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THE QUANTUM INVERSE METHOD AND GREEN'S FUNCTIONS  
FOR COMPLETELY INTEGRABLE FIELD THEORIES\*

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## I. INTRODUCTION

The development of the quantum inverse method<sup>1-6</sup> has provided new insight into the structure of solvable models in quantum field theory and statistical mechanics. It places the theory of completely integrable quantum systems in a unified framework and provides a powerful method for studying these systems. In this series of lectures, I will review some of these developments with particular emphasis on the study of Green's functions for integrable field theories. The approach to Green's functions which I will describe has been developed in collaboration with Dennis Creamer and David Wilkinson.<sup>6-9</sup> So far it has only been applied to the case of the nonlinear Schrödinger model, but it is reasonable to suspect that similar techniques can be applied to other models.

I'll begin in Section II by reviewing the direct scattering transform<sup>1-5</sup> by which a certain set of "scattering data" operators are defined as functionals of the local fields. I'll describe the connection between the direct transform and the more traditional Bethe's ansatz methods and briefly mention the relationship with transfer matrices in lattice statistical models. The treatment of Green's functions is built upon the inverse (Gel'fand-Levitan) transform by which the local fields are written as operator functionals of the scattering data. In Section III I'll review the derivation of the quantum Gel'fand-Levitan transform<sup>6</sup> for the nonlinear Schrödinger model and discuss some of its properties. Section IV sets up the general formalism for studying Green's functions via the Gel'fand-Levitan transform.<sup>9</sup> In Section V we'll use this formalism to study the strong coupling ( $c \rightarrow \infty$ ) limit of the two-point function. Finally, in Section VI I review the analysis of the  $c = \infty$  two-point function by Jimbo,

Miwa, Mori, and Sato<sup>12</sup> and show that the first two terms in a strong coupling (1/c) expansion can be expressed in closed form in terms of Painlevé functions.<sup>9,12</sup>

## II. BETHE'S ANSATZ AND THE DIRECT SCATTERING TRANSFORM

### The Nonlinear Schrödinger Model

The case we'll be considering is the nonlinear Schrödinger model, defined by the Hamiltonian

$$H = \int \left[ \frac{\partial \phi^* \partial \phi}{\partial x \partial x} + c \phi^* \phi^* \phi \phi \right] dx \quad (2.1)$$

where  $\phi(x)$  is a nonrelativistic boson field with equal time commutation relations

$$[\phi(x), \phi^*(y)] = \delta(x - y) \quad (2.2)$$

The second term in H corresponds to a two-body delta-function potential. We'll consider the repulsive case  $c > 0$ , for which the problem of interest is to determine the spectrum and Green's functions for a finite density ground state  $\langle \phi^* \phi \rangle \neq 0$ . This is analogous to the problem faced in relativistic models like sine Gordon/massive Thirring, where the physical vacuum is a many-body Bethe's ansatz state.

Before introducing the quantum inverse method, I'll review the Bethe ansatz approach to (2.1). In this approach we write down many-body states

$$|\Psi\rangle = \int dx_1 \dots dx_N \psi(x_1 \dots x_N) \phi^*(x_1) \dots \phi^*(x_N) |0\rangle \quad (2.3)$$

and try to choose the wave function  $\psi$  so that  $|\Psi\rangle$  is an exact eigenstate of  $H$ . The correct wave functions  $\psi$  have the characteristic Bethe ansatz form which I'll now describe. Consider first the two-particle state,

$$|\Psi(k_1, k_2)\rangle = \int dx_1 dx_2 e^{i(k_1 x_1 + k_2 x_2)} \left\{ \theta(x_1 - x_2) + S(k_{21}) \theta(x_2 - x_1) \right\} \phi^*(x_1) \phi^*(x_2) |0\rangle \quad (2.4)$$

where  $k_{21} \equiv k_2 - k_1$  and

$$S(k) = \frac{k - ic}{k + ic} \quad (2.5)$$

is the two-body phase shift. The fact that (2.4) is an eigenstate of  $H$  may be shown directly by applying the operator  $H$  and using integration by parts to bring the kinetic energy derivative  $-\partial^2/\partial x^2$  onto the two-body wave function. This gives

$$H|\Psi(k_1, k_2)\rangle = (k_1^2 + k_2^2)|\Psi(k_1, k_2)\rangle \quad (2.6)$$

In the derivation of this result, there is a leftover term proportional to  $\delta(x_1 - x_2)$  coming from kinetic energy derivatives acting on the step functions in (2.4). This term is exactly cancelled by the  $\delta$ -function interaction term. We can also write the two-body state in a different form by changing the normalization

$$\begin{aligned} |\Phi(k_1 k_2)\rangle &\equiv \left( 1 + \frac{ic}{k_{21}} \right) |\Psi(k_1 k_2)\rangle \\ &= \int dx_1 dx_2 e^{i(k_1 x_1 + k_2 x_2)} \left\{ 1 - \frac{ic}{k_{21}} \epsilon(x_{21}) \right\} \phi^*(x_1) \phi^*(x_2) |0\rangle \quad (2.7) \end{aligned}$$

where  $x_{21} \equiv x_2 - x_1$ .

The Bethe ansatz<sup>13</sup> for this model consists of a generalization of (2.4) to an arbitrary number of particles  $N$  such that

$$H|\Psi(k_1 \dots k_N)\rangle = \left( \sum_{i=1}^N k_i^2 \right) |\Psi(k_1 \dots k_N)\rangle \quad (2.8)$$

Just as in the two-body case (2.4) the wavefunction can be written as a sum over  $N!$  orderings of the coordinates  $x_{P_1} > x_{P_2} > \dots > x_{P_N}$  where  $(P_1, P_2, \dots, P_N)$  is some permutation of  $(1, 2, \dots, N)$ ,

$$\begin{aligned} |\Psi(k_1 \dots k_N)\rangle = & \int dx_1 \dots dx_N \exp \left( i \sum_{i=1}^N k_i x_i \right) \left\{ \sum_{P \in S_N} \theta(x_{P_1} > \dots > x_{P_N}) \right. \\ & \left. \times \prod_{\substack{i < j \\ P_i > P_j}} S(k_{P_i P_j}) \right\} \phi^*(x_1) \dots \phi^*(x_N) |0\rangle. \end{aligned} \quad (2.9a)$$

The unnormalized eigenstates analogous to (2.7) are written

$$|\Phi(k_1 \dots k_N)\rangle = \int dx_1 \dots dx_N \exp \left( i \sum_{i=1}^N k_i x_i \right) \prod_{i < j} \left\{ 1 - \frac{ic}{k_i - k_j} \varepsilon(x_i - x_j) \right\} \phi^*(x_1) \dots \phi^*(x_N) |0\rangle \quad (2.9b)$$

### Spectral properties of finite density states

The finite density system is traditionally studied by placing an  $N$ -body system in a periodic box of length  $L$  and letting  $N \rightarrow \infty$  with  $N/L = \text{density}$  fixed. The wave function

$$\Psi(x_1 \dots x_N) = \langle 0 | \phi(x_1) \dots \phi(x_N) | \Phi(k_1 \dots k_N) \rangle \quad (2.10)$$

is required to satisfy periodic boundary conditions (PBC's)

$$\psi(-L/2, x_2 \dots x_N) = \psi(L/2, x_2 \dots x_N) \quad (2.11)$$

which gives

$$e^{ik_i L} = \prod_{j \neq i} S(k_{ji}) \quad (2.12)$$

It is convenient to take the log of the PBC's

$$k_i L = \sum_{j \neq i} \theta(k_j - k_i) + 2\pi n_i \quad (2.13)$$

where

$$\theta(k) = -i \log S(k) \quad (2.14)$$

The choice of  $n_i$ 's in (2.13) is related to the choice of branch for the log in (2.14). The physical phase shift has a discontinuity of  $2\pi$  at  $k = 0$ . This phase shift vanishes as  $c \rightarrow 0$  and the description of the ground state of the system is bosonic, i.e.  $n_i = 0$  for all  $i$ . Instead it is convenient and conventional to choose the phase shift which is continuous at  $k = 0$  for finite  $c$  and becomes a step function as  $c \rightarrow 0$ . For this choice, the ground state has a fermionic description,  $n_{i+1} - n_i = 1$ .

By subtracting adjacent PBC's in the ground state we obtain

$$k_{i+1} - k_i = \frac{1}{L} \sum_j \left[ \theta(k_j - k_{i+1}) - \theta(k_j - k_i) \right] + \frac{2\pi}{L} \quad (2.15)$$

As  $L \rightarrow \infty$ ,  $N \rightarrow \infty$ ,  $N/L$  fixed

$$\rho(k_i) = \frac{1}{L(k_{i+1} - k_i)} \rightarrow \text{continuous function} \equiv \rho(k) \quad (2.16)$$

and the PBC's (2.15) reduce to an integral equation for the ground state density function

$$2\pi\rho(k) = 1 + \int_{-k_F}^{k_F} \Delta(k - k')\rho(k') dk' \quad (2.17)$$

where

$$\Delta(k) = \frac{\partial\theta(k)}{\partial k} = \frac{2c}{k^2 + c^2} \quad (2.18)$$

The ground state is a Fermi sea of closely packed modes between  $-k_F$  and  $k_F$ . Excited states are formed by removing modes from the sea and placing them above the surface, forming particle-hole pairs. The particle-hole spectrum was first worked out by Lieb.<sup>13</sup> In the formulation of Yang and Yang,<sup>14</sup> the spectrum is given by a single function  $\epsilon(k)$  which satisfies a linear integral equation

$$\epsilon(k) = k^2 - \mu + \int_{-k_F}^{k_F} \Delta(k - k')\epsilon(k') \frac{dk'}{2\pi} \quad (2.19)$$

where  $\mu$  is fixed by the requirement  $\epsilon(\pm k_F) = 0$ . The excitation energy of a particle at  $k_p$  and a hole at  $k_h$  is given by

$$E = \epsilon(k_p) - \epsilon(k_h) \quad (2.20)$$

In the Yang and Yang formulation, there is a similar excitation function  $\epsilon(k)$  at finite temperature, which satisfies a nonlinear equation

$$\epsilon(k) = k^2 - \mu - \frac{1}{\beta} \int \Delta(k - k') \log(1 + e^{-\beta\epsilon(k')}) \frac{dk'}{2\pi} \quad (2.21)$$

where  $\mu$  = chemical potential and  $\beta = 1/kT$ . This reduces to (2.19) for  $\beta \rightarrow \infty$ . The function  $\epsilon(k)$  also determines the equilibrium thermodynamics, e.g. the pressure of a gas as a function of  $\beta$  and  $\mu$  is

$$P = \frac{1}{\beta} \int \frac{dk}{2\pi} \log(1 + e^{-\beta\epsilon(k)}) \quad (2.22)$$

The function  $\epsilon(k)$  is of central importance in the model. (Similar functions can be constructed for other models, e.g. massive Thirring/sine Gordon.) It will reappear in the theory of Green's functions.

