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Origin of Cancellation of Infrared Divergences in Coherent State Approach: Forward Process $qq \to qq + \text{gluon}$

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ABSTRACT

The idea of Kulish and Faddeev for treating asymptotic dynamics in QED by a deductive construction of asymptotic states corresponding to the asymptotic behavior of the Hamiltonian operator for $|t| \to \infty$, in the interaction representation, is further extended in QCD to provide a more complete analysis of the combinatorial structure of the asymptotic states obtained by the analogous construction in QCD. In the processes so far considered in low orders in perturbative QCD, we find that with these asymptotic states, IR divergent contributions are systematically generated in the matrix element such as to cancel the leading-order IR divergences arising from the unitarity cuts of the standard covariant graph of perturbative QCD. This occurs separately for each topological type of covariant graph. The cancellation of the forward scattering, leading-order IR divergences in the matrix element in "$qq \to qq + \text{gluon}$" to lowest order in perturbation theory is discussed in detail.

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I. INTRODUCTION

In QED a constructive procedure leading to an infrared-finite asymptotic dynamics was proposed by Kulish and Faddeev[1,2] which produces a set of asymptotic states with an "S-matrix operator" with infrared-finite matrix elements. In QED and gravitation such states are essentially simple generalizations of the well-known coherent states and in the non-Abelian gauge theory, we adopt this constructive procedure for generating a set of asymptotic states[3] in the infrared region[4] of perturbative QCD. This procedure is an amplitude approach which is based on the asymptotic behavior of the Hamiltonian operator for $|t| \to \infty$ in the interaction representation. This specifies an asymptotic Hamiltonian and an associated time evolution operator, $U_{as}(t)$, which is used to generate an initial asymptotic states' space $\mathscr{H}_{as} = U_{as}(t)\mathscr{H}_F$, where $t \to -\infty$, from the usual Fock space $\mathscr{H}_F$.

Since this is a formal construction, explicit perturbative calculations are then required in QED and QCD to verify that the S-matrix elements $\langle\psi| S_D |\psi'\rangle$ for specific ψ and ψ' in the asymptotic $\mathscr{H}_{as}$ are indeed infrared-finite. Such calculations in the literature have usually exploited the relations between this approach and the standard cross section approach in order to demonstrate the IR cancellations in the matrix element. However, the basic mechanism underlying the cancellation of the leading-order IR divergences in the coherent state approach is apparently very simple. To be specific, we consider the topological set of diagrams of Fig. 1 which contribute to the bremstrahlung of one gluon in quark-quark scattering. The non-vanishing diagrams in this set are shown in Fig. 2. In these Fig. 2 diagrams, which are defined by the expansion of the S-matrix operator in Eq. (29) in Sec. II, on-mass-shell particles bridge the initial state, graph, and final state parts. The graph part is the usual Feynman graph, including disconnected graphs.

The observation is that for the processes which have been studied to date, with the asymptotic states constructed in Sec. II, IR divergent contributions are systematically generated in the matrix element such as to cancel the leading-order IR divergences arising from the unitarity cuts of the standard covariant graph of perturbative QCD. This occurs separately for each topological type of covariant graph.

An interesting implication of this observation is that the amplitude approach does give meaning to "qq $\rightarrow$ qq + gluon" and to "q $\rightarrow$ q in a color-singlet external potential" without summing over final color spins and without averaging over initial color spins. For example, in a Yang-Mills notation for color, pp $\rightarrow$ pnρ^+ is free of leading order IR divergences. However, here in the amplitude approach a massive asymptotic quark (for example) evolves thru the dynamics generated by the asymptotic Hamiltonian and so H_{as} provides an explicit "dressing" of the massive quark with a color-singlet set of self-interacting soft gluons. Hence, in a physical measurement with finite energy and momentum resolution, it is necessary to sum and average over gluon color spins. Nevertheless, for massive quarks, it appears consistent, as far as the perturbative QCD leading-order IR behavior is concerned, to discuss reactions involving a massive quark of a definite color, so long as the quark is accompanied by its associated color-singlet set of self-interacting soft gluons.

In Sec. III the asymptotic states obtained in Sec. II are applied to show the cancellation of the forward scattering IR divergences in separate topological sets in the matrix element in "qq $\rightarrow$ qq + gluon" to lowest order in perturbative QCD. These states are also applied to show that the leading order, forward scattering IR divergences cancel in the 4-gluon vertex diagram in "qq $\rightarrow$ qq + 2 gluons." Cancellation in sums of separate topological sets for non-Abelian class of graphs in quark scattering in a color singlet external potential to order g^5 is shown in Sec. IV and the Appendix.

