Fermi National Accelerator Laboratory

FERMILAB-Pub-80/83-THY
September 1980

Composite Gauge Fields

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ABSTRACT

In analogy with the SU(N) two dimensional σ-model we discuss models in which composite gauge fields arise. We show that it is possible to construct abelian as well as SU(2) composite gauge fields from the interactions of constituent scalar fields.

PACS Category Nos.: 11.10.Np, 11.30.Rd

I. INTRODUCTION

Soon after the introduction[1] of the two dimensional SU(N) σ-model as a close analogue to four dimensional nonabelian gauge theories, it was pointed out by several authors[2] that in an appropriate $N \to \infty$ limit the model also displays the interesting phenomenon of dynamical generation of an abelian gauge interaction. In other words, a composite abelian gauge field dominates the interactions in that limit and leads among other things to confinement. Whereas the gauge field may be introduced initially as a dependent quantity and without a kinetic term in the Lagrangian, such a term is generated dynamically in the $N \to \infty$ limit.

On the other hand, using some results from SO(8) extended super gravity[3] the suggestion has been made[4] that if the gauge fields that presumably mediate all interactions are not "elementary" but rather "composite," in a manner similar to the SU(2) two dimensional σ-model, then it becomes conceivable to construct a grand unified theory, from broken super gravity.

Independently from the above, however, the question of whether gauge fields can be thought of and obtained as composite fields from some more fundamental interaction is in itself of course interesting.

In this paper we consider this question and discuss models constructed by direct analogy with the SU(N) σ-model. Our emphasis is on whether a kinetic term for a dependent gauge field already introduced in these models can be generated as a result of the equations of motion of the model itself; and then whether this phenomenon survives quantization.

We discuss the problem in detail in Sec. II by reviewing briefly the SU(N) σ-model and then proceed to construct our model lagrangians for the abelian case in Sec. III and for SU(2) in Sec. IV. We then discuss some properties of the classical composite SU(2) gauge fields in Sec. V. We show in Sec. VI that the model survives quantization via a path integral approach and give in Sec. VII some concluding remarks.

II. COMPOSITE GAUGE FIELD IN THE TWO DIMENSIONAL SU(N) σ-MODEL

In this model in two dimensional Euclidean spacetime, one considers an N dimensional multiplet of complex scalar fields Z transforming as a vector under global SU(N) and satisfying the constraint $\bar{Z}Z = 1$. As this constraint restricts the fields only up to a general space dependent phase transformation

$$Z' = Ze^{i\Lambda(x)} \quad , \tag{2.1}$$

one is led to consider the "gauge invariant" Lagrangian density

$$\mathscr{L} = (\partial_\mu - iA_\mu)\bar{Z}(\partial_\mu + iA_\mu)Z \tag{2.2}$$

where A_μ is a "gauge field" transforming like

$$A'_\mu = A_\mu - \partial_\mu\Lambda(x) \tag{2.3}$$

under this phase transformation.

Clearly A_μ as introduced above is a dependent field and hence as it is determined through its equation of constraint (eqn. of motion!) its properties must be consistent with the above. In fact this is the case since one has

$$\frac{\partial\mathscr{L}}{\partial A_\mu} = 0 \rightarrow A_\mu = \frac{i}{2}(\bar{Z}\partial_\mu Z - \partial_\mu\bar{Z}\cdot Z) \quad . \tag{2.4}$$

Thus the classical constraint does give the required property (2.3) for A_μ and leads to a Lagrange density of the form

$$\mathscr{L} = \partial_\mu \bar{Z} \partial_\mu Z - i \frac{\bar{Z}\overleftrightarrow{\partial}_\mu Z}{2} \cdot i \frac{\bar{Z}\overleftrightarrow{\partial}_\mu Z}{2} \qquad (2.5)$$

showing a fundamental current current interaction between the constituent fields Z, $\bar{Z}$.