#### Quantum Inverse Method

In the classical inverse scattering method<sup>15-16</sup>, we solve the initial value problem for a nonlinear field equation by considering a linear "Lax pair":

$$\frac{\partial}{\partial x^\mu} \Psi(x, \zeta) = iQ_\mu(x, \zeta) \Psi(x, \zeta) \quad (2.23)$$

In the simplest applications,  $Q_\mu(x, \zeta)$  is a  $2 \times 2$  matrix which depends on the local field  $\phi(x, t)$ , and on an eigenvalue  $\zeta$ . If we think of the spatial component of the Lax pair as a time independent eigenvalue (scattering)

problem, the local field  $\phi(x)$  plays the role of a scattering potential, and (2.23) defines a one-to-one mapping between the field  $\phi(x)$  at a fixed time  $t$  and the scattering data associated with the linear eigenvalue problem. The key point is that, by judicious choice of the matrices  $Q_\mu(x, \zeta)$  we may interpret the original nonlinear equation of motion as the consistency (integrability) condition obtained by cross-differentiation of the Lax pair, which gives

$$F_{\mu\nu} = \partial_\nu Q_\mu - \partial_\mu Q_\nu + i[Q_\mu, Q_\nu] = 0 \quad . \quad (2.24)$$

With the particular choice

$$Q_0 = \begin{pmatrix} \frac{k^2}{2} - c\phi^*\phi & \sqrt{c}(k\phi + i\phi_x) \\ -\sqrt{c}(k\phi^* - i\phi_x^*) & \frac{k^2}{2} + c\phi^*\phi \end{pmatrix} \quad (2.25)$$

$$Q_1 = \begin{pmatrix} \frac{k}{2} & \sqrt{c}\phi \\ -\sqrt{c}\phi^* & -\frac{k}{2} \end{pmatrix} \quad (2.26)$$

then  $F_{\mu\nu} = 0$  becomes the nonlinear Schrödinger equation

$$i\partial_0\phi = -\partial_1^2\phi + c|\phi|^2\phi \quad . \quad (2.27)$$

From this result it follows that the scattering data  $a(k)$ ,  $b(k)$  (where  $1/a =$  transmission coefficient and  $b/a =$  reflection coefficient) have a trivial time dependence

$$a(k, t) = a(k, 0) \quad (2.28a)$$

$$b(k, t) = e^{-ik^2 t} b(k, 0) \quad (2.28b)$$

The inverse method solves the initial value problem much like Fourier transformation is used to solve a linear problem. The direct transform maps  $\phi(x) \rightarrow a(k), b(k)$  at time  $t = 0$ . The time evolution of  $a$  and  $b$  from  $t = 0$  to some later time  $t$  is given by (2.28). At time  $t$  we must perform an inverse transform which maps  $a(k, t), b(k, t)$  back into the field configuration  $\phi(x, t)$ . This last step is accomplished by the Gel'fand-Levitan equation.

In this section I'll discuss the quantum generalization of the direct transform, the significance of  $a(k)$  and  $b(k)$  as quantum operators, and the relationship with Bethe's ansatz. In the following section, I'll discuss the generalization of the Gel'fand-Levitan (inverse) transform, which is the centerpiece for the treatment of Green's functions in the remaining sections.

The quantum inverse method for the nonlinear Schrödinger model is based on a normal ordered operator version of the Zakharov-Shabat eigenvalue problem (2.23)

$$\frac{\partial}{\partial x} \Psi(x, k) = i : Q_1(x, k) \Psi(x, k) : \quad (2.29)$$

A particular solution is specified by choosing a boundary condition. Requiring  $\Psi(x_0, k) = I = \text{identity matrix}$ , we can write the solution to (2.29) formally as a path ordered exponential,

$$\Psi(x, k) = : P \exp i \int_{x_0}^x Q_1(y, k) dy : \quad (2.30)$$

The solution  $\Psi(x, k)$  is a nonlocal string functional of the field operator  $\phi(x)$ . If  $\phi(x) \rightarrow 0$  weakly as  $|x| \rightarrow \pm\infty$  we see that

$$\Psi(x, k) \xrightarrow{|x| \rightarrow \pm\infty} V(x, k) \times (\text{constant matrix}) \quad (2.31)$$

where

$$V(x, k) = \begin{pmatrix} e^{ikx/2} & 0 \\ 0 & e^{-ikx/2} \end{pmatrix} \quad (2.32)$$

The scattering data operators are defined by the asymptotic form of  $\Psi$ :

$$\begin{aligned} \mathcal{S}(k) &= \lim_{\substack{x \rightarrow \infty \\ x_0 \rightarrow -\infty}} V^{-1}(x, k) \Psi(x, k) V(x_0, k) \\ &= \begin{pmatrix} a(k) & b^*(k) \\ b(k) & a^*(k) \end{pmatrix} \end{aligned} \quad (2.33)$$

for real  $k$ . The central result of the quantum inverse method is a set of commutation relations among the scattering data operators. This is most elegantly derived by the method of Sklyanin,<sup>3</sup> which is patterned after earlier work of Baxter.<sup>17</sup> One uses the Zakharov-Shabat equation to derive  $4 \times 4$  matrix equations for the direct products  $H_{12}(x) \equiv \Psi(x, k_1) \otimes \Psi(x, k_2)$  and  $H_{21}(x) = \Psi(x, k_2) \otimes \Psi(x, k_1)$ . We get

$$\frac{\partial}{\partial x} H_{12} = i : \Gamma_{12} H_{12} : \quad (2.34)$$

$$\frac{\partial}{\partial x} H_{21} = i : \Gamma_{21} H_{21} : \quad (2.35)$$

where

$$\Gamma_{12} = Q(k_1) \otimes I + I \otimes Q(k_2) - ic \sigma^+ \otimes \sigma^- \quad (2.36)$$

The key observation is that the matrices  $\Gamma_{12}$  and  $\Gamma_{21}$  are equivalent under a c-number similarity transformation

$$\Gamma_{21} = \mathcal{R} \Gamma_{12} \mathcal{R}^{-1} \quad (2.37)$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.38)$$

with

$$\alpha = \frac{k_1 - k_2}{k_1 - k_2 - ic} \quad \beta = \frac{-ic}{k_1 - k_2 - ic} \quad (2.39)$$

This leads to the result that the direct products of the solutions are themselves related by

$$\Psi_2 \otimes \Psi_1 = \mathcal{R} [\Psi_1 \otimes \Psi_2] \mathcal{R}^{-1} \quad (2.40)$$

where the subscript denotes the eigenvalue.

Equation (2.40) gives a set of commutation relations among the elements of the solution matrices. At this point there are two somewhat different approaches we may follow to further investigate the model. Let me refer to these two possibilities as the finite volume approach and the infinite volume approach. In the finite volume approach we define the scattering data operators in a box by choosing  $x_0 = -L/2$  in (2.29) and defining

$$\Psi(L/2, k) = \begin{pmatrix} A(k) & C(k) \\ B(k) & D(k) \end{pmatrix} \equiv \mathcal{F}_L(k) \quad (2.41)$$

with the commutation relations

$$\left[ \mathcal{F}_L(k_2) \otimes \mathcal{F}_L(k_1) \right] \mathcal{R} = \mathcal{R} \left[ \mathcal{F}_L(k_1) \otimes \mathcal{F}_L(k_2) \right] \quad (2.42)$$

where  $\mathcal{R}$  is given by (2.38). By carefully taking the  $L \rightarrow \infty$  limit, we obtain a somewhat simpler infinite volume algebra

$$\left[ \mathcal{F}(k_2) \otimes \mathcal{F}(k_1) \right] \mathcal{R}_\infty = \mathcal{R}_\infty \left[ \mathcal{F}(k_1) \otimes \mathcal{F}(k_2) \right] \quad (2.43)$$

where

$$\mathcal{R}_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.44)$$

$$\alpha = \frac{k_1 - k_2}{k_1 - k_2 - ic} \quad \beta = \frac{k_1 - k_2 + ic}{k_1 - k_2} \quad (2.45)$$

In particular, we find

$$a(k)b(k') = \left(1 - \frac{ic}{k - k'}\right) b(k')a(k) \quad (2.46a)$$

$$a^*(k)b(k') = \left(1 + \frac{ic}{k - k'}\right) b(k')a^*(k) \quad (2.46b)$$

$$b^*(k)b(k') = \frac{(k - k')^2 + c^2}{(k - k')^2} b(k')b^*(k) + 2\pi a^*(k)a(k)\delta(k - k') \quad (2.46c)$$

$$[a, a^*] = [a, a] = [b, b] = 0 \quad (2.46d)$$

The commutators of  $a$  and  $b$  with the Hamiltonian may also be worked out,

$$[H, a(k)] = 0 \quad (2.47)$$

$$[H, b(k)] = k^2 b(k) \quad (2.48)$$

which is the quantum analog of (2.28). All of these results may be verified order by order using the normal ordered series expansions for  $a(k)$  and  $b(k)$ :

$$a(k) = 1 + c \int dx_1 dy_1 \theta(x_1 < y_1) e^{ik(x_1 - y_1)} \phi^*(x_1) \phi(y_1) + \dots \quad (2.49)$$

$$\begin{aligned} \frac{i}{\sqrt{c}} b(k) = & \int dx_1 e^{ikx_1} \phi^*(x_1) + c \int dx_1 dx_2 dy_1 \theta(x_1 < y_1 < x_2) e^{ik(x_1 + x_2 - y_1)} \\ & \times \phi^*(x_1) \phi^*(x_2) \phi(y_1) + \dots \quad (2.50) \end{aligned}$$

From (2.48) we see that the states

$$|\Phi(k_1 \dots k_N)\rangle = b(k_1) \dots b(k_N) |0\rangle \quad (2.51)$$

are exact eigenstates of  $H$ . From (2.50) it can be shown that the states (2.51) are precisely the unnormalized Bethe ansatz states (2.9b). The operator  $a(k)$  is diagonal on these states for all  $k$  and is the generator of an infinite number of conservation laws. In the infinite volume formalism, a particularly useful operator is the quantized reflection coefficient

$$R(k) = b(k) a^{-1}(k) \quad (2.52)$$

This operator and its conjugate obey a simple algebra

$$R(k)R(k') = S(k' - k)R(k')R(k) \quad (2.53)$$

$$R(k)R^\dagger(k') = S(k - k')R^\dagger(k')R(k) + 2\pi\delta(k - k') \quad (2.54)$$

where

$$S(k - k') = \frac{k - k' - ic}{k - k' + ic} = \text{2-body S-matrix} \quad (2.55)$$

States created by  $R^\dagger$ 's are also eigenstates of  $H$  but with a different normalization. They are in fact the properly normalized states  $|\Psi(k, \dots, k_N)\rangle$  defined in (2.9b). The  $R$  operators are of central importance in the theory of the inverse problem and Green's functions.

In the finite volume formalism it is also possible to construct Bethe's ansatz states, but this time the B-states diagonalize not  $A(k)$  but the trace of the monodromy matrix (2.41)

$$T(k) = \text{Tr } \mathcal{T}_L(k) = A(k) + D(k) \quad . \quad (2.56)$$

This quantity is precisely analogous to the transfer matrix in two-dimensional lattice statistics models. The states created by B's are not automatically eigenstates of  $T(k)$  as they are in the infinite volume case. Instead, a state  $B(k_1) \dots B(k_N) |0\rangle$  is an eigenstate of  $T(k)$  only if  $k_1, \dots, k_N$  satisfy periodic boundary conditions. In this approach, the PBC's follow directly from the algebra of the operators A, B, C, and D. On the other hand, in the finite volume formalism the R-operators do not have nice properties, and the Gel'fand-Levitan transform has not yet been constructed. For the remainder of these lectures we will use the infinite volume approach to study Green's functions. This will result in no loss of generality, since, as we will see, all the finite density results of Lieb and Liniger and Yang and Yang can be derived in this approach by studying finite temperature Green's functions.

Let me conclude this section with some remarks on the quantum inverse method for lattice models and its deep connection with Baxter's method for solving the eight-vertex model. This connection has been extensively developed by Faddeev and coworkers.<sup>10,18</sup> It leads to an elegant and general formulation of quantum integrability based on the "Yang-Baxter relation," which is a generalization of the similarity relation (2.37). Essentially, one views the Jost solutions as strings of vertices of the form

$$\Psi_m(k) = L_1(k)L_2(k)\dots L_m(k) \quad (2.57)$$

where  $L_j(k)$  is a matrix of local operators defined on lattice site  $j$ . Equation (2.57) is precisely analogous to the path-ordered exponential solution of the Zakharov-Shabat equation, Eq. (2.30). The Yang-Baxter relation is

$$\mathcal{R} \left[ L_n(k) \otimes L_n(k') \right] = \left[ L_n(k') \otimes L_n(k) \right] \mathcal{R} \quad (2.58)$$

For the nonlinear Schrödinger case,  $L_n$  is a  $2 \times 2$  matrix of field operators and  $\mathcal{R}$  is just (2.38). Equation (2.58) leads directly to the results (2.40) and (2.42). For further discussions of the Yang-Baxter relation and how it arises in various models I refer you to the literature and to the paper of Kulish and Sklyanin in these proceedings.

