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Colliding beams: What is an appropriate mathematical approach?

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Talk presented to seminar on the beam-beam interaction, at "Workshop on long-time prediction in nonlinear conservative dynamical systems," Austin, March 16-19, 1981.

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The perturbation on one beam due to the other in a colliding beam storage ring is concentrated in a finite number of points in the machine, so that the Hamiltonian describing the perturbation induces a number of explicit Dirac δ-functions. The most popular way to analyze such a system without recourse to a computer involves Fourier transformation, which amounts to replacing the δ-function $\delta(\theta - \theta_o)$ on the circle by $\frac{1}{2\pi} \sum_n e^{in(\theta-\theta_o)}$, with the intention of ultimately truncating the sum or concentrating only on a few particular terms.[1]

This approach has the disadvantage of obscuring the significance of the instantaneous kick-like nature of the interaction. This is easily remedied by observing that the instantaneous nature of the interaction allows the equations of motion (at least in the weak-beam/strong-beam idealization, assuming the strong-beam profile is known) to be explicitly integrated, at least for one revolution about the accelerator. In the case of e^+e^- or $\bar{p}p$ machines with one bunch of each type of particle, the equations of motion looked at in this way amount (ignoring effects due to the radio-frequency system) to applying a factorized transformation BLBL to the six-dimensional phase space describing the oscillation of a weak-beam particle about the closed design orbit. L represents the linear transformation corresponding to transit through half the ring; B represents the beam-beam kick that abruptly shifts momentum, leaving position unchanged. One must in addition include effectively instantaneous factors that describe radio-frequency impulses that give rise to phase stability. Whatever the RF modification, call the net single-revolution transformation (presumed calculable) F. Then study of long-time stability of the storage ring amounts to study of high iterates of the function F (i.e. sequences $\{x_n\}$ defined by $x_{n+1} = F(x_n)$, for large n).

I have restated this commonplace because a lot of progress has been made recently in precisely the area of highly iterated maps.[2] At Mel Month's suggestion I will review briefly those aspects of these developments which may have some suggestive power for the creation of an adequate theory of storage ring stability at high intersecting currents.

I might add, in passing, that the new developments, going under the name of "universality theory of iterated maps," is of interest to elementary particle theorists like myself not only because of the insight into experimental apparatus that it might lead to, but also because of its formal links with the theory of the renormalization group, one of the cornerstones of modern quantum field theory.

The bad news for accelerator theorists is that the universality theory at present only describes physical systems with (at large times) effectively one-dimensional phase-space; principally, dissipative systems (the one dimension corresponds to the least damped direction in the original multidimensional phase space)--indeed, only systems whose one-dimensional reduction amounts to iteration of a positive function F(x) on $x \in [0, 1]$ satisfying $F(0) = F(1) = 0$ and having a single maximum. (E.g. viscous fluids driven by periodic stirring or by an externally impressed temperature gradient. I will not go into the problem of how this reduction takes place, nor of how the existence of F is mathematically or physically ascertained.[3]) The beams in e^+e^- machines are damped by radiation emission, but they are not in addition subject to the kind of external pumping which makes possible the kind of effective one-dimensional F described above.

Nevertheless, although the setting of the universality theory and its results are manifestly not suitable for application to colliding-beam storage rings, the general style of argumentation may yet have something to teach us. I must say at the outset that although I have some obvious rough suggestions, I have no specific progress to report along such lines; the hope is that the kind of critical overview that follows may serve to stimulate deeper study.

I now proceed to sketch the most basic results of the universality theory. The theory's immediate practical goal (somewhat simplified) is to understand how the behavior of high iterates of $F_\lambda(x) \equiv \lambda f(x)$ depends on the parameter λ (assuming, as above, f is positive, vanishes at $x = 0$ and 1, and has a single extremum; the λ-dependence shown is chosen for simplicity only)--in particular to understand the behavior for λ near the value λ_∞ at which the distribution of high iterates becomes chaotic. This corresponds, for example, in a viscous fluid reducing to such a map, to the onset of turbulence at some critical Reynolds number.

<u>First basic fact</u>: There is a sequence of λ-values $\{\Lambda_n\}$, with n unbounded, such that for λ fixed in $[\Lambda_n, \Lambda_{n+1})$ the high iterates of F_λ approach a <u>unique</u> cycle (closed periodic orbit) or period 2^n, for almost all initial conditions.

[One of the main results of the theory is that the Λ_n cluster to λ_∞ from below, and that for large n they do so geometrically, $\Lambda_n \sim \Lambda_\infty - (\text{cnst})\delta^{-n}$, where δ is a number that depends only on the power with which f approaches its unique maximum at $x = x_m$ (i.e. ν in $f \sim f(x_m) - (\text{cnst})|x - x_m|^\nu$). I am not really concerned with this result in this review because in the storage ring setting, our cycles are never attractors; they are at best centers, at worst saddle points.

Nevertheless it may be useful to remark that the transition represented by $\lambda = \lambda_\infty$ is in some sense complementary to the Chirikov transition[4] of Hamiltonian systems, in that it seems to correspond to the <u>first</u> occurence of disorder (closed cycle of infinite period), rather than to some kind of takeover of all of a connected part of phase space by a stochasticity already existing (in a more localized way) at nearby parameters. This takeover transition occurs at some other value of λ above λ_∞; I will not dwell here on the phenomenology of $\lambda > \lambda_\infty$ (see ref. 3).]