II. TREATMENT OF NON-ABELIAN ASYMPTOTIC DYNAMICS IN INFRARED REGION

In this approach[1,3] the asymptotic S-matrix operator in the usual Fock space, $\mathscr{H}_F$, is defined as

$$S^g(t_1, t_2) = \lim_{\substack{t_1\to\infty \\ t_2\to-\infty}} U_{as}(t_1)^\dagger \exp\,[-iH(t_1 - t_2)]\,U_{as}(t_2) \tag{1}$$

where the operator $U_{as}(t)$ satisfies

$$i\,\frac{dU_{as}(t)}{dt} = H_{as}(t)U_{as}(t) \tag{2}$$

with $H_{as}(t)$ the Hamiltonian which describes the asymptotic dynamics in the interaction representation. This can be re-expressed as

$$S^g(t_1, t_2) = \lim_{\substack{t_1\to\infty \\ t_2\to-\infty}} e^{-\Omega(t_1)}\, S_D(t_1, t_2)\, e^{\Omega(t_2)} \tag{3}$$

$$\Omega^\dagger(t) = -\Omega(t) \quad , \quad \text{anti-Hermitian}$$

where $S_D = \exp\,(-iH(t_1 - t_2))$ is the usual Dyson S operator. Therefore, we consider the initial asymptotic states in the space

$$\mathscr{H}_{as} = \lim_{t\to-\infty} \exp\,[\,\Omega(t)\,]\,\mathscr{H}_F \tag{4}$$

where[5]

$$\Omega(t) = -i\int^{t} V_{as}(\tau)d\tau - \frac{1}{2}\int^{t} d\tau \int^{\tau} d\sigma\, [\, V_{as}(\tau), V_{as}(\sigma)\,]$$

$$+ (-i)^3 \frac{1}{4}\int^{t} d\tau \int^{\tau} d\sigma \int^{\sigma} d\rho \Big[V_{as}(\tau), [\, V_{as}(\sigma), V_{as}(\rho)\,] \Big]$$

$$+ (-i)^3 \frac{1}{12}\int^{t} d\tau \int^{\tau} d\sigma \int^{\tau} ds \Big[[\, V_{as}(\tau), V_{as}(\sigma)\,], V_{as}(s) \Big]$$

$$+ \ldots \tag{5}$$

and in $\mathscr{H}_{as}$ we use the usual Dyson S operator.

The asymptotic Hamiltonian $H_{as}(t)$ is obtained from the usual QCD Lagrangian

$$\mathscr{L}(x) = \mathscr{L}_o(x) + \mathscr{L}'(x)$$

$$= -\tfrac{1}{4}\left(\partial^{\mu} A^{\nu}_{a} - \partial^{\nu} A^{\mu}_{a} - g f_{abc} A^{\mu}_{b} A^{\nu}_{c} \right)^2 - \overline{\psi}\Big[\gamma_{\mu}\, (-i\,\partial^{\mu} + g t_a A^{\mu}_{a}) + m \Big] \psi$$

$$+ \ldots \tag{6}$$

where ". . ." denote the ghost and gauge-fixing terms. We work in the 't Hooft-Feynman gauge

$$\Big[a_{a\mu}(k), a^{\dagger}_{b\nu}(\ell) \Big] = -\, \delta_{ab} g_{\mu\nu} \delta(\overrightarrow{k - \ell}) \tag{7}$$

but will not explicitly display the ghost terms as their combinatorial character is the same as that of the gluons. In the interaction representation, the fields $\psi(x)$, $A_{\mu a}(x)$, and $\overline{\psi}(x)$ are decomposed in the usual momentum representation, then the limit $|t| \to \infty$ is taken and, as discussed in Ref. 1, terms with exp $[\pm it(q^o + p^o \pm k^o)]$

are discarded versus terms with $\exp[\pm it(q^o - p^o \pm k^o)]$ since $q^o, p^o \geq m$. This yields the asymptotic Hamiltonian

$$H_{as}(t) = H_o + V_{as}(t) \tag{8}$$

$$V_{as}(t) = V_f(t) + V_c(t) + V_q(t) \tag{9}$$

$$V_{as}(t)^\dagger = V_{as}(t)$$

where

$$V_f(t) = \frac{g}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_o}} \frac{d^3p}{p_o} p^\mu a_{a\mu}(k) \rho_a(p) e^{-i\frac{k\cdot p}{p_o}t} + \text{h.c.} \tag{10}$$

with

$$\rho_a(p) = \sum_{\pm s} \left[b^\dagger(p, s) t_a b(p, s) - d^\dagger(p, s) t_a^T d(p, s) \right] \quad . \tag{11}$$

Since the quark mass m is chosen not to vanish as $|t| \to \infty$, the asymptotic Hamiltonian $H_{as}(t)$ is somewhat simpler than that for the complete theory because the quark-gluon interaction does not contain $q\bar{q}$ creation or annihilation pieces. In QED and gravitation this and the Abelian structure enable an exact solution of $H_{as}(t)$. However, in QCD there is no simplification, only a time-ordering, of the self-gluon couplings from the cubic and quartic gauge field terms. We find

$$V_c(t) = i \frac{gf_{abc}}{4(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_o}} \frac{d^3\ell}{\sqrt{2\ell_o}}$$

$$\times \left\{ a^\dagger_{a\mu}(k) a_{b\nu}(\ell) a_{c\eta}(m) 2V_{\mu\nu\eta}(k, -\ell, -m)' \frac{1}{\sqrt{2m_o}} e^{i(k_o - \ell_o - m_o)t} \right.$$