### III. THE OPERATOR GEL'FAND-LEVITAN EQUATION

The Gel'fand-Levitan equation is a dispersion relation for a Jost solution to the Zakharov-Shabat eigenvalue problem,

$$\left( i \frac{\partial}{\partial x} + \frac{1}{2} \zeta \right) \Psi_1 = -\sqrt{c} \Psi_2 \phi \quad (3.1a)$$

$$\left( i \frac{\partial}{\partial x} - \frac{1}{2} \zeta \right) \Psi_2 = \sqrt{c} \phi^* \Psi_1 \quad (3.1b)$$

Consider two column vector solutions to (3.1) defined by the boundary conditions

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \xrightarrow{x \rightarrow -\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\zeta x/2} \quad (3.2)$$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \xrightarrow{x \rightarrow +\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\zeta x/2} \quad (3.3)$$

From these boundary conditions it is easily shown that both  $\psi$  and  $\chi$  admit analytic continuation into the lower half  $\zeta$ -plane. Here analyticity of an operator is taken to be equivalent to analyticity of all its physical matrix elements. We will also need the conjugate solutions

$$\tilde{\psi}(x, \zeta) = \begin{pmatrix} \psi_2^*(x, \zeta^*) \\ \psi_1^*(x, \zeta^*) \end{pmatrix} \quad \tilde{\chi} = \begin{pmatrix} \chi_2^*(x, \zeta^*) \\ \chi_1^*(x, \zeta^*) \end{pmatrix} \quad (3.4)$$

which are analytic in the upper-half  $\zeta$ -plane. The Gel'fand-Levitan equation is a dispersion relation for an analytic function  $\Phi(x, \zeta)$  which is constructed from these Jost solutions.

Classical case:

In the classical theory, for  $\zeta = k = \text{real}$  the Jost solution  $\psi$  can be written as a linear combination of  $\chi$  and  $\tilde{\chi}$ ,

$$\psi = a\tilde{\chi} + b\chi, \quad (3.5)$$

where  $a$  and  $b$  are the scattering coefficients defined previously. Equation (3.5) may be verified by taking the Wronskian of both sides with  $\chi$  and  $\tilde{\chi}$  and using

$$\psi_1 \chi_2 - \psi_2 \chi_1 = a \quad (3.6)$$

$$\psi_2 \tilde{\chi}_1 - \psi_1 \tilde{\chi}_2 = b \quad (3.7)$$

Thus, along the real axis, the function  $\tilde{\chi}$  which is analytic in the lower half-plane is related to the function  $\psi a^{-1}$  which is analytic in the upper half-plane by

$$\psi a^{-1} = \tilde{\chi} - i\sqrt{cR}^* \tilde{\chi} \quad (3.8)$$

(Note:  $a$  has no zeroes in the lower half-plane for repulsive coupling  $c > 0$ .) Equation (3.8) suggests that we define a function

$$\Phi(x, \zeta) = \tilde{\chi}(x, \zeta) e^{-i\zeta x/2} \quad \text{for } \text{Im}\zeta > 0 \quad (3.9a)$$

$$= \psi(x, \zeta) a^{-1}(\zeta) e^{-i\zeta x/2} \quad \text{for } \text{Im}\zeta < 0 \quad (3.9b)$$

This function has a discontinuity proportional to the reflection coefficient

$$\text{Disc } \Phi = i\sqrt{cR}^* \tilde{\chi} e^{-i\zeta x/2} \quad (3.10)$$

Also, from the Zakharov-Shabat equation we have

$$\Phi \rightarrow 1 \quad \text{as } \zeta \rightarrow \infty \quad (3.11)$$

Thus,  $\Phi$  can be reconstructed from its discontinuity,

$$\Phi(x, \zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{c}}{2\pi} \int dk \frac{R^*(k) \tilde{\chi}(x, k) e^{-ikx/2}}{k - \zeta} \quad (3.12)$$

Evaluating just above the real axis, we obtain a coupled pair of integral equations,

$$\tilde{\chi}(x, k)e^{-ikx/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} dk \frac{R^*(k')\chi(x, k')e^{-ik'x/2}}{k' - k - i\epsilon} \quad (3.13)$$

Quantum case:

In the quantum theory, the equation (3.8) which motivated the choice (3.9) for the  $\Phi$  function is not a valid operator relation. Instead we define a function

$$g(x, k) \equiv \tilde{\chi}(x, k) - i\sqrt{c}R^*(k)\chi(x, k) \quad (3.14)$$

and study the analytic continuation of  $g$  into the lower half-plane. From the Zakharov-Shabat equation, we find that  $g$  satisfies

$$(i \frac{\partial}{\partial x} + \frac{1}{2} k) g_1 = -\sqrt{c}g_2\phi \quad (3.15)$$

$$(i \frac{\partial}{\partial x} - \frac{1}{2} k) g_2 = \sqrt{c}\phi^* g_1 - ic[R^*(k), \phi^*(x)]\chi_1 \quad (3.16)$$

Note that the last term in (3.16) arises from quantum ordering. Without it we would conclude that  $g = \psi a^{-1}$  as in the classical case. But the commutator  $[R^*(k), \phi^*(x)]$  can be evaluated by writing  $R^* = ba^{-1}$  and using Wronskian relations for  $b$  and  $a^{-1}$ . This gives

$$\begin{aligned} [R^*(k), \phi^*(x)] &= (\tilde{\chi}_2 - i\sqrt{c}R^*\chi_2)\psi_2 a^{-1} \\ &= g_2\psi_2 a^{-1} \end{aligned} \quad (3.17)$$

Thus, the Z-S equation becomes a differential equation for  $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$  with coefficients which are analytic in the lower half  $k$  plane. The asymptotic form of  $g$  also has simple analytic properties. For  $x \rightarrow \infty$  we have

$$g(x, k) \rightarrow \begin{pmatrix} \tilde{a}(k) \\ 0 \end{pmatrix} e^{ikx/2} \quad (3.18)$$

where

$$\tilde{a}(k) = a^*(k) - cR^*(k)a^*(k)R(k) \quad (3.19)$$

$\tilde{a}(k)$  is diagonal on the Bethe ansatz states, and we may verify that it is analytic in the lower half  $k$ -plane by studying its eigenvalues. On a one particle state we get

$$\begin{aligned} \tilde{a}(k) |k_1\rangle &= \left[ 1 + \frac{ic}{k - k_1 + i\epsilon} - 2\pi c \delta(k - k_1) \right] |k_2\rangle \\ &= \left[ 1 + \frac{ic}{k - k_1 - i\epsilon} \right] |k_2\rangle \end{aligned} \quad (3.20)$$

More generally, the  $\delta$ -function terms in the eigenvalue of  $\tilde{a}(k)$  simply change the signs of all the  $i\epsilon$ 's,

$$\tilde{a}(k) |k_1 \dots k_N\rangle = \prod_{i=1}^N \left[ 1 + \frac{ic}{k - k_i - i\epsilon} \right] |k_1 \dots k_N\rangle \quad (3.21)$$

Thus, a function

$$\Phi(x, \zeta) = \tilde{X}(x, \zeta) e^{-i\zeta x/2} \quad \text{Im } \zeta > 0 \quad (3.22)$$

$$= g(x, \zeta) e^{-i\zeta x/2} \quad \text{Im } \zeta > 0 \quad (3.23)$$

is analytic in the full cut  $\zeta$ -plane with

$$\text{Disc } \Phi = i\sqrt{c} R^* X \quad (3.24)$$

and

$$\Phi \sim 1 + O\left(\frac{1}{\zeta}\right) \quad \text{as } \zeta \rightarrow \infty \quad (3.25)$$

This gives a pair of coupled integral equations for the operator Jost solutions  $X_1$  and  $X_2^*$ :

$$X_2^*(x, k) e^{-ikx/2} = 1 + \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} dk' \frac{R^*(k') X_1(x, k') e^{-ik'x/2}}{k' - k - i\epsilon} \quad (3.26a)$$

$$X_1(x, k) e^{ikx/2} = \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} dk' \frac{X_2^*(x, k') R(k') e^{ik'x/2}}{k' - k + i\epsilon} \quad (3.26b)$$

Solving these integral equations (e.g. by iteration) gives  $X_1$  and  $X_2^*$  as operator functionals of  $R$  and  $R^*$ :

$$\begin{aligned} X_1(x, k) e^{ikx/2} = & -\sqrt{c} \left\{ \int \frac{dk_0}{2\pi} \frac{e^{ik_0 x}}{k - k_0 - i\epsilon} R(k_0) \right. \\ & - c \int \frac{dp_1}{2\pi} \frac{dk_0}{2\pi} \frac{dk_1}{2\pi} \frac{e^{i(k_0 + k_1 - p_1)x}}{(k - k_0 - i\epsilon)(p_1 - k_0 - i\epsilon)(p_1 - k_1 - i\epsilon)} R^*(p_1) R(k_1) R(k_0) \\ & \left. + \dots \right\} \quad (3.27) \end{aligned}$$

$$\chi_2^*(x, k)e^{-ikx/2} = 1 - c \int \frac{dp_1}{2\pi} \frac{dk_1}{2\pi} \frac{e^{i(k_1 - p_1)x}}{(p_1 - k - i\epsilon)(p_1 - k_1 - i\epsilon)} R^+(p_1) R(k_1) \quad (3.28)$$

+ ...

The final step in the Gel'fand-Levitan procedure is to recover the local field operator  $\phi(x)$  by taking the  $k \rightarrow \infty$  limit of the Jost solution,

$$\chi_1(x, k)e^{ikx/2} \xrightarrow{k \rightarrow \infty} -\frac{\sqrt{c}}{k} \phi(x) + O(1/k^2) \quad (3.29)$$

The field is thus written as an infinite series,

$$\phi(x) = \sum_{n=0}^{\infty} \phi^{(n)}(x) \quad (3.30)$$

where

$$\phi^{(n)}(x) = (-c)^n \int \left( \prod_1^n \frac{dp_i}{2\pi} \right) \left( \prod_0^n \frac{dk_i}{2\pi} \right) \frac{e^{i(\sum_0^n k_i - \sum_1^n p_i)x}}{\prod_{i=1}^n [(p_i - k_{i-1} - i\epsilon)(p_i - k_i - i\epsilon)]} \times R^*(p_1) \dots R^*(p_n) R(k_n) \dots R(k_0) \quad (3.31)$$

The asymptotic expression for the other component of the Jost solution  $\chi_2$  yields a series for the charge density  $j_0(x) = \phi^*(x)\phi(x)$ .

Gel'fand-Levitan series as a generalized Jordan-Wigner transformation

The Gel'fand-Levitan transform (3.31) has a very interesting structure which can be studied term by term. Perhaps I should say at the outset that I'm not entirely satisfied with the style of analysis that I'll outline in this and subsequent sections. It would be nice if there were a more elegant way of studying Green's functions than term-by-term analysis

of series expansions. My general feeling is that a better approach would make more direct use of the Gel'fand-Levitan integral equation and the Jost solutions, but such an approach has not yet been devised. The situation is reminiscent of the direct problem, where the properties of the  $a$  and  $b$  operators were first discovered by studying their series expansions and then subsequently derived by more elegant means. I hope that this history will repeat itself for the inverse problem, but for now I must rely on the term-by-term approach.

