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# Fixed Points: Fermion Mass Predictions

CHRISTOPHER T. HILL
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

## ABSTRACT

The renormalization group equations for heavy fermion Higgs-Yukawa coupling constants possess low energy fixed points. We predict the masses of fourth generation quarks and leptons, or an ultra-heavy top quark. These also correspond to upper bounds on fermion masses in SU(5)-like theories.

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# FIXED POINTS: FERMION MASS PREDICTIONS

Christopher T. Hill

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

Familiar theoretical relationships exist amongst the masses of the light fermions in SU(5):[1]

$$\frac{m_d}{m_e} \simeq \frac{m_s}{m_\mu} \simeq 4 \pm \tfrac{1}{2} \qquad\qquad \frac{m_b}{m_\tau} \simeq 2.75 \pm .25 \qquad . \tag{1}$$

These result from essentially three ingredients: (a) the Higgs mechanism which relates the fermion mass to the Higgs-Yukawa (HY) coupling constant by $m_f = g_f\langle\phi^o\rangle/\sqrt{2} \simeq g_f \cdot (175 \text{ GeV})$; (b) the renormalization group (RG) equations which describe the evolution of the (HY) couplings down from $M_X$ to low energies and (c) the boundary conditions from SU(5), assuming a $\underset{\sim}{5}$ and $\underset{\sim}{24}$ of Higgs, which relate $g_{-1/3 \text{ quark}}(M_X) = g_{-1 \text{ lepton}}(M_X)$ within a given generation. Thus one obtains:

$$\frac{m_{-1/3}}{m_{-1}} = \frac{g_{-1/3}(\mu)}{g_{-1}(\mu)} \simeq \left( \frac{\alpha_3(\mu)}{\alpha_3(M_X)} \right)^{8/2b_o} + (\text{electroweak})\Big|_{\mu \simeq m_{\text{quark}}} \tag{2}$$

and the numerical results of eq. (1) follow to one loop accuracy with $\Lambda_s \backsim .4$ GeV. We note that ingredients (b) and (c) do not apply to models with composite Higgs bosons (e.g. ETC) or composite fermions on a scale of $M' < M_X$ since the RG equations demand pointlike particles over the entire range of the desert. Though the results for $m_d/m_e$ and $m_s/m_\mu$ are questionable we note that they are qualitatively correct. We take the "prediction" for $m_b/m_\tau$ to be a successful result of SU(5) and we seek to extend the above analysis to much heavier objects; either a very heavy top-quark or a heavy fourth generation of quarks and leptons.

For very heavy quarks and leptons the RG equations of the HY coupling constants, which lead to eq. (2), must be modified to include the effects of the (now large) HY couplings themselves. The equations thus become nonlinear in $g_f$ when $g_f \backsim (g_2$ or $g_3)$ and will thus fix the absolute scale of $g_f$. Hence, the relevant fermion mass scale for which these effects become important is expected to be, assuming a single $\underset{\sim}{5}$ of Higgs, $m_{\text{quark}} \backsim g_3(200)\cdot 175 \backsim 240$ GeV (with several isodoublet Higgs having VEVs this scale decreases).

Furthermore, as first emphasized by Frogatt and Nielsen[2] and more recently by Pendleton and Ross,[3] there will be "fixed points" which determine $g_f(\mu)$ <u>independent</u> of $g_f(M_X)$ (though in general dependent upon $\mu$, $M_X$, $g_3$, $g_2$, $g_1$, etc.). Thus for sufficiently large $g_f(M_X)$ (typically $g_f(M_X) \gtrsim 1$) we can hope to make predictions for heavy fermion masses from the fixed point structure of the RG equations without knowing any details of the initial conditions at $M_X$! However, the nature of these fixed points is slightly subtle,[4] as we now show.