$$\left. - \quad a_{a\mu}(k) a_{b\nu}(\ell) a_{c\eta}(n) \frac{2}{3} V_{\mu\nu\eta}(k, \ell, n)' \frac{1}{\sqrt{2n_o}} e^{-i(k_o + \ell_o + n_o)t} \right\}$$

$$+ \text{h.c.} \tag{12}$$

$$V_{\mu\nu\eta}(k,q,r)' = [g_{\mu\nu}(k - q)_\eta + g_{\nu\eta}(q - r)_\mu + g_{\mu\eta}(r - k)_\nu] \tag{13}$$

and the prime denotes three momentum conservation, $\vec{k} + \vec{q} + \vec{r} = 0$, which also specifies the dependence (in the $V_c(t)$ integrand) of the third four-momentum on the other two. Note that the time-ordering yields a factor of 1/3 for the weight of the three-gluon annihilation relative to that for two-gluon annihilation. Similarly we find

$$V_q(t) = - \frac{g^2}{4(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_o}} \frac{d^3p}{\sqrt{2p_o}} \frac{d^3q}{\sqrt{2q_o}}$$

$$\times \left\{ a^\dagger_{a\mu}(k) a^\dagger_{b\nu}(p) a_{c\sigma}(q) a_{d\rho}(r) V^{abcd}_{\mu\nu\sigma\rho}(k, p, -q, -r)' \frac{1}{\sqrt{2r_o}} e^{i(k_o + p_o - q_o - r_o)t} \right.$$

$$\left. + a^\dagger_{a\mu}(k) a_{b\nu}(p) a_{c\sigma}(q) a_{d\rho}(s) \frac{2}{3} V^{abcd}_{\mu\nu\sigma\rho}(k, -p, -q, -s)' \frac{1}{\sqrt{2s_o}} e^{i(k_o - p_o - q_o - s_o)t} \right\}$$

$$+ \text{h.c.}$$

$$+ a_{a\mu}(k) a_{b\nu}(p) a_{c\sigma}(q) a_{d\rho}(t) \frac{1}{6} V^{abcd}_{\mu\nu\sigma\rho}(k, p, q, t)' \frac{1}{\sqrt{2t_o}} e^{-i(k_o + p_o + q_o + t_o)t}$$

$$+ \text{h.c.} \tag{14}$$

$$V^{abcd}_{\mu\nu\sigma\rho}(k, p, q, r)' = f_{abe}f_{cde}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$$

$$+ f_{ace}f_{bde}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma})$$

$$+ f_{ade}f_{cbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}) \tag{15}$$

From these expressions, it is possible to easily calculate the first few terms in the series for $\Omega(t)$. As in the Abelian case,

$$\Omega_f(t) = -\int^t V_f(\tau)d\tau = R_f(t)$$

$$= \frac{g}{(2\pi)^{3/2}}\int \frac{p^\mu}{p\cdot k}\,\rho_a(p)\left[a^\dagger_{a\mu}(k)e^{i\frac{k\cdot p}{p_o}t} - a_{a\mu}(k)e^{-i\frac{k\cdot p}{p_o}t}\right]$$

$$\times d^3p d^2k/\sqrt{2k_o} \tag{16}$$

$$\Omega_{f^2}(t) = -\tfrac{1}{2}\int^t ds\int^s d\tau\,[\,V_f(s), V_f(\tau)\,]$$

$$\equiv R_{f^2}(t) + i\,\Phi_{f^2}(t) \tag{17}$$

$$\Phi_{f^2}(t) = \frac{g^2}{8\pi}\int :\,\rho_c(p)\,\rho_c(q):\,\frac{p\cdot q}{\left[(p\cdot q)^2 - m_p^{\,2}m_q^{\,2}\right]^{1/2}}\,\varepsilon(t)\ln\frac{|t|}{t_o}$$

$$\times\ d^3p\, d^3q\ + \text{(mass renorm. term, proportional to t)} \tag{18}$$

where $R_{f^2}(t)$ vanishes in the Abelian case, but not in the non-Abelian gauge theory. Note that because of the $\varepsilon(t)$, $\Phi_{f^2}(t)$ has the property

$$\lim_{t\to\infty} \Phi_{f^2}(t) = -\lim_{t\to-\infty} \Phi_{f^2}(t)$$

and so there is a lowest order (trivial) generalization of the cancellation[1] of such an IR divergent exponent with the well-known Coulomb phase in QED and gravitation. In the non-Abelian theory, with a mass λ(IR cut-off) for the gluon, to lowest-order

$$S^g = \lim_{\lambda\to 0}\left[1 + i\Phi_{f^2}(\lambda)\right] S_{F,} \left[1 + i\Phi_{f^2}(\lambda)\right] \tag{19}$$