The lowest order term in (3.30) is just the Fourier transform of the reflection coefficient

$$\phi^{(0)}(x) = \int \frac{dk_0}{2\pi} e^{ik_0 x} R(k_0) \equiv \tilde{R}(x). \quad (3.32)$$

The second term is

$$\phi^{(1)}(x) = \int \frac{dp_1}{2\pi} \frac{dk_0}{2\pi} \frac{dk_1}{2\pi} e^{i(k_0+k_1-p_1)x} \frac{(-c)R^*(p_1)R(k_1)R(k_0)}{(p_1-k_0)(p_1-k_1)}. \quad (3.33)$$

Hereafter, momentum denominators will be understood to have infinitesimal negative imaginary parts. By writing the denominator in (3.33) as

$$\frac{1}{(p_1-k_0)(p_1-k_1)} = \frac{1}{k_{10}} \left[ \frac{1}{p_1-k_0} - \frac{1}{p_1-k_1} \right], \quad (3.34)$$

making the change of variables  $k_1 \rightarrow k_0$  in the first term and using the commutation relation (2.53) we can replace the integrand in (3.33) by

$$\frac{(-c)}{(p_1-k_0)(p_1-k_1)} \rightarrow \frac{(-2c)}{(p_1-k_1)(k_{10}+ic)} = \frac{(-1)}{p_1-k_1} [S(k_{10}) - 1]. \quad (3.35)$$

Equation (3.33) can then be written very simply in coordinate space

$$\phi^{(1)}(x) = \int_x^\infty dz [\tilde{R}^*(z)\tilde{R}(x)\tilde{R}(z) - \tilde{R}^*(z)\tilde{R}(z)\tilde{R}(x)]. \quad (3.36)$$

To understand the general term  $\phi^{(n)}$  you should think of (3.36) as being obtained from  $\phi^{(0)}(x)$  [i.e.,  $\tilde{R}(x)$ ] by inserting the operators  $\tilde{R}^*(z)$  and  $\tilde{R}(z)$  in two different ways and then integrating over  $z$ . The first term in (3.36) is an "outside" insertion and appears with a plus sign, while the second term is an "inside" insertion and has a minus sign. This pattern repeats itself in a straightforward way for the higher terms in the series, with each term  $\phi^{(n)}$  being obtained from the previous term  $\phi^{(n-1)}$  by an "outside minus inside" insertion of  $\tilde{R}^*(z_n)$  and  $\tilde{R}(z_n)$ , with  $z_n$  integrated from  $z_{n-1}$  to  $\infty$ . For example, the next term is

$$\begin{aligned} \phi^2(x) = & \int_x^\infty dz_1 \int_{z_1}^\infty dz_2 \{ [\tilde{R}^*(z_2)\tilde{R}^*(z_1)\tilde{R}(x)\tilde{R}(z_1)\tilde{R}(z_2) \\ & - \tilde{R}^*(z_1)\tilde{R}^*(z_2)\tilde{R}(z_2)\tilde{R}(x)\tilde{R}(z_1)] \\ & - [\tilde{R}^*(z_2)\tilde{R}^*(z_1)\tilde{R}(z_1)\tilde{R}(x)\tilde{R}(z_2) - \tilde{R}^*(z_1)\tilde{R}^*(z_2)\tilde{R}(z_2)\tilde{R}(z_1)\tilde{R}(x)] \}, \end{aligned} \quad (3.37)$$

where the first two terms in (3.37) are obtained from the first term in (3.36) and the second two terms of (3.37) are obtained from the second term in (3.36). The general term can be written most easily in momentum space,

$$\begin{aligned} \phi^{(n)}(x) = & \int \prod_1^n \frac{dp_1}{2\pi} \prod_0^n \frac{dk_1}{2\pi} e^{ik_0 x} \int dz_1 \dots dz_n \theta(x < z_1 < \dots < z_n) \\ & \times \prod_{i=1}^n e^{i(k_i - p_i)z_i} (S_{10}^{-1}) (S_{20} S_{21}^{-1}) \dots (S_{n0} S_{n1}^{-1} \dots S_{n,n-1}^{-1}) \\ & \times R^*(p_1) \dots R^*(p_n) R(k_n) \dots R(k_0), \end{aligned} \quad (3.38)$$

where  $S^{ij} \equiv S(p_{ij})$  and  $S_{ij} \equiv S(k_{ij})$ . The factors  $(S \dots S-1)$  in (3.38) are the momentum space version of "outside minus inside" insertions. I will not give a complete derivation of (3.38) here. The only derivation I know involves a rather lengthy combinatorial analysis which is most easily handled by graphical techniques. It turns out that the Gel'fand-Levitan series for  $\phi(x)$  can be given a convenient graphical interpretation in terms of "factorized graphs," which were developed for this model several years ago.<sup>19</sup> This graphical formalism is very useful for handling the combinatorics involved in deriving formulas like (3.38), but it would take us too far afield to describe it here.

The form of the Gel'fand-Levitan series provided by Eq. (3.38) is particularly well suited to studying Green's functions in the strong coupling ( $c \rightarrow \infty$ ) limit. In fact, in the limit  $c \rightarrow \infty$ , the Gel'fand-Levitan transform reduces to the more familiar Jordan-Wigner transformation.<sup>7</sup> For  $c \rightarrow \infty$ ,  $S \rightarrow 1$ , and Eq. (3.38) reduces to

$$\phi^{(n)}(x) = \frac{(-2)^N}{N!} N_R \left[ \int_x^\infty dz \tilde{R}^*(z) \tilde{R}(z) \right] \tilde{R}(x), \quad (3.39)$$

where  $N_R$  specifies normal ordering with respect to the R operators. Note that the algebra of R operators (2.53)-(2.54) reduces to canonical anti-commutation relations, and thus  $\tilde{R}(x)$  is a local fermion field. The transform (3.30) reduces to

$$\phi(x) = N_R \exp \left[ -2 \int_x^\infty \tilde{R}^*(z) \tilde{R}(z) dz \right] \tilde{R}(x) \quad (3.40)$$

which may also be written

$$\phi(x) = \exp \left[ i\pi \int_x^\infty \tilde{R}^*(z) \tilde{R}(z) dz \right] \tilde{R}(x). \quad (3.41)$$

This can be recognized as the standard form of a Jordan-Wigner fermion-to-boson transformation. The Jordan-Wigner transformation is also used in the solution of other models (e.g., the 2-D Ising model and the X-Y spin chain) which have the algebraic structure of a free fermion theory. These free fermion models can be regarded as special cases of more general Bethe's ansatz models (e.g., the Ising model and XY spin chain are special cases of the Baxter model and the XYZ spin chain respectively). Whereas the theory of Green's functions for Bethe's ansatz models is not very well understood, the special free-fermion cases are rather well-studied. The Green's functions for the  $c=\infty$  non-linear Schrödinger model were first discussed by Schultz<sup>20</sup> and Lenard,<sup>21</sup> who related the  $2n$ -point functions to the  $n$ th Fredholm minor associated with an integral kernel  $K(x,y) = \sin(x-y)/(x-y)$ . I'll come back to this result in Section V, where I shall discuss the large  $c$  expansion of the Green's functions. Recently, in an elegant series of developments by Sato, Miwa, and Jimbo,<sup>22</sup> the Green's functions for the free-fermion models were found to be deeply related to the theory of isomonodromic deformations of linear differential equations, whose mathematical origins go back to the early part of this century. In particular, this connection allowed SMJ to express the two-point functions for these models in closed form in terms of Painlevé functions. I'll review some of these developments in the last lecture.

## IV. GREEN'S FUNCTIONS--GENERAL FORMALISM

I want to consider the two-point equal time correlation function  $\langle \phi^*(x) \phi(y) \rangle_\Omega$  where  $\langle \dots \rangle_\Omega$  represents either a ground state expectation value or a thermal average, depending on whether we're discussing the zero or finite-temperature Green's functions. The basic idea of the quantum inverse approach to Green's functions is to express  $\phi^*(x)$  and  $\phi(y)$  in terms of  $R^*$  and  $R$  operators and use this expression to compute expectation values. In order to do this we need two theorems, a reordering theorem and a trace theorem. The reordering theorem tells how to write the operator product  $\phi^*(x) \phi(y)$  as a normal ordered functional of  $R^*$  and  $R$ . The trace theorem tells how to compute the thermal average of a normal product of  $R^*$ 's and  $R$ 's.

Reordering Theorem

Beginning with the Gel'fand Levitan Series

$$\phi(y) = \sum_{N=0}^{\infty} \int \prod_1^N \frac{dp_i}{2\pi} \prod_0^N \frac{dk_i}{2\pi} g_N(p_i, k_i; y) \\ R^*(p_1) \dots R^*(p_N) R(k_N) \dots R(k_0) \quad (4.1)$$

$$\phi^*(x) = \sum_{N=0}^{\infty} \int \prod_0^N \frac{dp_i}{2\pi} \prod_1^N \frac{dk_i}{2\pi} g_N^*(k_i, p_i; x) \\ R^*(p_0) \dots R^*(p_N) R(k_N) \dots R(k_1), \quad (4.2)$$

we can form the operator product  $\phi^*(x) \phi(y)$  which must be rearranged into normal ordered form. For the present discussion, the integrand  $g_N$  may be

taken from either (3.31) or (3.38). Recall that in the derivation of the quantum Gel'fand-Levitan equation we made essential use of the fact that the commutator  $[R^*(k), \phi^*(x)]$ , Eq. (3.17), could be analytically continued into the lower half-plane. This same analyticity property can be used to derive a reordering theorem for  $\phi^*(x) \phi(y)$ . For definiteness, consider the case  $x > y$ . Write  $\phi^*(x)$  as a GL series but leave  $\phi(y)$ , giving

$$\phi^*(x) \phi(y) = \sum_{N=0}^{\infty} \int \prod_0^N \frac{dp_i}{2\pi} \prod_1^N \frac{dk_i}{2\pi} g_N^*(k_1, p_1; x) R^*(p_0) \dots R^*(p_N) R(k_N) \dots R(k_1) \phi(y). \quad (4.3)$$

For  $x > y$ , the analyticity of  $[R(k), \phi(y)]$  in the upper half-plane allows us to move  $\phi(y)$  to the left past all the  $R(k)$ 's. All the commutator terms vanish, since for each  $k_i$   $g_N^*(k_i, p_i)$  is also analytic in the upper half-plane and the integrand  $\rightarrow 0$  asymptotically. Thus  $\phi(y)$  can be placed between the  $R^*$ 's and the  $R$ 's in (4.3) and then expanded, yielding a normal ordered series for  $\phi^*(x) \phi(y)$ :

$$\phi^*(x) \phi(y) = \sum_{N=0}^{\infty} \int \prod_0^N \frac{dp_i}{2\pi} \prod_0^N \frac{dk_i}{2\pi} F_N(p_1, k_1; x, y) R^*(p_0) \dots R^*(p_N) R(k_N) \dots R(k_0), \quad (4.4)$$

where

$$F_N(p_1, k_1; x, y) = \sum_{\ell=0}^N g_{\ell}^*(k_0 \dots k_{\ell-1}; p_0 \dots p_{\ell}; x) g_{n-\ell}(p_{\ell+1} \dots p_N; k_{\ell} \dots k_N; y) \quad (4.5)$$

Temperature Green's Functions: Trace Theorem

As I discussed in Section II, the theory of Green's functions is being developed here in the infinite volume formalism, where the R operators have simple commutation relations (2.53)-(2.54) and also commute simply with the Hamiltonian

$$[H, R^*(k)] = k^2 R^*(k). \quad (4.6)$$

With these algebraic properties, the two-point function can be computed term by term from the series (4.4). Of course, in the infinite volume formalism we must be careful to define the correct prescription for handling the infrared singularities which arise from integrations over an infinite volume. The subtleties associated with these infrared singularities and their relevance to the formulation of statistical mechanics without a box were discussed some time ago in the language of factorized graphs. (See the second paper in Ref. 19.) The procedure which I will outline below evolved from these graphical studies.