Pendleton and Ross considered the possibility of a heavy t-quark (we will focus upon a fourth generation below which we expect to be a more realistic possibility; the t-quark will serve as a paradigm for the mathematics) for which the RG equations of $g_t$ and $g_3$ ($\equiv g_{QCD}$) become:

$$\left.\begin{aligned} 16\pi^2 \frac{d}{dt} \ln \, g_t &= \left( \frac{9}{2} g_t^{\,2} - 8 g_3^2 \right) \\ 16\pi^2 \frac{d}{dt} \ln \, g_3 &= -b_0 g_3^2 \quad ; \quad t = \ln \mu, \; b_0 = 7 \end{aligned}\right\} \quad . \tag{3}$$

Combining:

$$16\pi^2 \frac{d}{dt} \ln \, (g_t/g_3) = \frac{9}{2} g_t^{\,2} - (8 - b_0) g_3^2 = \frac{9}{2} g_t^{\,2} - g_3^2 \tag{4}$$

and hence, the Pendleton-Ross "quasi-stable fixed point" is the vanishing of the rhs of eq. (4):

$$g_t(\mu) = \sqrt{\frac{2}{9}}\, g_3(\mu) \quad . \tag{5}$$

If $g_t(\mu)$ is ever near the value given by eq. (5) in terms of $g_3(\mu)$, it will remain "locked in" to this relationship for all subsequent, decreasing $\mu$. Taking $\mu \sim 100$ GeV one has $m_{top} = g_t \cdot 175 = 110$ GeV, which becomes 135 GeV upon including electroweak corrections.

Physically we must ask, however, given an <u>arbitrary</u> initial $g_t(M_X)$ what is the <u>most probable</u> final result for $g_t(\mu)$? In fact, eq. (4) is just the "Bernoulli equation" and can be solved analytically:

$$g_t^{\,2}(\mu) = \frac{g_t^{\,2}(M_X) \left( g_3^{\,2}(\mu)/g_3^{\,2}(M_X) \right)^{8/b_0}}{\left\{ 1 + \dfrac{9 g_t^{\,2}(M_X)}{2 g_3^{\,2}(M_X)} \left( \left( \dfrac{g_3^{\,2}(\mu)}{g_3^{\,2}(M_X)} \right)^{1/b_0} - 1 \right) \right\}} \quad . \tag{6}$$

In the limit $\left(g_3^{\,2}(\mu)/g_3^{\,2}(M_X)\right)^{1/b_0} >> 1$ we reach the PR fixed point:

$$g_t^{\,2}(\mu) \rightarrow \frac{2}{9}\left(\frac{g_3^{\,2}(\mu)}{g_3^{\,2}(M_X)}\right)^{7/b_0} g_3^{\,2}(M_X) = \frac{2}{9}\, g_3^{\,2}(\mu)\Big|_{b_0=7} \quad . \tag{7}$$

However, to be at a fixed point in the sense that $g_t(M_X)$ no longer influences $g_t(\mu)$ it is sufficient that

$$\frac{9}{2}\,\frac{g_t^{\,2}(M_X)}{g_3^{\,2}(M_X)}\left(\left(\frac{g_3^{\,2}(\mu)}{g_3^{\,2}(M_X)}\right)^{1/b_0} - 1\right) >> 1 \tag{8}$$

which can easily occur for $\left(g_3^{\,2}(\mu)/g_3^{\,2}(M_X)\right)^{1/b_0} \backsim 1$, long before the limit leading to the PR fixed point. Defining $R = g_3^{\,2}(\mu)/g_3^{\,2}(M_X)$ we now assume $1/b_0 \ln R \backsim 0$. Expanding eq. (6) one finds:

$$g_t^{*2}(\mu) \equiv g_t^{\,2}(\mu) = \frac{2b_0}{9}\,\frac{g_3^{\,2}(\mu)}{\ln R}\left\{1 + \frac{1}{2b_0}\ln R + \frac{7}{12b_0^{\,2}}(\ln R)^2 + \ldots\right\} \tag{9}$$

Equation (9) defines a moving fixed point in $\mu$ and $M_X$ which we refer to as the "intermediate fixed point." It is the physically interesting asymptotic behavior for $g_t(\mu)$ as $\mu/M_X \rightarrow 0$, but sets in before the decoupling limit, $\mu \gtrsim m_t$.