A quantized theory may now be constructed by a path integral approach. One then finds that the constraint (2.4) is still satisfied. This may be done in several ways. The first approach is to integrate over the field A_μ around its classical minimum and show that quantum fluctuations around it are quadratic and may be integrated out as a constant multiplication factor. The second is to impose the constraint on to the functional integral by an appropriate δ-function and then show that its effect is to provide only an inessential multiplicative factor, too. We detail here the first approach as it will be used later and since in it the constraint is not imposed externally. We have[5]

$$\int dZ d\bar{Z} dA_\mu d\sigma \exp\left\{ i \int d^2x (\partial_\mu - iA_\mu)\bar{Z}(\partial_\mu + iA_\mu)Z - \sigma(\bar{Z}Z - 1) \right\} \qquad (2.6)$$

where the σ-field is used to impose the constraint $\bar{Z}Z = 1$ on the quantum fields. If we now integrate A_μ around its classical <u>minimum</u> by writing

$$A_\mu = \frac{i}{2}(\bar{Z}\overleftrightarrow{\partial}_\mu Z) + \eta_\mu \quad , \quad dA_\mu \to d\eta_\mu \qquad (2.7)$$

then we obtain

$$\int d\bar{Z} dZ d\eta_\mu d\sigma \exp\left\{ i \int d^2x (\partial_\mu + iJ_\mu)\bar{Z}(\partial_\mu - iJ_\mu)Z + \eta_\mu{}^2 - \sigma(\bar{Z}Z - 1) \right\} \qquad (2.8)$$

where $J_\mu = 1/2i(\bar{Z}\overleftrightarrow{\partial}_\mu Z)$.

The η_μ integration clearly is doable and leads to an inessential constant and the constraint is implemented. Note here that if the quadratic fluctuations $\eta_\mu^{\ 2}$ had Z dependent coefficients then besides the implementation of the constraint further Z interactions would result; we would expect this to happen in general when a theory is quantized in this manner. However, considering again Eq. (2.8), we are left with

$$\int d\bar{Z}dZd\sigma \exp\left\{ i\int d^2x(\partial_\mu + iJ_\mu)\bar{Z}(\partial_\mu - iJ_\mu)Z - \sigma(\bar{Z}Z - 1)\right\} \quad . \tag{2.9}$$

The effects of J_μ may be now isolated by introducing a field B_μ and the following integral into the expression (2.9)

$$\int dB_\mu\, \delta(B_\mu + J_\mu) = 1 \quad . \tag{2.10}$$

We then parametrize the δ function as

$$\int d\lambda_\mu \exp -i\int d^2x 2\lambda_\mu \cdot (B_\mu + J_\mu) \tag{2.11}$$

and obtain

$$\int d\bar{Z}dZd\sigma dB_\mu d\lambda_\mu \exp\left\{ i\int d^2x(\partial_\mu - iB_\mu)\bar{Z}\cdot(\partial_\mu + iB_\mu)Z - \sigma(\bar{Z}Z - 1) - 2\lambda_\mu(B_\mu + J_\mu)\right\} \tag{2.12}$$

where λ_μ plays the role of a Lagrange multiplier field. Here again by a change of variables to

$$B'_\mu = B_\mu + \lambda_\mu$$

one finds that Eq. (2.11) becomes

$$\int d\bar{Z}dZd\sigma dB'_\mu d\lambda_\mu \exp\left\{ i\int d^2x(\partial_\mu - iB'_\mu)\bar{Z}(\partial_\mu + iB'_\mu)Z - \sigma(\bar{Z}Z - 1) - \lambda_\mu{}^2 \right\} . \quad (2.13)$$

The λ_μ integration is now trivial and one thus ends up with an equation identical to (2.6). However, we know now that the constraint on B'_μ is obeyed and further that the integration over B'_μ may nevertheless be performed independently from that over the constituent Z, $\bar{Z}$. This is all achieved without introducing any further interaction terms into the exponent of the path integral. This may not happen in general and such extra terms may indeed show up.

Now that the quantum theory is well defined we note that the Z integration is quadratic and may be done exactly.[2] In fact when this is done and then the limit $N \to \infty$ is taken,[6] a perturbative treatment of the gauge field B'_μ may be performed. The interesting result shown in Refs. (2) is that although no kinetic term appears in the exponent of Eq. (2.13) for B'_μ, it nevertheless displays in this limit a behavior characteristic of a true gauge field. Thus a composite gauge field is dynamically generated.

Having seen the details above we are now in a position to clarify the question we address ourselves to in this paper.

We ask whether it is possible to modify the Lagrange density of Eq. (2.2) in a way where the fields Z, $\bar{Z}$ interact gauge invariantly with the gauge field A_μ in such a manner that the equation of constraint is still obeyed and further that when the constraint is satisfied a kinetic term is generated for the field A_μ. Moreover, we require that all of this survives a quantization of the theory in a manner similar to the model discussed above with the possibility of generating some extra interactions.