We will consider the finite temperature Green's function

$$G_{\beta, \mu}(x - y) = \frac{\left[ \text{Tr } \phi^*(x) \phi(y) e^{-\beta \Omega} \right]}{\text{Tr } e^{-\beta \Omega}}, \quad (4.7)$$

where  $\Omega = H - \mu N$ . In computing the trace in the numerator of (4.7) we must consider diagonal matrix elements of the operator in square brackets. In the calculation of these matrix elements, we encounter two basic types of infrared divergence. One type arises from the presence of disconnected graphs in the trace, which leads to momentum space delta-functions with vanishing argument (i.e.,  $\delta(0)$  factors). These disconnected graphs may

be easily removed by dividing out a factor of  $\text{Tr} e^{-\beta\Omega}$  as in (4.7). The other type of infrared divergence is more subtle. It arises in the connected part of a matrix element, e.g.,

$$\langle p_1 \dots p_N | \phi^*(x) \phi(y) e^{-\beta\Omega} | k_1 \dots k_N \rangle, \quad (4.8)$$

when we try to take the forward limit  $p_i \rightarrow k_i$ . Imagine computing the matrix element (4.8) from the many-body coordinate space wave functions. This involves integrating over the coordinates  $z_1, \dots, z_N$  of the  $N$  particles. The forward singularities arise from the asymptotic parts of this integration which become undamped in the limit  $p_i \rightarrow k_i$  (i.e., terms which behave like  $e^{i(p_i - k_i)z_i}$  as  $z_i \rightarrow \pm \infty$ ). If we considered a limit of (4.8) where some but not all of the  $p_i$ 's are set equal to the  $k_i$ 's, then these singularities are really there and the limit does not exist because of divergences of the form  $(p_i - k_i \pm i\epsilon)^{-1}$ . However, to compute a trace we need the diagonal (forward) matrix element, which is obtained by setting all momentum differences  $(p_i - k_i)$  to zero simultaneously with fixed ratio. For the connected part of the matrix element (4.8) this forward limit is finite, because each singular denominator  $(p_i - k_i \pm i\epsilon)$  is multiplied by a vanishing factor of the form  $[e^{i(\Theta - \Theta')} - 1]$ , where  $\Theta$  is a sum of Bethe's ansatz phase shifts depending on the relative momenta  $k_{ij}$  in the initial state, and  $\Theta'$  is the corresponding sum of phase shifts for the  $p_{ij}$ 's in the final state. In the forward limit the phase shifts in the initial and final states match up, i.e.  $\Theta' \rightarrow \Theta$ , rendering this limit finite. Pursuing this argument, it is now easy to see the correct procedure for calculating the forward matrix elements needed to compute the trace in (4.7). Consider the effect of any reasonable sort of cutoff on

the coordinate space integrations, e.g., a sharp cutoff (a box of length  $L$ ) or an adiabatic cutoff (i.e., keeping the  $i\epsilon$ 's finite in the singular denominators). This simply regularizes the singular denominators without affecting the vanishing numerators  $[e^{i(\theta-\theta')}-1]$ . So the correct prescription is to set to zero all terms in the matrix element which have singular factors of the form

$$\left[ \frac{e^{i(\theta-\theta')}-1}{p_i - k_i \pm i\epsilon} \right] \quad (4.9)$$

The connected forward matrix element is given by the remaining terms which have no singular ratios and hence have an unambiguous forward limit.

I will now introduce the basic device that will be used to correctly regularize the infrared singularities and compute the temperature Green's function (4.7). I will call this device the "infinitesimal boost method." Define the Galilean boost generator

$$K = \int x \phi^*(x) \phi(x) dx. \quad (4.10)$$

The  $R$  operators have a simple behavior under boosts:

$$e^{iqK} R(k) e^{-iqK} = R(k + q). \quad (4.11)$$

The basic assertion of the infinitesimal boost method is that the Green's function (4.7) is given by the formula

$$G_{\beta, \mu}(x-y) = \lim_{q \rightarrow 0} \text{Tr} \left[ \phi^*(x) \phi(y) e^{-\beta\Omega} e^{-iqK} \right]. \quad (4.12)$$

From the previous discussion it is easy to see why this method works. A forward N-body matrix element of the operator in square brackets in (4.12) will be of the form (4.8), where the  $k_i$ 's are shifted from the  $p_i$ 's by a small momentum  $q$ , i.e.,  $k_i = p_i - q$ . This does two things. First it eliminates disconnected graphs, since a disconnected subgraph is essentially an integrated matrix element of  $e^{-\beta\Omega}$  between states  $\langle p_1, \dots, p_\ell |$  and  $| p_1 - q, \dots, p_\ell - q \rangle$  which vanishes by momentum conservation. (In fact we could have divided (4.12) by a factor  $\text{Tr}(e^{-\beta\Omega} e^{-iqK})$ , which is unity because it only receives a contribution from the zero particle state.) The fully connected graphs do not vanish because the operator  $\phi^*(x) \phi(y)$  is there to absorb the momentum  $Nq$ . In addition to eliminating disconnected graphs, formula (4.12) also sets factors like (4.9) in the singular connected graphs to zero (which, as I have argued, is the correct thing to do). This happens because the phase shifts  $\Theta$  and  $\Theta'$  in the numerator depend only on the relative momenta  $k_{ij}$  and  $p_{ij}$  which are not affected by a Galilean boost. Thus the numerator is identically zero even for finite  $q$ .

Using the equation (4.12) along with the series (4.4) and the algebraic properties of the R-operators, we may compute the Green's function term by term in the series. To do this we will use a convenient theorem for evaluating traces of the form  $\text{Tr}\{R^*(p_0) \dots R^*(p_N) R(k_N) \dots R(k_0) e^{-\beta\Omega} e^{-iqK}\}$ . Consider first the simplest case  $N=0$ ,

$$\text{Tr}\{R^*(p) R(k) e^{-\beta\Omega} e^{-iqK}\}. \quad (4.13)$$

Using the properties

$$R(k) e^{-\beta\Omega} = e^{-\beta(k^2 - \mu)} e^{-\beta\Omega} R(k) \quad (4.14)$$

$$R(k) e^{-iqK} = e^{-iqK} R(k + q) \quad (4.15)$$

$$R(k) R^*(p) = 2\pi\delta(p - k) + S(k - p) R^*(p) R(k), \quad (4.16)$$

and the cyclic property of the trace, we generate a fugacity series for (4.13) in the limit  $q \rightarrow 0$ :

$$\text{Tr} \{R^*(p)R(k) e^{-\beta\Omega} e^{-iqK}\} \xrightarrow{q \rightarrow 0} - \sum_{n=1}^{\infty} (-z)^n e^{-n\beta k^2} \times \langle k-nq|p \rangle \{1+O(q)\}, \quad (4.17)$$

where  $z \equiv e^{\beta\mu}$  = fugacity. By an inductive argument, this result can be generalized to the following trace theorem:

$$\begin{aligned} & \text{Tr} \{R^*(p_0) \dots R^*(p_N) R(k_N) \dots R(k_0) e^{-\beta\Omega} e^{-iqK}\} \\ &= (-1)^N \sum_{n_0, n_1, \dots, n_N=1}^{\infty} \prod_{i=0}^N \left[ (-z)^{n_i} e^{-n_i \beta k_i^2} \right] \end{aligned} \quad (4.18)$$

$$\langle k_0 + n_0 q, \dots, k_N + n_N q | p_0 \dots p_N \rangle \times \{1 + O(q)\},$$

where

$$|p_0 \dots p_N \rangle \equiv R^*(p_0) \dots R^*(p_N) | 0 \rangle. \quad (4.19)$$

In normal ordered GL series, such as (4.1) or (4.4), the integrands  $g_N$  or  $F_N$  are not uniquely specified. This is clear, for example, from

the equivalence of expressions (3.31) and (3.38). What is uniquely specified is the "R-symmetrized" function  $g_N^{(S)}$  or  $F_N^{(S)}$  which is obtained by symmetrizing over the  $p$ 's and over the  $k$ 's and using the commutation relations of the R-operators. Thus, for example

$$F_N^{(S)}(p, k) = \frac{1}{[(N+1)!]} \sum_{P, Q} F_N^{(Pp, Qk)} \prod_{\substack{i < j \\ p_i > p_j}} S(p_{ij}) \prod_{\substack{i < j \\ Q_i > Q_j}} S(k_{ji}). \quad (4.20)$$

Any two functions which lead to the same R-symmetrized function will give equivalent operator expressions. Using the trace theorem (4.18) and the GL series (4.4), and writing the inner product of R-states in (4.18) as sums of products of  $\delta$ -functions and S-matrices we get the result for the Green's function,

$$G_{\beta, \mu}(x-y) = \lim_{q \rightarrow 0} \sum_{N=0}^{\infty} \sum_{n_0, \dots, n_N=1}^N (-1)^N \int \prod_{i=0}^N \left[ (-z)^{n_i} e^{-\beta n_i p_i^2} \frac{dp_i}{2\pi} \right] \times \quad (4.21)$$

$$\times F_N^{(S)}(p, p - nq; x, y).$$

In order to proceed further, we must derive some properties of the functions  $F_N^{(S)}$ . These are obtained from (4.5) and (4.20). Note that the integrands  $g_n$  in the GL series for  $\phi(x)$  can be written in many different ways (only the R-symmetrized function has meaning), in particular, as in (3.31) or (3.38). Let us first consider the  $n_i$

dependence of  $F_N^{(S)}(\mathbf{p}, \mathbf{p}-\mathbf{m}q; \mathbf{x}, \mathbf{y})$ . Since the denominators in (3.31) involve only single momentum differences ( $p_i - k_j$ ), we conclude that

$$F_N^{(S)}(\mathbf{p}, \mathbf{p}-\mathbf{m}q; \mathbf{x}, \mathbf{y}) \xrightarrow{q \rightarrow 0} \frac{H_N(n_0, \dots, n_N; \mathbf{p}, \mathbf{x}, \mathbf{y})}{n_0 n_1 \dots n_N}, \quad (4.22)$$

where  $H_N$  is a homogeneous  $(N+1)^{\text{th}}$  order multinomial in  $n_0, \dots, n_N$  which is symmetric under simultaneous permutation of  $p_i$ 's and  $n_i$ 's. (The finiteness of  $F_N^{(S)}$  in the  $q \rightarrow 0$  limit follows from the inductive argument outlined below.) Thus each term in  $H_N$  is of the form

$$n_0^{\lambda_0} n_1^{\lambda_1} \dots n_N^{\lambda_N} \times \text{function of } (\mathbf{p}, \mathbf{x}, \mathbf{y}), \quad (4.23)$$

where

$$\sum_{i=0}^N \lambda_i = N + 1. \quad (4.24)$$

Let us pick out the "nonsingular" (i.e.,  $n_i$  independent) term in (4.22) by writing

$$H_N(n_0 \dots n_N; \mathbf{p}, \mathbf{x}, \mathbf{y}) = n_0 n_1 \dots n_N f_N(\mathbf{p}; \mathbf{x}, \mathbf{y}) + \tilde{H}_N(n_0 \dots n_N; \mathbf{p}, \mathbf{x}, \mathbf{y}), \quad (4.25)$$

where  $\tilde{H}_N$  contains only terms where one or more of the  $\lambda_i$ 's is zero. The point of making this separation is that now  $\tilde{H}_N$  may be obtained by symmetry in the  $n_i$ 's from its value with one of the  $n_i$ 's set equal to zero, e.g.,  $\tilde{H}_N|_{n_N=0} = 0$ . The function  $\tilde{H}_N|_{n_N=0}$  is determined by the residue of the pole in  $F_N^{(S)}(\mathbf{p}, \mathbf{k}; \mathbf{x}, \mathbf{y})$  at  $p_N = k_N$  which can be related to the lower order function  $F_{N-1}^{(S)}$ . This is the essential inductive step which

allows us to sum up all the  $n_i$ -dependent terms in (4.21) and express the Green's function entirely in terms of the functions  $f_N$  in (4.25). To study the residue of the pole in  $F_N(S)$  at  $p_N = k_N$ , it is convenient to use (4.20) with an unsymmetrized function  $F_N$  which is obtained from (3.38):

$$\begin{aligned}
 F_N(p, k; x, y) &= e^{ik_0 y - p_0 x} \sum_{\ell=0}^N \int dz_1 \dots dz_N \theta(y < z_1 < \dots < z_\ell < x < z_{\ell+1} < \dots < z_N) \\
 &\times \left[ \prod_{i=1}^N e^{i(k_i - p_i) z_i} \right] (S_{10}^{-1}) \dots (S_{\ell 0} S_{\ell 1}^{\ell} S_{\ell 1}^{\ell-1} \dots S_{\ell-1, \ell}^{\ell-1, \ell} S_{\ell, \ell-1}^{-1}) \\
 &\times (S_{\ell+1, 0}^{0, \ell+1} S_{\ell+1, 0}^{\ell, \ell+1} \dots S_{\ell+1, \ell}^{\ell, \ell+1} S_{\ell+1, \ell}^{-1}) \dots (S_{N0}^{0N} \dots S_{N, N-1}^{N-1, N} S_{N, N-1}^{-1}).
 \end{aligned} \tag{4.26}$$

