider the grading of the O(4) two dimensional space-time conformal algebra. The procedure differs from the four dimensional case where we must append a chiral U(1) symmetry to the SU(2,2) conformal algebra so that the associated SU(2,2) $\otimes$ U(1) group is a symmetric subgroup of SU(2,3) and hence can be graded. No such additional symmetry is needed in two dimensions since the O(4) conformal algebra is already embedded as a symmetric subalgebra in the superconformal O(5) algebra.

The generators for the conformal algebra are the usual conformal generators: P_μ, $M_{\mu\nu}$, D, and K_μ generating space-time translations, Lorentz transformations, dilatation transformations, and special conformal transformations, respectively. The grading representation is carried by the two component Majorana, Grassmann spinor generators Q_a, a = 1,2, for translational supersymmetry and S_a, a = 1,2, for conformal supersymmetry. The graded Lie algebra these generators satisfy is known as the superconformal algebra. The conformal Lie algebra commutation relations are as usual

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda})$$

$$[M_{\mu\nu}, K_\lambda] = i(K_\mu g_{\nu\lambda} - K_\nu g_{\mu\lambda})$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma})\,(=0)$$

$$[D,P_\mu] = -iP_\mu$$
$$[D,K_\mu] = +iK_\mu$$
$$[P_\mu, K_\nu] = 2i(g_{\mu\nu}D - M_{\mu\nu})$$
$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = [D, M_{\mu\nu}] = [D,D] = 0 \; . \tag{B.1}$$

The commutation relations between the spinor generators and the conformal generators yield the grading representation of the conformal algebra

$$[P^\mu, Q_a] = 0 \qquad [P^\mu, S_a] = \gamma^\mu_{ab}Q_b$$
$$[M^{\mu\nu}, Q_a] = -\frac{i}{2}\sigma^{\mu\nu}_{ab}Q_b \qquad [M^{\mu\nu}, S_a] = -\frac{i}{2}\sigma^{\mu\nu}_{ab}S_b$$
$$[D,Q_a] = -\frac{i}{2}Q_a \qquad [D,S_a] = +\frac{i}{2}S_a$$
$$[K^\mu, Q_a] = \gamma^\mu_{ab}S_b \qquad [K^\mu, S_a] = 0 \; . \tag{B.2}$$

The anti-commutation relations among the spinor generators are

$$\{Q_a,Q_b\} = -2(\gamma^\mu\gamma^o)_{ab}P_\mu$$
$$\{S_a,S_b\} = -2(\gamma^\mu\gamma^o)_{ab}K_\mu$$
$$\{Q_a,S_b\} = -2i\gamma^o_{ab}D - i(\gamma^o\sigma^{\mu\nu})_{ab}M_{\mu\nu} \; . \tag{B,3}$$

These generators can be represented by linear superspace differential operators acting on superfields ϕ. Classically this representation gives the intrinsic variation of the classical superfields ϕ as discussed in Section 2. In quantum field theory the generators are operators whose commutation relations with the quantum fields ϕ give the above mentioned representation, i.e. $[G, \phi] = -i\hat{\delta}^G\phi$, where G is the quantum symmetry generator and $\hat{\delta}^G$ the linear differential operator. The linear differential operators are denoted by $\hat{\delta}^G$ where $G \epsilon \{P^\mu, M^{\mu\nu}, D, K^\mu, Q_a, S_a\}$. Listing these, we have

$$
\begin{array}{lcl}
G & : & \hat{\delta}^G \\
\hline
P_\mu & : & \hat{\delta}^P_\mu = \partial_\mu \\
M_{\mu\nu} & : & \hat{\delta}^M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + \frac{1}{2}\bar{\theta}\,\sigma_{\mu\nu}\frac{\partial}{\partial\bar{\theta}} \\
D & : & \hat{\delta}^D = d + x^\lambda \partial_\lambda + \frac{1}{2}\bar{\theta}\frac{\partial}{\partial\bar{\theta}} \\
K_\mu & : & \hat{\delta}^K_\mu = 2x_\mu \hat{\delta}^D + 2x^\nu \hat{\delta}^M_{\mu\nu} + x^2 \partial_\mu - 2x_\mu x^\lambda \partial_\lambda \\
Q_a & : & \hat{\delta}^Q_a = \frac{\partial}{\partial\bar{\theta}_a} + i(\not{\partial}\theta)_a \\
S_a & : & \hat{\delta}^S_a = 2d\theta_a - i(\not{x}\hat{\delta}^Q)_a ,
\end{array}
\tag{B.4}
$$

where d is the scale dimension of the superfield on which $\hat{\delta}^G$ is acting.

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