In Figure 1 we illustrate how an arbitrary initial $g_t(M_X)$ tends to be swept toward $g_t^*(\mu)$ (we also plot the decoupling limit. $\mu$ = 175 $g_t(\mu)$) provided $g_t(M_X) \gtrsim 1$. This is a valid perturbative estimate provided $g_t^{\,2}(\mu)/16\pi^2 \lesssim 1$ or $g_t \lesssim 4\pi$. Hence, there is a large perturbative domain of attraction corresponding to $g_t^*(\mu) \backsim 1.3$ for $\mu$ = 200 GeV. The resulting "prediction" for our hypothetical single heavy t-quark is 240 GeV (± 10%) including full electroweak corrections.[4] This is also equivalent to the absolute upper bound of Cabibbo et al.[5] Is this a reasonable value for the mass of the physical t-quark? Recently Buras[6] has obtained a limit of $m_t \lesssim$ 33 GeV and we remark that a t-quark heavier than $\backsim$ 200 GeV will destroy the quantitatively successful $m_b/m_\tau$ relationship.[7] Otherwise, this is consistent with all bounds.

Probably more interesting and relevant to the real world are the consequences for a fourth SU(5) generation. Here we must numerically integrate the equations for $g_E$, $g_T$, $g_B$ (where T, B, E refer to +2/3,