Second basic fact: For all but one λ in $[\Lambda_n, \Lambda_{n+1})$ the transients in the approach to the 2^n-period cycle decay like μ^{-k}, where the scale μ depends on λ but not on the initial x, and where k is the number of iterations of F_λ. This fact is introduced so that I can draw attention to the unique value $\lambda_n \in [\Lambda_n, \Lambda_{n+1})$ at which μ vanishes ("superstability"). When $\lambda = \lambda_n$, the transients decay like $(\mu')^{-2^k}$, where μ' now does depend on the initial condition. It turns out that x_m (the location of the maximum of F_λ) belongs to the superstable 2^n-cycle of F_λ (i.e. it's one of the 2^n fixed points of $(F_\lambda)^{2^n}$, where raising F_λ to a power refers to multiple functional composition) when $\lambda = \lambda_n$. [As before, I do not mean to dwell on the notion of superstability per se since in our conservative storage ring systems there are no completely attracting orbits.]

Main result: Here is the point that may have some inspirational power for accelerator theorists. According to Feigenbaum, there exists an α (> 1) such that the following limit exists for all integers r:

$$\lim_{n\to\infty} (-\alpha)^{n-r} \left(F_{\lambda_n} \right)^{2^{n-r}} \left[(x - x_m)/(-\alpha)^{n-r} \right] \equiv g_r(x) \qquad .$$

Moreover the limit g_r is the same for all original functions f having a single maximum with the same power ν characterizing the approach to that maximum. Incidently, the limit of the g_r as $r \to \infty$ (call it g) also exists and satisfies the functional equation $g(x) = -\alpha g(g(x/\alpha))$.

In words, the existence of these limits means that certain high iterates of F_λ, at certain values of λ, when rescaled in a universal way relative to a certain value of x (i.e. x_m) <u>look alike</u>. In the viscous fluid context, such results enable one to quantitatively account for the way the power spectrum of a stirred or heated fluid changes as the Reynolds number comes close to the critical value for turbulence.[2]

It's natural to ask whether an analogous self-similarity law characterizes high iterates of the measure-preserving map correspoonding to the passage of the weak beam through a single revolution of a colliding-beam storage ring, in the presence of an unperturbed counterrotating strong beam. For example, are there matrices α_n and β_n, iterate numbers k_n, parameter values λ_n (specifying machine energy, beam-beam tune shifts, machine tunes, RF characteristics, etc.) and cycle points x_n such that

$$\lim_{n\to\infty} \alpha_n \left(F_{\lambda_n}\right)^{k_n} [\beta_n(x - x_n)]$$

exists? One might not want to preclude the possibility that $\alpha_n\beta_n \neq 1$, that $\alpha_{n+1}\alpha_n^{-1}$ is <u>not</u> independent of n, that the x_n are not all the same, or even that all the λ_n <u>are</u> the same. [In this last possibility I have in mind computer-generated pictures showing a two-dimensional phase space subject to the iterations of a fixed area-preserving map, revealing an apparently endless cascade of resonance islands surrounded by smaller islands surrounded by smaller ones, and so on.[5]]

Could such a limit be universal in some sense? What kinds of gross features would such an F_λ have to satisfy to ensure its membership in some particular universality class? In the one-dimensional case, as analyzed by Feigenbaum, the exponent ν characterizing the approach of the one-dimensional F_λ to its unique maximum x_m determines the universal limits by appearing explicitly in the boundary conditions under which one solves the functional equation $g(x) = -\alpha g(g(x/\alpha))$. Specifically (this is in fact how α is determined, by looking for such a solution) $g(x)$ must look near $x = 0$ like $a - b\ |x|^{\nu}$. The g_r are determined approximately by the eigenvectors of the linearization of the operator $\psi \to -\alpha\psi(\psi(x/\alpha))$ about the fixed point g, with ψ also subject to boundary conditions fixed by ν. What characteristics of a multidimensional conservative F_λ could provide boundary conditions for such a functional calculus?

The beam-beam force reaches its maxima near the periphery of the strong beam. Can this be enough to determine universal scaling limits of higher iterations? From a functional point of view the location of the maximum force is not especially interesting since F is still one-to-one there, (by virtue of its measure-preserving properties) and none of the eigenvalues of its linearization vanish (in contrast with the one-dimensional case where the derivative of F is zero at its maximum, and F is one-to-one only at x_m, two-to-one everywhere else). This would have been the most obvious possibility, but apparently it is too naive.

Still, if there were nontrivial aspects of self-similarity in the high iterates of the multidimensional Hamiltonian maps characterizing motion in storage rings, what kind of physics would this correspond to? The statement that some kind of nontrivial structure repeats itself, in some region, at ever-decreasing scales suggests motion of a chaotic nature, although it does not suggest in any way that this chaos has encroached upon the whole of phase space. The most popular theoretical ideas currently available for analysis of stochasticity in the colliding-beam context are those involving resonance overlap,[1] which concern transitions beyond which almost all of phase space is stochastic. A self-similarity theory could provide us with new information on the opposite limit--the local onset of stochasticity.

This could be quite useful: First, in view of the known existence of Hamiltonian systems exhibiting stochastic diffusion that ranges wide in phase space while occupying a small total volume[6]; Second, in view of the results of Kheifets'[7] and Ruggiero's[8] semiphenomenological analyses of the beam-beam effect in e^+e^- machines, which attribute to weak-beam phase space a diffusive fraction that (with radiation noise separated out) is typically very small, according to some measurements.

To summarize: The success of notions of self-similarity in the analysis of certain one-dimensional iterative systems related to turbulent flow encourages us to seek applications of such notions to the analysis of the higher-dimensional conservative systems corresponding to colliding-beam storage rings. This may yield information inaccessible to the resonance-overlap criterion, and may be of greater relevance to observed phenomena.

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