$$\Phi_{f^2}(\lambda) = -\frac{g^2}{4\pi}\ln(\lambda t_o)\sum_{i\neq j}\left(v^{-1}(p_i p_j) t_c^{(i)} t_c^{(j)} + v^{-1}(q_i q_j) t_c^{T(i)} t_c^{T(j)}\right.$$
$$\left. - v^{-1}(p_i q_j) t_c^{(i)} t_c^{T(j)}\right) \tag{20}$$

whereas the virtuals (that are responsible for the divergent Coulomb phase) yield a factor,[7] for each divergent pair of fermions in the initial or final state,

$$\left[1 + \frac{i}{4\pi} v^{-1}(p_n p_m) g_n^c g_m^c \ln(\lambda/\Lambda)\right] \tag{21}$$

$$g_n^c = \begin{cases} \eta_n g t_c^{(n)} & , \text{ quark} \\ -\eta_n g t_c^{T(n)} & , \text{ antiquark} \end{cases}$$

with $\eta_n = \eta_m = \pm 1$ for outgoing, ingoing, fermion respectively. Here, the relative velocity

$$v(p, q) = \left[1 - m_{p^2} m_{q^2}/(p\cdot q)^2\right]^{1/2}$$

and $\Lambda = 1/t_o$ (UV cut-off). In the non-Abelian theory, in $\Omega(t)$, as given by Eqs. (5) and (6), we find that generalizations of this type of term, but with a more complicated t structure occur in the case of states with several charged quarks or antiquarks. These appear as contributions from the Lie elements of third degree or higher in Eq. (5).

In the "R(t) terms" we assume that the contributions from the lower limits of integration must vanish (in the Abelian case this is necessary so that $\Omega(t)$ commutes asymptotically with the total momentum operator).[1] In these terms, we adopt as a working "ansatz" the replacement of

$$a^{\dagger}_{a\mu}(k) e^{i\frac{k\cdot p}{p_o}t} \rightarrow a^{\dagger}_{a\mu}(k) = \sum_m e^{(m)}_{\mu} a^{\dagger}_i \tag{22}$$

where $e_{\mu}^{(m)}(k)$, m = 1,2 are transverse polarization vectors with "i" denoting "a" for the color gauge-group index, "k" for the gluon four-momentum, and "m" for the polarization, and we make a similar replacement in the terms arising from the self-gluon couplings (i.e. replace the exponential of [i (energy difference) t] by one). Again, in the Abelian case this "ansatz" has been justified and shown to be relativistically and gauge invariant, and $\mathcal{H}_{as}$ has been shown to be t independent and to contain a Lorentz and gauge invariant subspace with nonnegative metric.[1]

Therefore, we obtain

$$\Omega_f = -\int d^3p\, d^3k\, \rho_a(p) \left[\sum_{\ell=1}^{2} \tilde{S}^{(\ell)}(k) a_i(k) + \sum_{\ell=1}^{2} \tilde{S}^{(\ell)}(k)^* a_i^{\dagger}(k) \right]$$

$$\tilde{S}^{(\ell)}(k) \equiv \frac{g\, p\cdot e^{(\ell)}}{\left[2(2\pi)^3 k_o\right]^{1/2} p\cdot k} \tag{23}$$

as in the Abelian case, but now obtain also higher-order terms due to the non-Abelian structure of the theory.

$$R_f^{\,2} = -\int d^3p d^3k d^3 \ell \rho_c(p) \sum_{\ell,m} \Big[a_j^{\dagger}(\ell)a_i(k)\bar{\gamma}_{ji}(p) + a_j(\ell)a_i^{\dagger}(k)\bar{\gamma}_{ji}(p) + a_j(\ell)a_i(k)\bar{\beta}_{ji}(p) + a_j^{\dagger}(\ell)a_i^{\dagger}(k)\bar{\beta}_{ji}(p) \Big]$$

$$\bar{\beta}_{ji} = \frac{1}{2}\, i f_{cba} \tilde{S}_i^{(\ell)}(k)\tilde{S}_j^{(m)}(\ell)\frac{p\cdot k}{p\cdot(k+\ell)}$$

$$\bar{\gamma}_{ji} = \frac{1}{2}\, i f_{cba} \tilde{S}_i^{(\ell)}(k)\tilde{S}_j^{(m)}(\ell)\frac{p\cdot k}{p\cdot(\ell-k)} \tag{24}$$

The simplest terms arising from the self-gluon couplings are

$$\Omega_c = i\int d^3k d^3 \ell \sum_{\ell mn} \Big(\tilde{S}_{ijk}(k,-\ell,-m)2a_i^{\dagger}(k)a_j(\ell)a_k(m) + \tilde{S}_{ijk}(k,\ell,n)\frac{2}{3}\, a_i(k)a_j(\ell)a_k(n) + \text{a.c.} \Big) \tag{25}$$

where

$$\tilde{S}_{ijk}(k,\ell,n) = \frac{g}{4(2\pi)^{3/2}}\frac{f^{abc}}{\sqrt{8k_o\ell_o m_o}}\frac{e_\mu^{(\ell)}e_\nu^{(m)}e_\eta^{(n)}V_{\mu\nu\eta}(k,\ell,n)'}{[\,k_o+\ell_o+n_o\,]} \tag{26}$$