We discuss this question for both abelian and SU(2) gauge fields. The classical Lagrangians are introduced in the following two sections and some of their properties discussed in Section V, whereas the quantization via path integrals is left then to Section VI.

III. KINETIC TERM FOR COMPOSITE GAUGE FIELDS-- ABELIAN CASE

We start our discussion by considering the abelian case started in the previous section except now we do not restrict ourselves to two space time dimensions.

If we define the covariant derivative

$$D_\mu Z = (\partial_\mu + iA_\mu)Z \tag{3.1}$$

then aside from the term already used before, namely $\overline{D_\mu Z} D_\mu Z$ we can construct the following gauge and SU(N) invariant combination

$$\mathscr{L}_K = \tfrac{1}{4} K_{\mu\nu} K_{\mu\nu}$$

with

$$K_{\mu\nu} = \frac{1}{i}\left[\overline{D_\mu Z} D_\nu Z - \overline{D_\nu Z} D_\mu Z\right] \quad . \tag{3.2}$$

In fact when the constraint $\bar{Z}Z = 1$ is imposed in Eq. (3.2) we find that the A_μ field drops out and one obtains

$$K_{\mu\nu} = \frac{1}{i}\left[\partial_\mu \bar{Z} \partial_\nu Z - \partial_\nu \bar{Z} \partial_\mu Z\right] \quad . \tag{3.3}$$

Thus if we take as a Lagrangian density

$$\mathscr{L} = \mathscr{L}_K + \overline{D_\mu Z} D_\mu Z \tag{3.4}$$

then the constraint equation on A_μ is unchanged, namely $A_\mu = \frac{i}{2} \bar{Z} \overleftrightarrow{\partial}_\mu Z \equiv -J_\mu$. Remarkably now, however, if this constraint is implemented in the Lagrangian density of Eq. (3.4) we find

$$\mathscr{L} = ¼ \left[(\partial_\mu J_\nu - \partial_\nu J_\mu)^2 \right] + (\partial_\mu + iJ_\mu)\bar{Z}(\partial_\nu - iJ_\nu)Z \quad . \tag{3.5}$$

Thus $\mathscr{L}_K$ of Eqs. (3.2) and (3.3) plays the role of a "kinetic term" for the composite field J_μ. We shall show in Section VI that implementing the constraint on A_μ into the quantized theory is unchanged by our addition; however isolating the effects of the composite gauge fields $A_\mu = -J_\mu$ requires the introduction of a Lagrange multiplier field λ_μ that does not integrate out completely but introduces further interactions into the Lagrangian.

IV. KINETIC TERM FOR COMPOSITE GAUGE FIELDS-SU(2)

The source of the abelian gauge freedom in the SU(N) σ-model discussed above is the invariance under a general phase transformation of the restriction $\bar{Z}Z = 1$. Thus in order to build a σ-model with an SU(2) gauge invariance one must generalize the space of Z fields in such a way that the phase invariance group of the restriction becomes the full SU(2). As emphasized by Gürsey and Tze[7], Z most naturally then belongs to quaternionic projective space. Thus instead of the elements Z_i, $i = 1 \ldots N$ be complex numbers we shall now consider them as quaternionic numbers. If we use as quaternionic basis vectors the quantities $\sigma^a = (1, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ with $\vec{\sigma}$ being the Pauli 2×2 matrices we then have

$$Z_i = q_i^a \sigma^a \quad , \quad \begin{array}{l} a=0,\ldots,3 \\ i=1,\ldots,N \end{array} \tag{4.1}$$

where the 4N quantities q_i^a are real. We also define

$$\bar{Z}_i = \sigma_2 Z_i^T \sigma_2 \tag{4.2}$$

where Z_i^T is the transpose of Z_i. Thus

$$Z_i = q_i^o - i\vec{q}_i \cdot \vec{\sigma}$$

and

$$\bar{Z}_i = q_i^o + i\vec{q}_i \cdot \vec{\sigma}$$

and hence

$$\sum_i \bar{Z}_i Z_i = \sum_i \left[(q_i^o)^2 + \vec{q}_i \cdot \vec{q}_i \right] \qquad (4.3)$$

is a quaternionic scalar quantity.