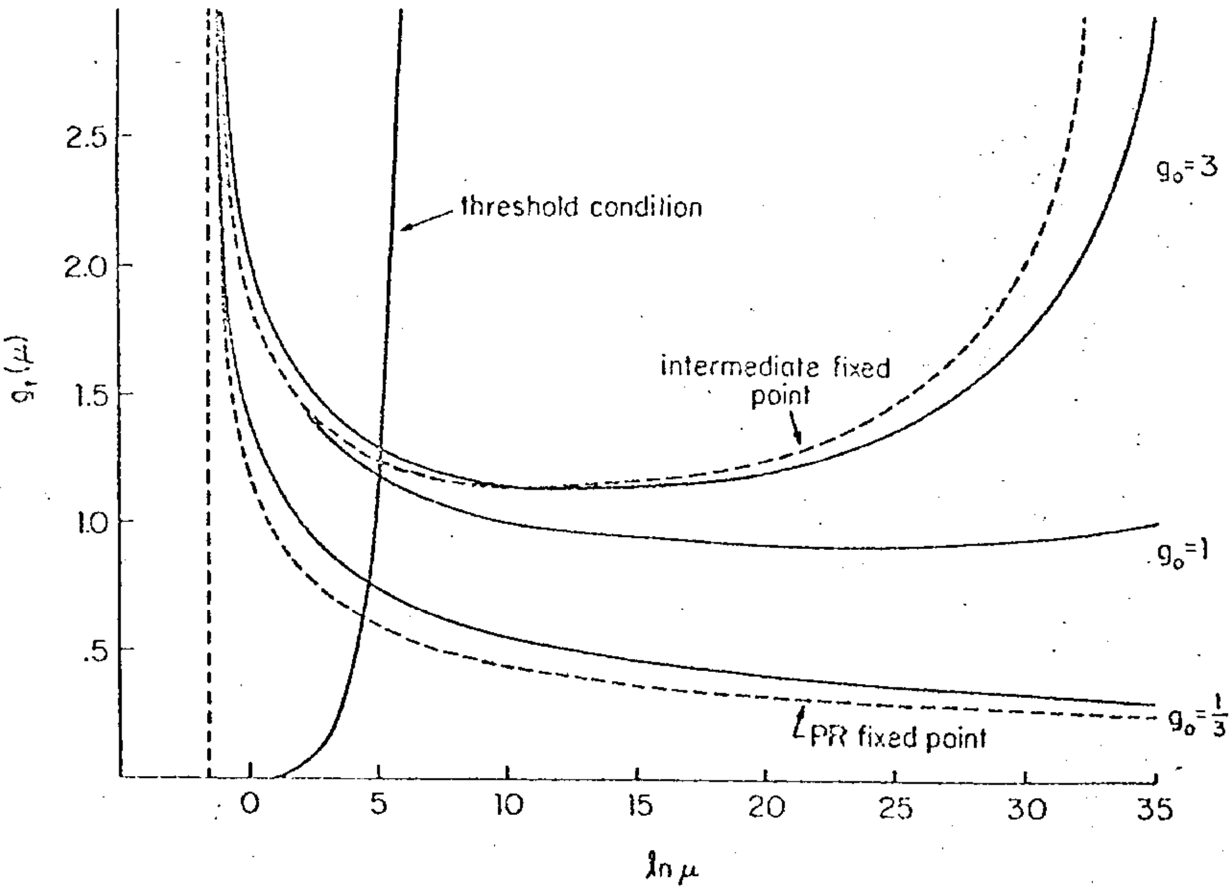


Figure 1. The evolution of $g_t(\mu)$ for three initial values, $g_t(M_X)$. Only the portion to the right of the "threshold condition" is physical. The intermediate fixed point of eq. (9) is the dotted line.

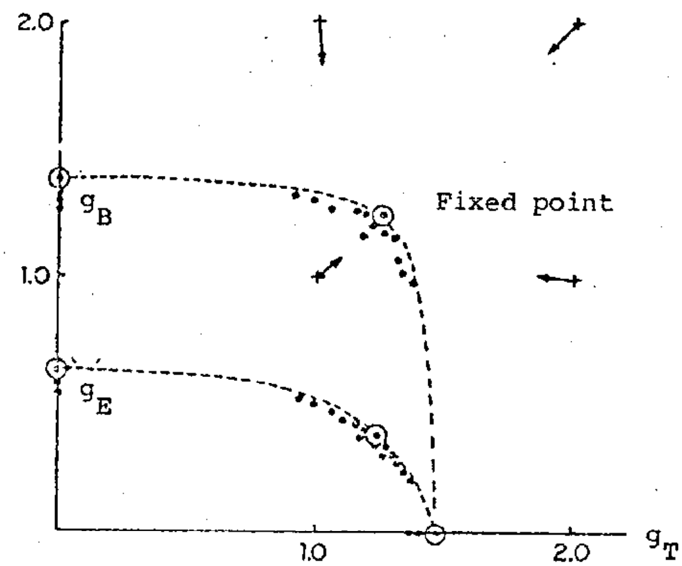


Figure 2. The results of the numerical integration of eq. (10) for a 5 × 5 integer array of $(g_T(M_X), g_B(M_X) = g_E(M_X))$. Points cluster about the fixed points and boundary curves.

$-1/3$ quarks and the $-1$ lepton respectively; we assume a light neutrino here):

$$16\pi^2 \frac{d}{dt} \ln g_T = \frac{9}{2} g_T^2 + \frac{3}{2} g_B^2 + g_E^2 - 8g_3^2 - \frac{9}{4} g_2^2 - \frac{17}{12} (g')^2$$

$$16\pi^2 \frac{d}{dt} \ln g_B = \frac{9}{2} g_B^2 + \frac{3}{2} g_T^2 + g_E^2 - 8g_3^2 - \frac{9}{4} g_2^2 - \frac{5}{12} (g')^2$$

$$16\pi^2 \frac{d}{dt} \ln g_E = \frac{5}{2} g_E^2 + 3g_T^2 + 3g_B^2 - \frac{9}{4} g_2^2 - \frac{15}{4} (g')^2 \quad . \quad (10)$$

The results of this analysis have been detailed in ref. (4). For lack of space we quote only the essential results.

(I) There is a nontrivial fixed point for these equations with all $g_i$ nonvanishing analogous to the intermediate fixed point discussed above. It corresponds to:

$$m_T = 220 \text{ GeV} \qquad m_B = 215 \text{ GeV} \qquad m_E = 60 \text{ GeV} \qquad (11)$$

to within 10% uncertainty at one loop. We've also assumed no Cabibbo mixing to lighter generations (the results are quite insensitive to this; rotating say the B quark maximally ($90^\circ$) out of the weak current for this generation displaces the fixed point 27%).