From this expression it is easy to show that the residue at  $p_N = k_N$  is given by

$$\begin{aligned}
 F_N(S) \xrightarrow{p_N \rightarrow k_N} \frac{(-1)}{p_N - k_N} \left\{ \prod_{i=0}^{N-1} S(p_i - p_N) S(p_N - k_i) - 1 \right\} F_{N-1}^{(S)} \\
 \xrightarrow{k_i = p_i - n_i q} \left\{ \left( \frac{-1}{n_N} \right) \sum_{i=0}^{N-1} n_i \Delta(p_i - p_N) \right\} F_{N-1}^{(S)}.
 \end{aligned} \tag{4.27}$$

where  $\Delta(k)$  is given in (2.18). The relation (4.27) allows us to sum up the  $n_i$  dependent terms in (4.21). To understand the result, it is instructive to first consider the result of summing only the nonsingular ( $n_i$ -independent) terms in  $F_N(S)$  (i.e., keeping only the first term in (4.25). This would give

$$\sum_{N=0}^{\infty} \prod_{i=0}^N \left[ \frac{1}{e^{\beta(p_i^2 - \mu)} + 1} \frac{dp_i}{2\pi} \right] f_N(p; x, y). \quad (4.28)$$

Using the induction (4.27), it can be shown that the sole effect of the singular terms is to replace  $\left[ e^{\beta(p_i^2 - \mu)} + 1 \right]^{-1}$  by  $\left[ e^{\beta \epsilon(p_i)} + 1 \right]^{-1}$  where  $\epsilon(p)$  is the excitation energy function of Yang and Yang, Eq. (2.21). Thus, the full Green's function is reduced to

$$G_{\beta, \mu}(x-y) = \sum_{N=0}^{\infty} \int \prod_{i=0}^N \left[ \tilde{\rho}(p_i) \frac{dp_i}{2\pi} \right] f_N(p; x, y), \quad (4.29)$$

where

$$\tilde{\rho}(p) = \frac{1}{e^{\beta \epsilon(p)} + 1}. \quad (4.30)$$

We have now isolated all the  $\beta$  and  $\mu$  dependence of  $G_{\beta, \mu}(x-y)$  in terms of a single particle distribution function  $\tilde{\rho}(p)$ . The familiar thermodynamics of Yang and Yang can be recovered from (4.29) in the limit  $(x-y) \rightarrow 0$ . In this case the polynomial  $H_N$  in (4.25) can be shown to have an overall factor of  $\prod_{i=0}^N n_i$ . This fact allows us to determine  $f_N$  as well as  $\tilde{H}_N$  from the induction (4.27), giving

$$G_{\beta, \mu}(0) = \sum_{N=0}^{\infty} \int \prod_{i=0}^N \left[ \tilde{\rho}(p_i) \frac{dp_i}{2\pi} \right] \prod_{i=0}^{N-1} \Delta(p_i - p_{i+1}). \quad (4.31)$$

This can be recognized as the expansion of the Yang and Yang integral equation for the density function  $\rho(k)$

$$\frac{\rho(k)}{\tilde{\rho}(k)} = 1 + \int \frac{dq}{2\pi} \Delta(k - q) \rho(q). \quad (4.32)$$

Thus, the zero separation Green's function, which is just  $\langle \phi^*(0)\phi(0) \rangle_{\Omega}$  = particle density, is given by

$$G_{\beta\mu}(0) = \int dk \rho(k), \quad (4.33)$$

where  $\rho(k)$  is defined by (4.32), (4.30), and (2.21). From the density as a function of  $\beta$  and  $\mu$ , other thermodynamic quantities may be derived.

Finally, let me note for later reference that the Green's function expression (4.29) has a simple zero temperature limit

$$G(x-y) = \lim_{\beta \rightarrow \infty} G_{\beta, \mu}(x-y) = \sum_{N=0}^{\infty} \int_{-k_F}^{k_F} \frac{N}{i} \frac{dp_1}{2\pi} f_N(p; x, y). \quad (4.34)$$

To summarize, we compute the Green's functions as follows: Begin with the functions  $F_N^{(S)}$ , the R-symmetrized GL integrands for  $\phi^*(x)\phi(y)$  defined by (4.20) with some suitable unsymmetrized integrands  $F_N$ , e.g., Eq. (4.26). Then calculate the functions  $f_N(p; x, y)$  by Eqs. (4.22) and (4.25). The zero-temperature and finite temperature Green's functions are given by (4.34) and (4.29) respectively.

It would be nice to write the Green's function in closed form, but so far this has not been done. But recently it was shown that the first two terms in a strong coupling (large  $c$ ) expansion of  $G(x-y)$  may be expressed in closed form in terms of Painlevé functions. The large

coupling results and their connection with the work of Sato, Miwa, and Jimbo will be discussed in the last two lectures.

#### V. LARGE $c$ EXPANSION OF THE TWO-POINT FUNCTION

The Green's functions for impenetrable bosons ( $c=\infty$ ) were extensively studied, first by Schultz and Lenard and more recently by Jimbo et al. To make contact with these results we will consider a large  $c$  expansion of the two-point function

$$G = G^{(0)} + G^{(1)} + G^{(2)} + \dots, \quad (5.1)$$

where  $G^{(n)}$  is of order  $(1/c)^n$ . The form of the GL integrands  $F_N$  given in (4.26) is well-suited for studying the  $c \rightarrow \infty$  limit. Note that the  $z_1$  integrations are ordered,  $z_1 < z_2 < \dots < z_N$ , and that  $l$  of the integrations are "trapped" between  $y$  and  $x$ , and  $N-l$  of them are "untrapped" between  $x$  and  $\infty$ . With each trapped  $z$ -integration is associated a factor

$$(\text{odd number of } S\text{'s} - 1), \quad (5.2)$$

while each untrapped  $z$ -integration has a factor

$$(\text{even number of } S\text{'s} - 1). \quad (5.3)$$

In the limit  $c \rightarrow \infty$ ,  $S(p_{ij}) \rightarrow -1$ , and the factors (5.2) and (5.3) become  $-2$  and  $0$  respectively. Thus, only the terms in (4.26) with all  $z$ -integrations trapped contribute at  $c = \infty$ . (The Jordan-Wigner "tails" in

(3.41) cancel exactly outside the interval  $y < z < x$ .) This gives

$$F_N(p, k; x, y) = (-2)^N \int dz_1 \dots dz_N \theta(y < z_1 < \dots < z_N < x) \prod_{i=1}^N e^{i(k_i - p_i)z_i}. \quad (5.4)$$

Since the R-operators anticommute for  $c = \infty$ , we may symmetrize (5.4) over simultaneous permutations of  $p_i$ 's and  $k_i$ 's, allowing us to make the replacement

$$F_N(p, k, x, y) \rightarrow \frac{(-2)^N}{N!} \int_y^x dz_1 \dots \int_y^x dz_N \prod_{i=1}^N e^{i(k_i - p_i)z_i}. \quad (5.5)$$

Note that because there are no untrapped integrations, there are no poles at  $p_i = k_i$  and we may set  $q = 0$  from the start. Since  $S = -1$ , the R-symmetrized function  $F_N^{(S)}$  involves a determinant. In this way we get the expression for the  $c = \infty$  Green's function

$$G_{\beta, \mu}^{(0)}(x-y) = \sum_{N=0}^{\infty} \frac{(-2)^N}{N!} \int \left[ \prod_0^N \tilde{\rho}(p_i) \frac{dp_i}{2\pi} \right] \int_y^x dz_1 \dots \int_y^x dz_N \mathcal{D}_N(x, y; z, p), \quad (5.6)$$

where  $\mathcal{D}_N$  is an  $(N+1) \times (N+1)$  determinant of exponentials, e.g.,

$$\mathcal{D}_0 = 1 \quad (5.7a)$$

$$\mathcal{D}_1 = \begin{vmatrix} e^{-ip_0(x-y)} & e^{-ip_0(x-z_1)} \\ e^{-ip_1(z_1-y)} & 1 \end{vmatrix} \quad (5.7b)$$

$$\mathcal{D}_2 = \begin{vmatrix} e^{-ip_0(x-y)} & e^{-ip_0(x-z_1)} & e^{-ip_1(x-z_2)} \\ e^{-ip_1(z_1-y)} & 1 & e^{-ip_1(z_1-z_2)} \\ e^{-ip_2(z_2-y)} & e^{-ip_2(z_2-z_1)} & 1 \end{vmatrix} \quad (5.7c)$$

etc.

By carrying out the  $p_i$  integrations in (5.6) we may write

$$G_{\beta, \mu}^{(0)}(x-y) = \frac{1}{2} \left\{ \lambda K(x,y) - \int_y^x dz_1 \begin{vmatrix} K(x,y) & K(x,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix} \right. \\ \left. + \frac{\lambda^3}{2!} \int_y^x dz_1 \int_y^x dz_2 [3 \times 3] - \dots \right. \quad (5.8)$$

where  $\lambda = 2/\pi$ , and the kernel  $K$  is the Fourier transform of a Fermi-Dirac distribution,

$$K(x,y) = \frac{1}{2} \int e^{-ip(x-y)} \tilde{\rho}_0(p) dp \quad (5.9)$$

with  $\rho_0(p) = [e^{\beta(p^2 - \mu)} + 1]^{-1}$ . At zero temperature the kernel reduces to

$$K(x-y) = \frac{1}{2} \int_{-k_F}^{k_F} e^{-ip(x-y)} dp = \frac{\sin(x-y)}{x-y}. \quad (5.10)$$

(Hereafter, I will set  $k_F = 1$ .)

The  $c = \infty$  Green's function (5.8) is essentially a Fredholm minor associated with the integral kernel  $K(x,y)$ . Let me remind you how an integral equation is solved by Fredholm determinants. Consider the integral equation for a function  $R(x,y)$ ,

$$R(x,y) = \lambda K(x,y) + \lambda \int_a^b dz K(x,z) R(z,y). \quad (5.11)$$

By a continuum version of Kramer's rule,  $R(x,y)$  may be written as a ratio of determinants,

$$R(x,y) = \frac{D_1(x,y;a,b)}{D(a,b)}, \quad (5.12)$$

where  $D_1$  is the Fredholm minor

$$D_1(x,y;a,b) = \lambda K(x,y) - \lambda^2 \int_a^b dz_1 \begin{vmatrix} K(x,y) & K(x,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix} + \frac{\lambda^3}{2!} \int_a^b dz_1 \int_a^b dz_2 |3 \times 3| - \dots \quad (5.13)$$

and  $D$  is the Fredholm determinant,

$$D(a,b) = 1 - \lambda \int_a^b K(z,z) dz + \frac{\lambda^2}{2!} \int_a^b dz_1 \int_a^b dz_2 \begin{vmatrix} K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} - \dots \quad (5.14)$$

$$= \text{Det} (1 - \lambda K).$$

The  $c = \infty$  Green's function (5.8) is thus a Fredholm minor with its arguments evaluated at the endpoints of the integration region,

$$G^{(0)}(a-b) = \frac{1}{2} D_1(a,b;a,b). \quad (5.15)$$

All of these  $c = \infty$  results have been known since the work of Lenard. Here we see how they follow as a special case of the quantum inverse formalism. Moreover, we can now go on to consider finite  $c$  corrections.