and also

$$\Omega_q = \int d^3k d^3\ell d^3q \sum_{\ell mno} \Big(\tilde{S}_{ijk\ell}(k, p, -q, -r) a_i^\dagger(k) a_j^\dagger(p) a_k(q) a_\ell(r)$$

$$+ \tilde{S}_{ijk\ell}(k, -p, -q, -s) \tfrac{2}{3} a_i^\dagger(k) a_j(p) a_k(q) a_\ell(r) + \text{a.c.}$$

$$+ \tilde{S}_{ijk\ell}(-k, -p, -q, -t) \tfrac{1}{6} a_i(k) a_j(p) a_k(q) a_\ell(r) + \text{a.c.} \Big) \tag{27}$$

where

$$\tilde{S}_{ijk\ell}(k, p, q, r) = \frac{g^2}{4(2\pi)^3} \frac{1}{\sqrt{16 k_o q_o p_o r_o}} \frac{e_\mu^{(\ell)} e_\nu^{(m)} e_\sigma^{(n)} e_\rho^{(o)} V^{abcd}_{\mu\nu\sigma\rho}(k, p, q, r)'}{[k_o + p_o + q_o + r_o]} . \tag{28}$$

The diagrams associated with the matrix element of interest, such as those displayed in Figs. 2-4, are given meaning by expanding

$$S^g = \exp\, [-\Omega(\infty)] S_D \exp\, [\Omega(-\infty)] \tag{29}$$

inserted between the chosen initial and final states. The significance of the $\exp[-\Omega(+\infty)]$ and $\exp[\Omega(-\infty)]$ operators is that they automatically generate the set of graphs[3] with the desired signs and weights such that the matrix-element is infrared-finite. This is discussed in detail in Secs. III and IV. Because of the time-ordering, the state contributions resemble that of time-ordered perturbation theory but with the contributions of one end of the time integration omitted.

In the non-Abelian gauge theory a simple "c-number" exponentiation of the IR structure will only occur in special cases, unlike in QED where it and the associated Poisson distribution is a more general IR property (but c.f. Ref. 1). Chung's states, Ref. 2, but with eA_μ in QED replaced by $gt_a A^a_\mu$ have been used in some phenomenological calculations in QCD in Ref. 8. This is only the first term in the expression for Ω given by Magnus' Theorem in Eqs. (5) and (6) above. In the cross section approach, Appelquist and Carazzone have emphasized[5] the importance of grouping together the IR divergent contributions from the different unitarity cuts of each distinct topological diagram. That IR cancellation might occur in distinct topological diagrams in the amplitude approach was observed earlier[3,6] but the only complete proof was[3] that in order g^5 the leading order non-Abelian class of IR divergences cancel off separately in quark scattering in a color-singlet external potential.

III. FORWARD PROCESS $qq \rightarrow qq$ + gluon

The treatments of forward scattering IR divergences in $qq \rightarrow qq$ + gluon, or in the process of single gluon bremsstrahlung by a quark scattering in a color singlet external potential, are quite different in the cross section and in the amplitude approaches. Unlike the treatment of quark scattering in a color singlet external potential, the two approaches to cancelling the leading order IR divergences in the calculation for this process are not related by transformations in the perturbation expansions between the diagrams of one approach and those of the other where both sets of diagrams are of the same order in the coupling constant. In the cross section approach Sachrajda showed[9] that in lowest order perturbative QCD, ignoring contributions from virtual soft gluons, the cross section for single-gluon radiation in quark-quark scattering is quadratically divergent, i.e. $\sim 1/\lambda^2$, where λ is a gluon mass IR cut-off. Radiation from an internal gluon line via the 3-gluon vertex was found to contribute to this divergence. However, in the cross section approach, Matsson and Meuldermans later showed[10] that the forward quark-quark cross section including second order virtual and real radiative corrections in QCD is not IR divergent. Our purpose here is to show, using the asymptotic states constructed in Sec. II for the non-Abelian gauge theory, that there is a cancellation of the leading-order forward scattering IR divergences in the matrix element to lowest order in perturbation theory, and that this occurs separately for each topological set of diagrams.

The possible contributing diagrams in the topological set with the 3-gluon vertex which could contribute to "qq → qq + gluon" can be grouped according to the number of vertices contributed by the states. Examples of such diagrams are shown in Fig. 3. Explicit expressions for these graphs are obtained by expanding S^g of Eq. (29). For no state vertices, to lowest-order in g there is only one diagram, that of Fig. (3i) and it is the usual Feynman graph. We find

$$\begin{aligned} T_i = -2g^3 f_{abc} t_c t_b' \Big\{ & e\cdot(p_2 - p_4)\bar{u}(p_3)\gamma^\nu u(p_1)\bar{u}(p_4)\gamma_\nu u(p_2) \\ & + \bar{u}(p_3) e\cdot\gamma u(p_1)\bar{u}(p_4)\not{k} u(p_2) - \bar{u}(p_3)\not{k} u(p_1)\bar{u}(p_4) e\cdot\gamma u(p_2) \Big\} \\ & \times \left([p_1 - p_3]^2 - \lambda_\ell^2\right)^{-1}\left([p_2 - p_4]^2 - \lambda_m^2\right)^{-1} \end{aligned} \tag{30}$$

where this amplitude has been regulated by insertion of $\lambda_\ell^2 = \lambda_m^2 = \lambda^2$ with $k^2 = \lambda^2 > 0$ for the radiated gluon.