The restriction $\bar{Z}Z = 1$ therefore allows a general SU(2) "phase" transformation of the quaternionic elements Z_i. Thus it is invariant under

$$Z' = ZU \quad , \quad U = e^{i\,\vec{\sigma}/2\,\vec{\Lambda}(x)} \in SU(2) \quad . \qquad (4.4)$$

With this choice for the field space Z, $\bar{Z}$ we now proceed along lines similar to the previous sections. We introduce a quaternionic gauge field A_μ which under the gauge transformation U transforms as

$$A'_\mu = U^{-1}A_\mu U + iU^{-1}\partial_\mu U \qquad (4.5)$$

and consider the covariant derivative

$$D_\mu Z = Z(\overleftarrow{\partial}_\mu + iA_\mu) \quad .$$

We now construct the Lagrange density

$$\mathscr{L} = \mathrm{Tr}\left\{ \mathscr{L}_K + (\partial_\mu - iA_\mu)\bar{Z} \cdot Z(\overleftarrow{\partial}_\mu + iA_\mu) \right\} \qquad (4.6)$$

where

$$\mathscr{L}_K = \tfrac{1}{4}\,\mathrm{Tr}\, K_{\mu\nu} \cdot K_{\mu\nu}$$

and

$$K_{\mu\nu} = \frac{1}{i}\left[(\partial_\mu - iA_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu + iA_\nu) - (\partial_\nu - iA_\nu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu + iA_\mu)\right] \quad . \tag{4.7}$$

$\overleftarrow{\partial}_\mu$ acts only on the field Z preceding it and the order in which the quantities are written allows for thinking of them as appropriate matrices with all operations as matrix multiplication.

Clearly again the field A_μ is introduced as a dependent quantity and in this case $K_{\mu\nu}$ is dependent on A_μ in contrast to the abelian case. The constraint equation for A_μ may be worked out and both $\mathscr{L}_K$ and the other term in Eq. (4.6) give the same result, namely

$$A_\mu = \frac{i}{2}\bar{Z}\overleftrightarrow{\partial}_\mu Z = -J_\mu \quad . \tag{4.8}$$

This is of the same form as before except that all quantities are quaternionic.

One notices however immediately that A_μ is purely a "vector" quaternion, namely that it is of the form

$$A_\mu = \vec{A}_\mu\cdot\frac{\vec{\sigma}}{2} \quad . \tag{4.9}$$

Therefore there are only three independent quantities A_μ^a, precisely what is required of an SU(2) gauge field. Furthermore the constraint on A_μ satisfies the assumed gauge transformation properties of Eq. (4.5). Most importantly when the constraint is implemented into the Lagrangian of Eq. (4.6) we get

$$\mathscr{L} = A\,\mathrm{Tr}\left\{\partial_\mu J_\nu - \partial_\nu J_\mu - i[J_\mu, J_\nu]\right\}^2 + (\partial_\mu + iJ_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu - iJ_\mu) \quad . \tag{4.10}$$

Thus $\mathscr{L}_K$ plays again the role of a "kinetic" term for the composite SU(2) gauge field J_μ, just as in the abelian case.

V. SOME PROPERTIES OF THE CLASSICAL LAGRANGIANS

One may consider the models introduced above simply as σ-models by eliminating the A_μ fields altogether. In this case a unified notation may be used for both the abelian as well as the SU(2) case. We have

$$\begin{aligned}\mathscr{L} = \; & \frac{1}{4}\,\mathrm{Tr}\left\{\frac{1}{i}[\,\partial_\mu \bar{Z}\cdot(1 - Z\bar{Z})\partial_\nu Z - \partial_\nu \bar{Z}\cdot(1 - Z\bar{Z})\partial_\mu Z\,]\right\}^2 \\ & + \mathrm{Tr}\left(\partial_\mu - \frac{\bar{Z}\overleftrightarrow{\partial}_\mu Z}{2}\right)\bar{Z}\cdot Z\left(\overleftarrow{\partial}_\mu + \frac{\bar{Z}\overleftrightarrow{\partial}_\mu Z}{2}\right)\end{aligned} \tag{5.1}$$

where Z is an N component complex field in the abelian case and an N component quaternionic field for SU(2). In both cases $\bar{Z}Z = 1$ is understood. The "gauge" invariance of $\mathscr{L}$ is now implicit. One may fix the gauge by taking Z_N to be a real scalar in the abelian case and a quaternionic scalar, i.e. a real constant, in the case of SU(2). Furthermore classical equations of motion are best studied in terms of independent unconstrained variables t_i, $i = 1,\ldots,N-1$. There are various choices for these but to illustrate our points we choose two specific ones to facilitate contact with other works. These points are first that a Gribov type of ambiguity exists in these models even after a choice of gauge[8] and the second[7] is that the kinetic term of the SU(2) model allows instanton solution with a maximum winding number N, in contrast to a full-fledged SU(2) gauge field.