An expansion in powers of  $(1/c)$  can be obtained by collecting the terms in (4.26) according to the number of untrapped  $z$ -integrations. Each untrapped integration is accompanied by a factor (even number of  $S$ 's - 1), which is of order  $(1/c)$ . Here I'll consider the Green's function up to order  $(1/c)$ , so only terms with zero or one untrapped integration must be kept.

$$G^{(1)} = G_0^{(1)} + G_1^{(1)}. \quad (5.16)$$

For the terms with no untrapped integrations, we use expansions of the form

$$\begin{aligned}
(S_{10}^- - 1) &\sim (-2) \left(1 - \frac{k_{10}}{ic}\right) \\
(S_{10}^- - 1)(S_{20} S_{21}^{12} S_{21}^- - 1) &\sim (-2)^2 \left[1 - \frac{(k_{10}^+ k_{20}^+ k_{21}^+ p_{12})}{ic}\right] \quad (5.17)
\end{aligned}$$

etc. By this approach we obtain the  $1/c$  correction

$$\begin{aligned}
G_0^{(1)} = \left(\frac{1}{ic}\right) \sum_{N=1}^{\infty} \frac{(-2)^N}{N!} \int \left[ \prod_{i=0}^N \tilde{\rho}(p_i) \frac{dp_i}{2\pi} \prod_{j=1}^N (p_0^- p_j) \right] \\
\times \int_y^x dz_1 \dots \int_y^x dz_N \mathcal{D}_N(x, y; z, p), \quad (5.18)
\end{aligned}$$

and  $\mathcal{D}_N$  is the same determinant of exponentials which appeared in the  $c = \infty$  Green's function. By writing out the determinant in (5.18) and expressing the factor  $\Sigma(p_0^- p_j)$  as derivatives with respect to  $x, y$ , and  $z_1$ , it is possible to write (5.18) in terms of the Fredholm resolvent and minor (5.12) and (5.13). Let us define the functions  $R(t)$  and  $D_1(t)$  by

$$\begin{aligned}
R(t) = \lambda K(t, 0) + \lambda^2 \int_0^t dz_1 K(t, z_1) K(z_1, 0) \\
+ \lambda^3 \int_0^t dz_1 \int_0^t dz_2 K(t, z_1) K(z_1, z_2) K(z_2, 0) + \dots \quad (5.19)
\end{aligned}$$

$$\begin{aligned}
D_1(t) = \lambda K(t, 0) - \lambda^2 \int_0^t dz_1 \begin{vmatrix} K(t, 0) & K(t, z_1) \\ K(z_1, 0) & K(z_1, z_1) \end{vmatrix} + \frac{\lambda^3}{2!} \int_0^t dz_1 \int_0^t dz_2 |3 \times 3| - \dots \quad (5.20)
\end{aligned}$$

After some manipulation, (5.18) may be written

$$G_0^{(1)}(t) = \frac{1}{\pi c} D_1 \left\{ \left( \frac{\partial^2 \ln R}{\partial t \partial \lambda} - \frac{\partial \ln R}{\partial t} \frac{\partial \ln R}{\partial \lambda} \right) - \left( \frac{\partial^2 \ln D_1}{\partial t \partial \lambda} - \frac{\partial \ln D_1}{\partial t} \frac{\partial \ln D_1}{\partial \lambda} \right) \right\} \quad (5.21)$$

Next we must consider the contribution of the terms in (4.26) with one untrapped  $z$ -integration. The corresponding  $S$ -matrix factor may be expanded,

$$\begin{aligned} \left( S_{NO}^{ON} S_{NO} \dots S_{N,N-1}^{N-1,N} S_{N,N-1} \right) &\sim \frac{2}{ic} \sum_{i=0}^{N-1} [(k_i - k_N) - (p_i - p_N)] \\ &= \frac{2}{ic} \sum_{i=0}^N (k_i - p_i) + \frac{2N}{ic} (p_N - k_N). \end{aligned} \quad (5.22)$$

Since this is already of order  $1/c$ , the rest of the expression can be evaluated at  $c = \infty$ . In particular, the  $R$ 's can be taken to anti-commute. The second term in (5.22) is found to vanish by antisymmetry, while the first term gives a contribution proportional to the  $c = \infty$  Green's function,

$$G_1^{(1)} = \frac{2}{\pi c} G^{(0)}. \quad (5.23)$$

To summarize, the first two terms in a  $1/c$  expansion of  $G(t)$  are

$$G^{(0)} = \frac{1}{2} D_1 \quad (5.24)$$

$$G^{(1)} = \frac{2}{\pi c} G^{(0)} \left\{ 1 + \left( \frac{\partial^2 \ln R}{\partial t \partial \lambda} - \frac{\partial \ln R}{\partial t} \frac{\partial \ln R}{\partial \lambda} \right) - \left( \frac{\partial^2 \ln D_1}{\partial t \partial \lambda} - \frac{\partial \ln D_1}{\partial t} \frac{\partial \ln D_1}{\partial \lambda} \right) \right\}, \quad (5.25)$$

where  $R(t)$  and  $D_1(t)$  are given by (5.19) and (5.20). In the last lecture, I'll discuss the method for treating Green's functions developed by Sato, Miwa, and Jimbo<sup>22</sup> and applied to the  $c = \infty$  nonlinear Schrödinger model by Jimbo et al.<sup>12</sup> This method gives closed forms for  $R(t)$  and  $D_1(t)$  in terms of Painlevé transcendents.

## VI. GREEN'S FUNCTIONS AS PAINLEVÉ FUNCTIONS

Simple Derivation Of Painlevé Equation

Consider the resolvent  $R(x,y)$  at zero temperature defined by (5.11),

$$R(x,y) = \lambda K(x,y) + \lambda^2 \int_a^b K(x,z) K(z,y) dz + \dots = \left[ \frac{\lambda K}{1 - \lambda K} \right] (x,y), \quad (6.1)$$

where the kernel  $K(x,y) = \sin(x-y)/(x-y)$ . The  $x$  and  $y$  dependence of (6.1) may be written in a factorized form by defining

$$R_{\pm}(x) = e^{\pm ix} + \lambda \int_a^b dz_1 K(x, z_1) e^{\pm iz_1} + \lambda^2 \int_a^b dz_1 \int_a^b dz_2 K(x, z_1) K(z_1, z_2) e^{\pm iz_2} + \dots = \left[ \frac{1}{1 - \lambda K} E_{\pm} \right] (x), \quad (6.2)$$

where

$$E_{\pm}(x) = e^{\pm ix}. \quad (6.3)$$

The quantity  $R_+(x) R_-(y) - R_-(x) R_+(y)$  can be worked out term by term using

$$E_+(z) E_-(z') - E_-(z) E_+(z') = 2i(z-z') K(z, z'), \quad (6.4)$$

and

$$(x-z_1) + (z_1-z_2) + \dots + (z_m-y) = (x-y). \quad (6.5)$$

This gives the factorized expression

$$R(x,y) = \frac{\lambda \left[ R_+(x) R_-(y) - R_-(x) R_+(y) \right]}{2i(x-y)}. \quad (6.6)$$

The series (6.2) may be differentiated term by term to obtain  $\partial R_{\pm}/\partial x$ . Since  $K$  is a difference kernel, we may replace  $\frac{\partial}{\partial x} \rightarrow -\frac{\partial}{\partial z_1}$ , integrate by parts, then replace  $\partial/\partial z_1 \rightarrow -\partial/\partial z_2$ , integrate by parts again, etc., until the derivative is acting on  $e^{\pm iz_N}$ . The surface terms can also be summed, and we find

$$\frac{\partial R_{\pm}(x)}{\partial x} = \pm i R_{\pm}(x) + R(x,a) R_{\pm}(a) - R(x,b) R_{\pm}(b). \quad (6.7)$$

Using the factorization property (6.6), it is seen that the column vector

$$y(x) = \begin{pmatrix} R_+(x) \\ R_-(x) \end{pmatrix}, \quad (6.8)$$

satisfies a first order equation

$$\frac{\partial y(x)}{\partial x} = \left[ \frac{A(a)}{x-a} - \frac{A(b)}{x-b} + C \right] y(x), \quad (6.9)$$

where

$$A(a) = \lambda \begin{pmatrix} R_+(a) R_-(a) & -R_+^2(a) \\ R_-^2(a) & -R_+(a) R_-(a) \end{pmatrix}, \quad (6.10)$$

and

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.11)$$

We may also derive an equation by differentiating  $R_{\pm}$  with respect to  $a$  or  $b$ . The derivative acting on the upper or lower limit of integration gives terms which sum up in the same way as the surface terms in (6.7).

In this way we get

$$\frac{\partial y}{\partial b} = \frac{A(b)}{x-b} y, \quad (6.12)$$

$$\frac{\partial y}{\partial a} = \frac{-A(a)}{x-a} y, \quad (6.13)$$

Eqs. (6.9), (6.12), and (6.13) can be written as a total differential relation,

$$dy = \Omega y, \quad (6.14)$$

where  $\Omega$  is a differential form given by

$$\Omega = A(a) d\ln(x-a) - A(b) d\ln(x-b) + Cdx. \quad (6.15)$$

The linear system of equations (6.14) is the fundamental property of the series (6.1) with kernel  $\sin(x-y)/(x-y)$  which was discovered by Jimbo, et al. Later on, I will discuss how (6.14) follows directly from an isomonodromy property. But first let me derive the Painlevé expressions for  $R(t)$  and  $D(t)$ , using (6.14). Let

$$x = b = t/2 \quad (6.16)$$

$$a = -t/2,$$

Then  $d \ln(x-a) = d \ln t = dt/t$

$$d \ln(x-b) = 0 \quad (6.17)$$

$$dx = 1/2 dt.$$

Denote by  $r_{\pm}(t)$  the quantities (6.2) evaluated at (6.16). Then (6.14) can be written

$$r'_+ = \frac{1}{2} r_+ + \frac{\lambda}{2it} (r_+^2 - r_-^2) r_- \quad (6.18a)$$

$$r'_- = -\frac{1}{2} r_- + \frac{\lambda}{2it} (r_+^2 - r_-^2) r_+ \quad (6.18b)$$

Now introduce two functions  $r(t)$  and  $\psi(t)$  by

$$r_+ = e^{i\pi/4} r \cosh \frac{1}{2} \left( \psi - \frac{i\pi}{2} \right) \quad (6.19a)$$

$$r_- = e^{i\pi/4} r \sinh \frac{1}{2} \left( \psi - \frac{i\pi}{2} \right). \quad (6.19b)$$

Then Eqs. (6.18) reduce to

$$r^2 = \frac{t}{\lambda} (\psi' + \cosh \psi), \quad (6.20)$$

and

$$\psi'' + \frac{1}{t} \psi' + \frac{1}{t} \cosh \psi - \frac{1}{2} \sinh 2\psi = 0. \quad (6.21)$$

Eq. (6.21) is equivalent to a Painlevé equation of the fifth kind. It is convenient to make another change of variables

$$\sinh \psi = \cot \phi. \quad (6.22)$$

Then Eq. (6.21) becomes

$$\phi'' = [(\phi')^2 - 1] \cot \phi + \frac{(1 - \phi')}{t}. \quad (6.23)$$

The function  $\phi(t)$  is completely specified by (6.23) along with the boundary condition

$$\phi(t) \sim t - \lambda t^2 + O(t^3) \text{ as } t \rightarrow 0, \quad (6.24)$$

which follows from the series expression (6.2).