(II) The bound of Cabibbo et al.[5] generalizes to an allowed region in the space of $g_T$, $g_B$ and $g_E$. Only points within this region are physical (have finite values of $g_T$, $g_B$, $g_E$ over the entire desert). Moreover, the boundary of this region acts somewhat like a "generalized fixed point" since arbitrary, large initial points $g_T(M_X)$, $g_B(M_X)$ and $g_E(M_X)$ are mapped preferentially to the fixed point or the boundary, the fixed point lying on the boundary. Figure 2 illustrates the distribution of values of $g_B$(200 GeV) and $g_E$(200 GeV) vs. $g_T$(200 GeV) resulting from the numerical integration of eq. (10) for an initial $5 \times 5$ array of points $(g_B(M_X), g_T(M_X))$ and $(g_E(M_X) \equiv g_B(M_X), g_T(M_X))$. We see that the points cluster near the fixed point along the boundary curve (the fixed point corresponds to the mass values of eq. (11)). Hence, in addition to a relationship between $m_B$ and $m_E$, we also obtain a further relationship between all three masses $m_B$, $m_E$ and $m_T$ for a sufficiently heavy fourth generation! These results can be generalized to many succeeding heavy generations.

(III) These heavy masses are fully consistent with known bounds on fermion masses, e.g. the $\rho$-parameter,[8] unitarity[9] and the stability of the Higgs potential.[10]

Recently we have considered the effects upon the standard evolution of $g_1$, $g_2$ and $g_3$ in SU(5) from large $g_f$, at the two loop level, and thus the effects upon $M_X$ and $\sin^2\theta_W$.[11] These are found to be miniscule with:

$$\frac{\Delta M_X}{M_X} \lesssim \quad 2.5\% \text{ per heavy fermion} \tag{12a}$$

$$0 > \frac{\Delta \sin^2\theta_W}{\sin^2\theta_W} \gtrsim \quad -.15\% \text{ per heavy fermion} \tag{12b}$$

where eq. (12a) is the change in $M_X$ from the large HY coupling effects but <u>does not include</u> the change from a new flavor threshold. Surprisingly, the change in $M_X$ coming from the addition of a fourth generation neglecting HY effects, but with $m_T$, $m_B$ $\sim$ 200 GeV is found to be a factor of $\sim$ 1.25, much less than the 1.8 quoted earlier[12] but consistent with recent estimates of Marciano.[13] The effects of a fourth generation on $m_b/m_\tau$ are not known at present.

Searches for heavy quarks in the 200 to 240 GeV region and leptons in the vicinity of 60 GeV may make interesting grist for the Tevatron and collider mills of the future.

REFERENCES


1. A. Buras, J. Ellis, M.K. Gaillard and D.V. Nanopoulos, Nucl. Phys. B155, 189 (1979).

2. C.D. Frogatt and H.B. Nielsen, Nucl. Phys. B136, 115 (1978).

3. D. Pendleton and G.G. Ross, Phys. Letters 98B, 291 (1981).

4. C.T. Hill, Fermilab-Pub-80/97-THY, Dec. 1980, to be published in Phys. Rev. D.

5. N. Cabibbo, L. Maiani, G. Parisi, R. Petronzio, Nucl. Phys. B158, 295 (1979).

6. A.J. Buras, Fermilab-Pub-81/27-THY, March 1981.

7. M. Machacek, M. Vaughn, Northeastern Univ. Preprint, NUB-2493.

8. M. Veltman, Nucl. Phys. B123, 89 (1979).

9. M. Chanowitz, M. Furman, I. Hinchcliffe, Nucl. Phys. B153 402, (1979).

10. P.Q. Hung, Phys. Rev. Lett. 42, 873 (1979); H.D. Politzer and S. Wolfram, Phys. Lett. 82B, 242 (1979).

11. M. Fischler and C.T. Hill, Fermilab-Pub-81/43-THY, May 1981.

12. T. Goldman, D. Ross, Nucl. Phys. B171, 273 (1980); J. Ellis, M. Gaillard, D. Nanopoulos, S. Rudaz, Nucl. Phys. B176, 61 (1980).

13. W. Marciano, Talk presented at 2nd Workshop on Grand Unification and private communication.