A. "Gribov Ambiguity"

One way to see that a Gribov ambiguity exists in our models even after a choice of gauge is to follow the method of Ref. 8 and change to the independent unconstrained variables ϕ_i defined by

$$Z_i = \frac{2\phi_i}{(1+\bar{\phi}\phi)} \qquad ; \qquad i = 1,\ldots,N-1$$

$$|Z_N| = \frac{(1-\bar{\phi}\phi)}{(1+\bar{\phi}\phi)} \qquad . \tag{5.2}$$

In this case the Lagrangian becomes

$$\begin{aligned}
\mathscr{L} = \ & \tfrac{1}{4}\,\mathrm{Tr}\Big\{ \tfrac{1}{i}\Big[\frac{4}{(1+\bar{\phi}\phi)^2}\,[\partial_\mu\bar{\phi}\partial_\nu\phi - \partial_\nu\bar{\phi}\partial_\mu\phi] \\
& - \frac{16}{(1+\bar{\phi}\phi)^4}\,[\partial_\mu\bar{\phi}\cdot\phi\bar{\phi}\partial_\nu\phi - \partial_\nu\phi\cdot\phi\bar{\phi}\cdot\partial_\mu\phi] \\
& + \frac{16(1-\bar{\phi}\phi)}{(1+\bar{\phi}\phi)^4}[\,\bar{\phi}\overleftrightarrow{\partial}_\mu\phi\cdot\partial_\nu(\bar{\phi}\phi) + (\bar{\phi}\overleftrightarrow{\partial}_\nu\phi)\partial_\mu(\bar{\phi}\phi)\,]\Big]\Big\}^2 \\
& + \mathrm{Tr}\left[4\,\frac{\partial_\mu\bar{\phi}\partial_\mu\phi}{(1+\bar{\phi}\phi)^2} + 16\,\frac{\bar{\phi}\overleftrightarrow{\partial}_\mu\phi\cdot\bar{\phi}\overleftrightarrow{\partial}_\mu\phi}{(1+\bar{\phi}\phi)^4} \right]
\end{aligned} \tag{5.3}$$

unless of course $\bar{\phi}\phi = 1$ for in that case the Lagrangian may be rewritten as

$$\begin{aligned}
\mathscr{L} = \ & \tfrac{1}{4}\,\mathrm{Tr}\Big\{ \tfrac{1}{i}[\,\partial_\mu\bar{\phi}(1-\bar{\phi}\phi)\partial_\nu\phi - \partial_\nu\bar{\phi}(1-\phi\bar{\phi})\partial_\mu\phi\,]\Big\}^2 \\
& + \mathrm{Tr}\left(\partial_\mu - \frac{\bar{\phi}\overleftrightarrow{\partial}_\mu\phi}{2}\right)\bar{\phi}\cdot\phi\left(\overleftarrow{\partial}_\mu + \frac{\bar{\phi}\overleftrightarrow{\partial}_\mu\phi}{2}\right)
\end{aligned} \tag{5.4}$$

which is of the same form as Eq. (5.1) with ϕ now an N-1 dimensional field! Thus for all configurations $\bar{\phi}\phi = 1$ one has to remake a choice of gauge and so on. This of course is so, clearly because for $\bar{\phi}\phi = 1$, $|Z_N| = 0$ vanishes and a specification of reality becomes meaningless then. This of course is simply the same phenomenon that appears in the simple CP^{N-1} model as discussed in Ref. 6.

B. The Kinetic Term and Instanton Solutions

An expected property of the kinetic term for the SU(2) case is to possess instanton solutions. To see that this is so we now choose the following set of unconstrained coordinates t_i:

$$t_i = \frac{Z_i}{|Z_N|} \quad , \quad i = 1,\ldots,N-1 \tag{5.5}$$

where

$$|Z_N| = \frac{1}{\sqrt{1 + \bar{t}t}} \quad .$$

This choice is made mainly to make contact with the work of Gürsey and Tze in Ref. 7. The kinetic term may be written now simply as