If the gluon masses are allowed to vanish, then $\ell^2 = [p_1 - p_3]^2$ vanishes only when ℓ_μ vanishes, so $\vec{p}_3$ is parallel to $\vec{p}_1$ and ℓ_μ is "soft." But then, $k_\mu = m_\mu$ and since $k^2 = 0$, $m^2 = [p_2 - p_4]^2$ also vanishes, so m_μ vanishes and $\vec{p}_4$ is parallel to $\vec{p}_2$. To study this forward scattering IR divergence, we adopt the procedure of assuming a finite mass for the virtual and radiated gluons to regulate the contributing amplitudes in the topological set. By letting regulator masses vanish, we then consider each of the physical unitarity cuts of T_i and show for each that any leading IR divergence has been cancelled by contributions from additional amplitudes involving state vertices, i.e. from amplitudes for Figs (3ii) thru (3v). As discussed in Sec. II, the amplitude approach is based on the behavior of the total Hamiltonian operator for $|t| \to \infty$ so it is reasonable to use unitarity cuts to separate and collect contributions to the leading IR divergences. However, the assumption of a finite mass for the virtual and radiated gluons excludes some diagrams with state vertices (see next paragraph), so for consistency, in collecting the contributions at the unitarity cuts the regulator masses must be allowed to vanish in such a manner as to not produce unitarity cuts which would correspond to any of these excluded diagrams.

For one and two state vertices, there are 18 possible contributing diagrams (including diagrams with gluon self-coupling effects in the states). However, energy-momentum conservation for the covariant graph part implies that it is impossible to satisfy the mass-shell constraints for all 3 particles at a qq-gluon trilinear vertex ($2m^2 > \lambda^2 > 0$) and similarly at a 3-gluon vertex (equal finite masses). Hence, only the 4 diagrams displayed in Fig. (3ii) to Fig. (3v) are non-vanishing. We find

$$
\begin{aligned}
T_{ii} = -2g^3 f_{abc} t_c t'_b \Big\{ & e\cdot(p_2 - p_4)\bar{u}(p_3)u(p_1)\bar{u}(p_4)\not{p}_3 u(p_2) \\
& + e\cdot p_3 \bar{u}(p_3)u(p_1)\bar{u}(p_4)\not{k}u(p_2) \\
& - p_3\cdot k\bar{u}(p_3)u(p_1)\bar{u}(p_4)e\cdot\gamma u(p_2) \Big\} \\
& \times \frac{\sqrt{p_{1,0}p_{3,0}}}{m} [2\ell'_0 p_3\cdot \ell']^{-1} \theta\left([p_3 - p_1]^0 \right) \left([p_2 - p_4]^2 - \lambda_m^2 \right)^{-1} \quad ; \quad \vec{\ell}' = \overrightarrow{p_3 - p_1}
\end{aligned}
\tag{31}
$$

$$T_{iv} = -2g^3 f_{abc} t_c t'_b \Big\{ e\cdot(p_2 - p_4)\bar{u}(p_3)u(p_1)\bar{u}(p_4)\not{p}_1 u(p_2)$$

$$+ e\cdot p_1 \bar{u}(p_3)u(p_1)\bar{u}(p_4)\not{k} u(p_2)$$

$$- p_1\cdot k\bar{u}(p_3)u(p_1)\bar{u}(p_4)e\cdot\gamma u(p_2) \Big\}$$

$$\times \frac{\sqrt{p_{1,0}p_{3,0}}}{m} [2\ell_o p_1 \cdot \ell]^{-1} \theta\left([p_1 - p_3]^o \right) \left([p_2 - p_4]^2 - \lambda_m^2 \right)^{-1} \; ; \; \vec{\ell} = \overrightarrow{p_1 - p_3} \qquad (32)$$

and similarly for T_{iii} and T_v. Here $\ell_o = \ell'_o = \sqrt{\lambda_\ell^2 + (\overrightarrow{p_1 - p_3})^2} = \alpha$.

For three state vertices, $S_D = 1$ so the anti-Hermitian property of Ω implies that the total contribution from diagrams with three state vertices is zero.