All the functions needed to express the  $c = \infty$  Green's function and the  $1/c$  correction can be expressed in terms of  $\phi(t)$ . The function  $R(t)$  defined by (5.19) is

$$R(t) = \frac{1 - \phi'}{2 \sin \phi}, \quad (6.25)$$

while  $D_1(t)$ , Eq. (5.20) satisfies

$$\frac{\partial^2 D_1}{\partial t^2} = \frac{t [(\phi')^2 - 1]}{4 \sin^2 \phi} + \cot \phi - \frac{1}{t}. \quad (6.26)$$

Eq. (6.26) along with the condition

$$D_1(0) = \lambda, \quad (6.27)$$

specifies  $D_1(t)$  completely. By numerical integration of Eq. (6.23), the functions  $G^{(0)}(t)$  and  $G^{(1)}(t)$ , Eqs. (5.24) and (5.25), may be easily

plotted. The long distance behavior of the Green's function may be studied using the asymptotic expansion of  $\phi(t)$ . The behavior of  $\phi(t)$  as  $t \rightarrow \infty$  depends critically on the value of  $\lambda$ . For  $-\infty < \lambda < 1/\pi$ , the asymptotic behavior is  $\phi(t) \sim t + O(\lambda \ln t)$  while for  $1/\pi < \lambda < \infty$ , it is  $\phi(t) \sim -t + O(\lambda \ln t)$ . At the critical value  $\lambda = 1/\pi$ ,  $\phi(t)$  goes to a constant,  $\phi(t) \sim \pi/2 + O(1/t)$ . The  $c = \infty$  Green's function  $G^{(0)}$  is given in terms of  $\phi(t)$  at  $\lambda = 2/\pi$ , while the  $1/c$  correction  $G^{(1)}$  involves  $\phi(t)$  and its first  $\lambda$ -derivative at that point. It is amusing to note that the critical value  $\lambda = 1/\pi$  also arises in a physical problem, that of determining the eigenvalue distribution of random matrices.

To study the long distance behavior of the two-point function, we use the asymptotic expansion

$$\phi(t) \sim -t + t_0 + k \lambda \ln t + O(1/t), \quad (6.28)$$

where  $k$  and  $t_0$  are  $\lambda$ -dependent constants. For a detailed discussion of the asymptotic analysis, I refer you to the literature.<sup>9</sup> Here, I will simply mention that the dominant effect of the  $1/c$  correction is to alter the power-law falloff of the Green's function. The full Green's function  $G(t)$  behaves like

$$G(t) \sim \text{const} \times t^{-\nu} [1 + O(1/t)], \quad (6.29)$$

where  $\nu$  is a  $c$ -dependent constant. At  $c = \infty$ , Vaidya and Tracy showed that  $\nu = 1/2$ . Our result for the  $1/c$  correction gives

$$\nu = \frac{1}{2} - \frac{2k_F}{\pi c} + O\left(\frac{1}{c^2}\right). \quad (6.30)$$

This agrees with a recent result of Haldane<sup>23</sup> and also of Popov, who obtained the value of  $v$  for arbitrary  $c$ ,

$$v = \frac{1}{2} [\rho(k_F)]^{-2}, \quad (6.31)$$

where  $\rho(k_F)$  is the Lieb-Liniger density function at the Fermi surface.

### Monodromy and Isomonodromic Deformation Theory<sup>22</sup>

The connection between Green's functions and Painlevé functions is particularly fascinating because of the elegant mathematical structure which can be associated with the Painlevé equations. This mathematical structure forms the basis of the analysis of Sato, Miwa, and Jimbo. Before introducing these ideas, let me explain what Painlevé did to get his name attached to these functions. In 1902 Painlevé<sup>25</sup> studied and solved the problem of classifying all second order ordinary differential equations of the form

$$y'' = f(y, y', t) \quad (6.32)$$

where  $f$  is an algebraic function, and with the requirement that the solutions should have no movable singularities. A movable singularity is one whose position depends on integration constants (i.e., on boundary conditions) and not just on the parameters in the equation. For example, the equation  $y' = 1/2y$  has a movable singularity because its solution is  $y = (t-a)^{1/2}$  where  $a$  is an arbitrary constant which doesn't appear in the differential equation. Painlevé showed that there were six kinds of equations of this form whose solutions could not be expressed in terms of

elementary functions. These equations are known as Painlevé I-VI. Somewhat later Schlesinger<sup>26</sup> and Garnier<sup>27</sup> showed that all six Painlevé equations were obtained in a natural way as integrability conditions in the deformation theory of ordinary differential equations. The relevance of Painlevé equations to the theory of Green's functions was first exhibited by Wu et al.,<sup>28</sup> who showed that the spin-spin correlation function of the two-dimensional Ising model in the scaling limit could be expressed in terms of a solution to Painlevé III. Motivated by this result, Sato, Miwa, and Jimbo discovered a very elegant derivation of the correlation function which exploited the monodromy property of a certain expectation value of order and disorder operators. The result of Jimbo et al.,<sup>12</sup> for the  $c = \infty$  nonlinear Schrödinger model was obtained by a similar technique.

To introduce the idea of monodromy and isomonodromic deformations, let us consider a linear problem of the form

$$\frac{dY}{dx} = \sum_{\nu=1}^N \frac{A_{\nu}}{x-a_{\nu}} Y, \quad (6.33)$$

where  $A_{\nu}$ ,  $\nu = 1, \dots, N$ , are  $x$  independent  $M \times M$  matrices, and  $Y(x)$  is an  $M \times M$  matrix solution satisfying some specified boundary condition, e.g.,  $Y(x_0) = I$ .  $Y(x)$  is not generally single-valued as we continue around the singularities at  $a_1, a_2, \dots, a_N$ . In general, traversing a closed curve around the singularity at  $x = a_{\nu}$  will produce a linear transformation

$$Y(x) \longrightarrow Y(x) M_{\nu}. \quad (6.34)$$

where  $M_\nu$  is an  $x$ -independent matrix called the monodromy matrix. Near  $x = a_\nu$  we can write

$$Y(x) = \hat{Y}(x) (x-a_\nu)^{L_\nu}, \quad (6.35)$$

where  $L_\nu$  is a constant matrix and  $\hat{Y}(x)$  is nonsingular at  $x = a_\nu$ . The monodromy matrices are related to the  $L_\nu$ 's by

$$M_\nu = e^{2i\pi L_\nu}. \quad (6.36)$$

Looking at the  $x = a_\nu$  pole of  $\frac{\partial Y}{\partial x} Y^{-1}$  we see that

$$A_\nu = \hat{Y}(a_\nu) L_\nu \hat{Y}(a_\nu)^{-1}. \quad (6.37)$$

The problem of constructing the monodromy matrices from the differential equation (6.33) is somewhat analogous to the direct problem of scattering theory. The analog of the inverse problem, i.e., reconstructing the function  $Y(x)$  and the differential equation (6.33) from the monodromy "data," is known as the Riemann-Hilbert problem. Schlesinger<sup>26</sup> addressed this question by studying the behavior of the equation (6.33) under a variation of the positions of the singularities  $a_\nu$ . Specifically, he allowed the coefficient matrices  $A_\nu$  to depend on the  $a_\nu$ 's and asked what conditions would lead to monodromy matrices  $M_\nu$  which were independent of the variation. Such an "isomonodromic deformation" leads to a linear system of equations for  $Y$ :

$$\frac{\partial Y}{\partial x} = \sum_{\nu=1}^N \frac{A_\nu}{x-a_\nu} Y \quad (6.38a)$$

$$\frac{\partial Y}{\partial a_\nu} = - \frac{A_\nu}{x-a_\nu} Y. \quad (6.38b)$$

In differential forms, this reads

$$dY = \Omega Y \quad (6.39)$$

where

$$\Omega = \sum_{\nu=1}^N A_\nu d \ln(x-a_\nu). \quad (6.40)$$

The dependence of the  $A_\nu$ 's on the  $a_\mu$ 's is given by the nonlinear consistency (integrability) conditions for the linear system (6.39). Using Poincare's Lemma,  $d^2(\text{anything}) = 0$  (i.e., mixed partial derivatives taken in reverse order are equal), we get the integrability condition

$$d\Omega = \Omega \wedge \Omega. \quad (6.41)$$

This is analogous to  $F_{\mu\nu} = 0$ , Eq. (2.24) in the inverse scattering method. In explicit form, the dependence of the  $A_\nu$ 's on the  $a_\mu$ 's which yields fixed monodromy data is given by "Schlesinger's equations,"

$$\frac{\partial A_\nu}{\partial a_\mu} = \frac{[A_\nu, A_\mu]}{a_\nu - a_\mu} \quad (\mu \neq \nu) \quad (6.42a)$$

$$\frac{\partial A_\nu}{\partial a_\nu} = - \sum_{\nu' \neq \nu} \frac{[A_\nu, A_{\nu'}]}{a_\nu - a_{\nu'}}. \quad (6.42b)$$

In the simplest nontrivial case of Schlesinger's equations, the  $A_\nu$ 's are  $2 \times 2$  matrices and there are  $N=4$  singularities. This case reduces to an ordinary nonlinear differential equation which is just Painlevé VI. Garnier<sup>27</sup> showed that Painlevé I thru V could also be obtained as monodromy preserving deformation equations. For this one must consider the linear equation

$$\frac{\partial Y}{\partial x} = \left\{ \sum_{\nu=1}^N \frac{A_{\nu}}{x-a_{\nu}} + C \right\} Y, \quad (6.43)$$

which allows for exponential behavior at infinity,

$$Y(x) \sim e^{Cx} [1 + O(1/x)]. \quad (6.44)$$

For (6.43) the simplest nontrivial case is  $N=2$ , and the deformation equations reduce to Painlevé V. This is the case which is relevant to the nonlinear Schrödinger model. The result expressing the functions  $R(t)$  and  $D_1(t)$ , Eqs. (5.19) and (5.20) in terms of Painlevé function can be derived from a monodromy argument. To see how this works, separate the kernel  $K(x,y) = \sin(x-y)/(x-y)$  into two pieces,  $K = K_+ + K_-$ , with

$$K_{\pm}(x,y) = \pm \frac{e^{\pm i(x-y)}}{2i(x-y)}. \quad (6.45)$$

Now define  $R_{\pm}(x)$  as in (6.2), and also define the series

$$\bar{R}_{\pm} = E_{\pm} + \lambda K_- E_{\pm} + \lambda^2 K_- K E_{\pm} + \lambda^3 K_- K^2 E_{\pm} + \dots = E_{\pm} + \lambda K_- \left[ \frac{1}{1-\lambda K} \right] E_{\pm}. \quad (6.46)$$

We see that  $R_{\pm}(x)$  is nonsingular at  $x=a$  and  $x=b$ , while  $\bar{R}_{\pm}(x)$  has a cut in the  $x$ -plane from  $a$  to  $b$ . Using the discontinuity of the kernel,

$$\text{Disc } K_-(x,y) = \pi \delta(x-y), \quad (6.47)$$

we get

$$\text{Disc } \bar{R}_{\pm}(x) = \pi \lambda R_{\pm}(x). \quad (6.48)$$

Thus, the matrix

$$Y(x) = \begin{pmatrix} R_+(x) & \bar{R}_+(x) \\ R_-(x) & \bar{R}_-(x) \end{pmatrix}, \quad (6.49)$$

has an isomonodromy property. Defining closed curves  $\gamma_1$  and  $\gamma_2$  around  $a$  and  $b$  respectively, it is easy to show that

$$Y(x) \xrightarrow{\gamma_1} Y(x)M_1 \quad (6.50a)$$

$$Y(x) \xrightarrow{\gamma_2} Y(x)M_2, \quad (6.50b)$$

where the monodromy matrices are

$$M_1 = \begin{pmatrix} 1 & \pi\lambda \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & -\pi\lambda \\ 0 & 1 \end{pmatrix}. \quad (6.51)$$

The monodromy is independent of the positions of the singularities  $a$  and  $b$ , and hence, by Schesinger's result, it follows that  $Y(x)$  obeys the linear relation

$$dY = \Omega Y \quad (6.52)$$

where

$$\Omega = A(a) d\ln(x-a) - A(b) d\ln(x-b) + C. \quad (6.53)$$

The first column of (6.52) is just Eq. (6.9), but now elegantly derived from monodromy properties. The Painlevé  $V$  equation follows from the integrability condition  $d\Omega = \Omega\Lambda\Omega$ .

The SMJ analysis of Green's functions adds some substance to the connection between integrability and duality. This is especially clear

in the case of the two-dimensional Ising model, where the function which exhibits a monodromy property is constructed from expectation values of order and disorder fields, and the monodromy is a direct consequence of the algebraic properties of these fields. It is encouraging to note that very similar ideas have emerged in recent studies of four-dimensional gauge fields.<sup>29</sup>

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