$$\begin{aligned} \mathscr{L}_K = \; & \frac{1}{4} \mathrm{Tr} \left\{ \frac{1}{i} \left[\partial_\mu \bar{t} \left(\frac{1}{1 + \bar{t}t} - \frac{t\bar{t}}{(1 + \bar{t}t)^2} \right) \partial_\nu t \right. \right. \\ & \left. \left. - \partial_\nu \bar{t} \left(\frac{1}{1 + \bar{t}t} - \frac{t\bar{t}}{(1 + \bar{t}t)^2} \right) \partial_\mu t \right] \right\}^2 \end{aligned} \tag{5.6}$$

with

$$A_\mu = \frac{i}{2} \frac{\bar{t} \overleftrightarrow{\partial}_\mu t}{1 + \bar{t}t} \quad .$$

This is the same as

$$\mathscr{L}_K = \tfrac{1}{4} \mathrm{Tr} \left\{ [illegible]_\mu [illegible] \partial_\nu [illegible]_\mu A_\nu [illegible] \right\}^2$$

and is precisely the parametrization used by Gürsey and Tze to study instanton solutions for SU(2) gauge theory. For the equations of motion for t_i derived from Eq. (5.6) automatically minimize the action derived from $\mathscr{L}_K$ and are equivalent to the duality or self-duality of the field tensor derived from A_μ. Such solutions are studied in detail by these authors and we refer the reader to their paper for details. We make here however the following observations:

1. Our composite gauge field A_μ is expressed in such a way that instanton solutions for the kinetic Lagrangian exist and are known in terms of the constituent fields t_i.

2. These instanton solutions however can carry a maximum winding number of N-1. For as shown in Ref. 7 the solutions to the equations following from the Lagrangian of Eq. (5.6), the instantons have 8(N-1)-3 parameters. This is precisely the maximum independent number of parameters for a winding number N-1 instanton in SU(2). Thus our composite gauge fields do not embody the full topological structure of a true gauge field unless N, the number of components in the constituent fields Z, is allowed to go to infinity. Thus it seems that the natural choice for the construction of composite gauge fields is to look at an appropriate $N \to \infty$ limit of the Lagrangian densities we have constructed.

3. As a corollary of the above, our Lagrangians represent then gauge theories with a truncated topological, and hence classical ground state, structures. These may be of interest by themselves as approximations of increasing complexity (with increasing N) to the true gauge theory. Thus for example an $N = 2$ model would represent a gauge theory with only one instanton solution which may be interpreted as a theory with a doubly degenerate classical ground state! Larger N involves an increased degeneracy until the full gauge theory is reached.

VI. PATH INTEGRAL QUANTIZATION

In defining a quantum theory out of the Lagrangians discussed above, special attention should be paid to the constraint equation on A_μ. For while classically A_μ is forced to take on the value $i/2\, \bar{Z}\overleftrightarrow{\partial}_\mu Z$ and consequently a kinetic term for this composite object is acquired, one must show that quantum fluctuations of A_μ around this extremum do not wash out this effect. In other words, one must show that this extremum is a stable minimum (and not say a saddle point!) and that preferably the fluctuations are integrable with minimal effects. We saw that this is indeed the case for the SU(N) σ-model discussed in Section II. We proceed now to discuss our models along similar lines.

Keeping to a simplified notation and dropping gauge fixing and other terms we are interested in the following path integral

$$\int dZ d\bar{Z} dA_\mu d\sigma \exp i \int (\mathscr{L} + \sigma(\bar{Z}Z - 1))dx \qquad . \tag{6.1}$$

The integration over the σ field is needed to enforce the condition $\bar{Z}Z = 1$ and the Lagrangian is of course

$$\mathscr{L} = \tfrac{1}{4}\, \mathrm{Tr}\left\{ \frac{1}{i} \left[(\partial_\mu - iA_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\nu + iA_\nu) - (\partial_\nu - iA_\nu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu + iA_\mu) \right] \right\}^2$$

$$+ \mathrm{Tr}\left[(\partial_\mu - iA_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu + iA_\mu) \right] \qquad . \tag{6.2}$$