We find that T_{ii} and T_{iv} serve to cancel the leading IR divergences of T_i which arise from the unitarity cuts associated with the $\left([p_1 - p_3]^2 - \lambda_\ell^2 \right)$ denominator vanishing in forward scattering as $\lambda^2 \to 0$, then $\lambda_\ell^2 \to 0$ with λ_m^2 finite since

$$\left([p_1 - p_3]^2 - \lambda_\ell^2 \right)^{-1} = \left\{ [(p_1 - p_3)^o - \alpha]^{-1} - [(p_1 - p_3)^o + \alpha]^{-1} \right\} / 2\alpha$$

$$\simeq \left\{ - p_{3,0}\theta\left([p_3 - p_1]^o \right) \Big/ 2\ell'_o p_3 \cdot \ell' - p_{1,0}\theta\left([p_1 - p_3]^o \right) \Big/ 2\ell_o p_3 \cdot \ell \right\}$$

$$\times \left\{ 1 + 0\, [\lambda E / 2(E^2 - m^2)] \right\} \qquad . \qquad (33)$$

in center of mass frame where $E = p_{1,0} = p_{2,0}$. Similarly, T_{iii} and T_v are found to cancel the leading IR divergences of T_i which arise from the unitarity cuts associated with the $\left([p_2 - p_4]^2 - \lambda_m^2 \right)$ denominator vanishing in forward scattering as $\lambda^2 \to 0$, then $\lambda_m^2 \to 0$ with λ_ℓ^2 finite.

For quark-antiquark scattering with the antiquark the 1 to 3 line, both T_{ii} and T_{iv} obtain an opposite overall sign since Ω_f has an opposite sign for antiquark absorption or emission, relative to the sign for a quark. However, in the forward scattering region, the helicity conserving piece at the antiquark vertex in T_i contributes and it also has an opposite sign in the case of an anti-quark line.

We next consider the non-vanishing diagrams in the topological set of Fig. 1 which contribute to the bremsstrahlung of one gluon in quark-quark scattering. These diagrams are shown in Fig. 2 and, when evaluated and regulated by inserting $\delta m_q^2, \lambda_m^2 > 0$ with $k^2 = \lambda^2 > 0$ for the radiated gluon, give

$$T_i = -t_a t_b t_b' \bar{u}(p_3) e\cdot\gamma(\not{p}_3 + \not{k} + m)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2) \times \left(2p_3\cdot k + \delta m_q^2\right)^{-1}\left([p_4 - p_2]^2 - \lambda_m^2\right)^{-1} \tag{34}$$

$$T_{ii} = t_a t_b t_b' \bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)\left(-\frac{e\cdot p_3}{p_3\cdot k}\right)\left(-([p_4 - p_2]^2 - \lambda_m^2)^{-1}\right) \tag{35}$$

$$T_{iii} = -t_a t_b t_b' \bar{u}(p_3) e\cdot\gamma(\not{p} + \not{k} + m)\gamma_\mu u(p_1)\bar{u}(p_4) u(p_1)\frac{1}{2p_3\cdot k} \times \frac{\sqrt{p_{4,0}p_{2,0}}}{m}[2k_o p_4\cdot k]^{-1}\theta([p_4 - p_2]^o) \;;\; \vec{k} = \overrightarrow{p_4 - p_2} \tag{36}$$

$$T_{iv} = -t_a t_b t_b' \bar{u}(p_3) e\cdot\gamma(\not{p}_3 - \not{k}' + m)\gamma_\mu u(p_1)\bar{u}(p_4) u(p_1)\frac{1}{2p_3\cdot k} \times \frac{\sqrt{p_{4,0}p_{2,0}}}{m}[2k_o' p_2\cdot k']^{-1}\theta([p_2 - p_4]^o) \;;\; \vec{k}' = \overrightarrow{p_2 - p_4} \quad . \tag{37}$$

The forward scattering leading IR divergence of T_i due to the unitarity cut associated with the quark propagator ($\lambda^2 \to 0$, then $\delta m_q^2 \to 0$ with $\lambda_m^2 > 0$) is cancelled by T_{ii}, and the IR divergences arising from the unitarity cuts associated with $([p_4-p_2]^2 - \lambda_m^2)$ vanishing are cancelled by the T_{iii} and T_{iv} contributions.

Finally, as a simple extension, we consider those lowest order diagrams for the radiation of two gluons in quark-quark scattering which are in the topological set with the 4-gluon vertex. In this set there is one additional diagram, that shown in Fig. 4, besides the analogues to those in Fig. 3 which are obtained by replacing the 3-gluon vertex in Fig. 3 with a 4-gluon vertex in which both real gluons are radiated into the final state. The contribution from diagram (4i) is now needed because the IR divergence of (3i), the usual covariant graph, in the region where both virtual gluons are near their mass shell, with $l_o = (p_1 - p_3)_o > 0$ and $m_o = (p_2 - p_4)_o > 0$, is cancelled both by the (3iv) contribution and again by the (3v). From Eq. (29) it again follows that the (4i) contribution is of the proper sign and weight so as to correctly compensate for this, and thereby remove the IR divergence arising from this allowed kinematic region.

IV. QUARK SCATTERING IN AN EXTERNAL POTENTIAL

We next show that a cancellation of leading IR divergences occurs in separate topological sets of diagrams, or at least in the sum of such sets, in low orders in perturbative QCD for the non-Abelian class of graphs for quark scattering in a color-singlet external potential. Usage of Magnus' Theorem in Sec. II to construct the asymptotic states has summed over state contributions to different topological sets and so except for special cases, as in Sec. III, it is simpler to explicitly verify the cancellation for sums of such separate topological sets.