We consider now fluctuations of A_μ around its classical minimum $i/2\, \bar{Z}\overleftrightarrow{\partial}_\mu Z$. We may write

$$A_\mu = \frac{i}{2}\bar{Z}\overleftrightarrow{\partial}_\mu Z + \eta_\mu \quad ; \quad dA_\mu \to d\eta_\mu \tag{6.3}$$

in which case the integrand in the exponent becomes

$$\mathscr{L} = \tfrac{1}{4}\,\mathrm{Tr}\left[F_{\mu\nu}(J) - i\,[\eta_\mu, \eta_\nu]\right]^2 + \mathrm{Tr}\left[(\partial_\mu + iJ_\mu)\bar{Z}\cdot Z(\partial_\nu - iJ_\nu)\right.$$

$$\left. + \eta_\mu{}^2 - \sigma(\bar{Z}Z - 1)\right]$$

where

$$F_{\mu\nu}(J) = \partial_\mu J_\nu - \partial_\nu J_\mu - i[J_\mu, J_\nu]$$

$$J_\mu \equiv \frac{1}{2i}\,\bar{Z}\overleftrightarrow{\partial}_\mu Z \qquad . \tag{6.4}$$

Clearly in the abelian case $[\eta_\mu, \eta_\nu]$ vanishes and the only fluctuations in η_μ are quadratic as in the two dimensional case. These can be integrated out leading only to a multiplicative factor. In the non-abelian case the situation is somewhat more complicated. Nevertheless one immediately notices that the fluctuations are quadratic in any one component of η_μ and at most quartic in this field. In fact the η_μ integral can be done formally exactly. To see how this comes about let us first expand the equation for the integrand above. We get

$$\mathscr{L} = \tfrac{1}{4}\left\{ \mathrm{Tr}\,[F_{\mu\nu}(J)]^2 + 2\eta_\mu^a \epsilon^{abc} f_{\mu\nu}^c \eta_\nu^b + \tfrac{1}{2}\eta_\mu^a \eta_\nu^b \eta_\mu^d \eta_\nu^e (\delta^{ad}\delta^{be} - \delta^{ae}\delta^{bd})\right\}$$

$$+ 2\eta_\mu^a \delta^{ab} \delta_{\mu\nu} \eta_\nu^b + \mathrm{Tr}\,(\partial_\mu + iJ_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu - iJ_\mu) - \mathrm{Tr}\,\sigma(\bar{Z}Z - 1) \tag{6.5}$$

where we have used the definitions

$$F_{\mu\nu} = f_{\mu\nu}^a \frac{\sigma^a}{2}$$

and

$$\eta_\mu = \eta_\mu^a \frac{\sigma^a}{2} \quad . \qquad (6.6)$$

If for the moment consider small η_μ and neglect the quartic terms the integrand has a term quadratic in η_μ and of the form

$$\eta_\mu^a \left(2\delta^{ab}\delta_{\mu\nu} - ½\,\epsilon^{abc} f^c_{\mu\nu} \right) \eta_\nu^b \quad . \qquad (6.7)$$

The quantity in parenthesis can be diagonalized in internal symmetry space by a global SU(2) rotation. The resulting three by three matrix has diagonal elements:

$$\lambda_1 = 2\delta_{\mu\nu}$$

$$\lambda_2 = 2\delta_{\mu\nu} - \sqrt{¼\,\vec{f}^{\,2}_{\mu\nu}}$$

$$\lambda_3 = 2\delta_{\mu\nu} + \sqrt{¼\,\vec{f}^{\,2}_{\mu\nu}} \quad . \qquad (6.8)$$

If we call this diagonal matrix M then the η_μ integration can be performed leading to an effective Lagrangian of the form

$$\mathscr{L} = ¼\,\mathrm{Tr}\,(F_{\mu\nu}(J))^2 - \mathrm{tr}\,\ln\det M(\vec{f}^{\,2}_{\mu\nu})$$

$$+ \mathrm{Tr}\,(\partial_\mu + iJ_\mu)\bar{Z}\cdot Z(\overleftarrow{\partial}_\mu - iJ_\mu) - \mathrm{Tr}\,\sigma(\bar{Z}Z - 1) \quad . \qquad (6.9)$$

Therefore an extra term symbolized by M is introduced by the quantum fluctuations.

If we now consider the quartic term we note that we can write

$$\sum_{a=1}^{3} \eta_\mu^a \; \eta_\nu^a = |\eta_\mu| \, |\eta_\nu| \; \cos \Theta_{\mu\nu} \tag{6.10}$$

where $\Theta_{\mu\nu}$ is an "angle" in the internal symmetry space between the μ and ν components of η_μ. This angle is of course invariant under rotations in this space. The quartic term thus becomes

$$|\eta_\mu|^2 \, |\eta_\nu|^2 \left(1 - (\cos \theta_{\mu\nu})^2 \right) \tag{6.11}$$

and is invariant under rotations that diagonalize the quadratic term. In fact since $\cos^2 \theta \leq 1$ this term is always greater than or equal to zero, that is of the same sign as the quadratic $F_{\mu\nu}$ dependent term Λ, so that at best it can modify the η_μ integral but not change its character. Thus including the quartic term would only complicate the η_μ integral without changing the main result, namely that the kinetic term persists in the quantized theory with extra interactions symbolized by a modified matrix M.

In order to isolate the effects of J_μ we introduce the following δ-functional into the functional integral

$$\int dB_\mu \, \delta(J_\mu + B_\mu) = 1$$

and finally get the model in the form

$$\int d\bar{Z} dZ d\sigma dB_\mu d\lambda_\mu \exp \Big[i\mathrm{Tr} \int \Big\{ \tfrac{1}{4}[F_{\mu\nu}(B)]^2 - \ln \det M(\vec{f}^{\,2}_{\mu\nu}(B)) + (\partial_\mu - iB_\mu)\bar{Z} \cdot Z(\overleftarrow{\partial}_\mu + iB_\mu) - \sigma(\bar{Z}Z - 1) - \lambda_\mu \cdot (B_\mu + \tfrac{i}{2}\bar{Z}\overleftrightarrow{\partial}_\mu Z) \Big\} \Big] \quad . \tag{6.12}$$

The Z, $\bar{Z}$ integral now is quadratic with coefficients depending on the fields B_μ, σ and the Lagrange multiplier field λ_μ. This may be done and a perturbation expansion in terms of the remaining σ, λ_μ and B_μ fields can be attempted. This and further study of the quantized version of this model will be left for future publications. We note however here that the presence of λ_μ is in the spirit of introducing collective variables by Jevicki and Sakita,[9] something that becomes more justifiable in $N \to \infty$ limit to be taken in our model as the natural limit required for a gauge field with full topological structure.

VII. CONCLUSIONS AND DISCUSSION

We have seen in the above that it is possible to construct Lagrangians where composite abelian and nonabelian SU(2) gauge fields follow as a result of the equations of motion. Other non-abelian composite gauge fields can be constructed in a similar manner and will be dealt with later. Thus we may conclude that not only is it possible to "parametrize" a gauge field in a composite manner but that this compositeness follows from ordinary equations of motion. In path integral language this means that the integral over the dependent field A_μ picks up its leading contribution when $\mathscr{L}$ is minimum at $A_\mu = i/2\, \bar{Z} \overleftrightarrow{\partial}_\mu Z$ and in that case this composite object appears with its own kinetic term in $\mathscr{L}$. Quantum fluctuations of A_μ around this minimum are then argued to introduce further interactions but not to eliminate this leading contribution.

One important observation is that as long as we consider σ-models of finite N then the composite gauge field does not have the full topological structure, thus indicating that it is then natural to consider the $N \to \infty$ limit of our models. Furthermore it also shows that the composite gauge fields considered by the authors of Ref. 4 may be such "truncated" gauge fields. This of course raises the question of whether this in any way diminishes the value of their suggestion. In other words, if the gauge fields are elementary they presumably carry the full topological structure; but if they are composite that structure may be limited by the symmetry group of the underlying constituents unless the $N \to \infty$ limit is considered. Which, if any, of all these possibilities does nature choose?

Finally if we consider other fields in the Lagrangian, such as Dirac fields, coupled to the Z fields via the currents $J_\mu = \frac{1}{2i}\bar{Z}\overleftrightarrow{\partial}_\mu Z$ such as $\bar{\psi}\gamma_\mu\psi J_\mu$ then clearly the fact that J_μ becomes effectively a composite gauge field one expects that the J mediated interactions between the ψ's become identical to the interaction expected from a gauge field. In particular all of instanton physics can be incorporated here. For the path integral over Z, $\bar{Z}$ does cover, for any N, all gauge theory instantons with winding number up to N.

Of course many problems remain to be considered concerning the above models and for example the exploration of the various properties of the "truncated" composite gauge fields seems also to be of interest particularly as concerns questions of CP violation. These questions and others are currently being pursued.

ACKNOWLEDGMENTS

This work was done while I was visiting Fermilab. I wish to thank C. Quigg and W.A. Bardeen for their hospitality during that visit and W.A. Bardeen and C. Hill for some discussions related to this work.

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