We consider orders g^3 and g^5 and use the asymptotic states listed in Eqs. (8) and (9) of Ref. 3 where the technique of dimensional regularization was employed. To order g^3 the IR divergences cancel in the vertex-correction set as the equal contributions from the virtual and one gluon disconnected diagrams cancel the one real gluon diagram's contribution. Here "real gluon" refers to a mass-shell gluon bridging the graph and an initial, or final, state part. Similarly for the initial (final) quark line self-energy-correction set, the equal contributions from the virtual and g^2 initial (final) state diagrams cancel the one real gluon diagram's contribution.

In order g^5, in the appendix we consider the four non-overlapping summations of the amplitudes for the diagrams of the separate topological sets. These are respectively the sum of the higher-order self-energy-correction sets for the initial quark line, the sum of the sets with g^2 self-energy-corrections for both the initial and final quark lines, the sum of the sets with a vertex-correction and a self-energy-correction on the initial quark line, and the sum of the sets with two gluons connecting the initial and final quark lines. The leading IR divergences are found to cancel in each summation.

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APPENDIX

For completeness, we list the contributions from the various diagrams to show that there is a cancellation of the leading-order IR divergences in each of the four summations for quark scattering in a color-singlet external potential. We give the leading IR divergences in the amplitudes as an ordered pair of numbers in units of $C_F^2[g^4M_o/\varepsilon^2(2\pi)^4]$ and $C_FC_{YM}[g^4M_o/\varepsilon^2(2\pi)^4]$, with M_o the order-g basic interaction.

In the higher-order self-energy-correction summation for the initial quark line: there is (1/8, 1/4) from g^4 virtuals and again from g^4 states, (-1/2, -1/4) from one real gluon and g^2 states and again from one real gluon and g^2 virtuals, with (1/4, 0) from g^2 virtuals and g^2 states and (1/2, 0) from two real gluons. In the summation of g^2 self-energy corrections for both lines: there is (1/4, 0) from each of g^4 virtuals, g^4 states, g^2 virtual final line and g^2 initial state, and g^2 virtual initial line and g^2 final state; (-1/2, 0) from each of the four combinations of one real initial (final) gluon and g^2 virtuals (states); with (1, 0) from two real gluons. The summation of the sets with a vertex-correction and a self-energy correction on the initial quark line is to be multiplied by the factor

$$(1 + r^2)\ln\,[(1 + r)/(1 - r)]\,/2r \tag{A1}$$

where $r = 1/[(1 - 4m^2/q^2)]^{1/2}$ with fermion mass m and four-momentum transfer q. In this case there is (-1/2, -1/4) from g^4 virtuals and again from g^4 states, (1/2, 1/4) from one real gluon to final state (order g^3 graph) and again from one real gluon from final line of graph (order g^3 initial state), (3/2, -1/4) from one real gluon and g^2 virtuals and again from one real gluon from initial line of graph (order g^2 initial state), (-1/2, 0) from g^2 virtuals and g^2 states, (-1, 1/4) from two real gluons in from initial state, with (-3/2, 1/4) from one gluon disconnected (g^2 virtuals) plus one real gluon in from initial state and another from final state.

The summation of the sets with two gluons connecting the initial and final quark lines is to be multiplied by the square of the factor in Eq. (A1). In this summation there is (1/2, 0) from g^4 virtuals and again from g^4 states, (-2, 1/2) from sum of order g initial state (so g^3 in graph) plus order g final state (so g^3 in graph) and again (-2, 1/2) from sum of order g^2 initial state with order g final state plus vice versa, (2, 0) from initial and final states each of order g, with (1, -1) from sum of order g^2 initial state (so g^2 graph) plus order g^2 final state (so g^2 graph).

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FIGURE CAPTIONS

Fig. 1: A topological set of diagrams contributing to the bremsstrahlung of one gluon in quark-quark scattering.

Fig. 2: Non-vanishing diagrams in the topological set of Fig. 1 which contribute to the bremsstrahlung of one gluon in quark-quark scattering. On-shell particles bridge the initial-state, graph, and final-state parts.

Fig. 3: Non-vanishing diagrams in the topological set with the 3-gluon vertex which contribute to the radiation of one gluon in quark-quark scattering.

Fig. 4: Additional non-vanishing diagram in the topological set with the 4-gluon vertex, besides the analogues to those in Fig. 3, which contributes to the radiation of two gluons in quark-quark scattering.

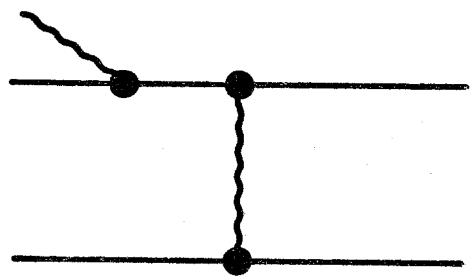

Fig. 1

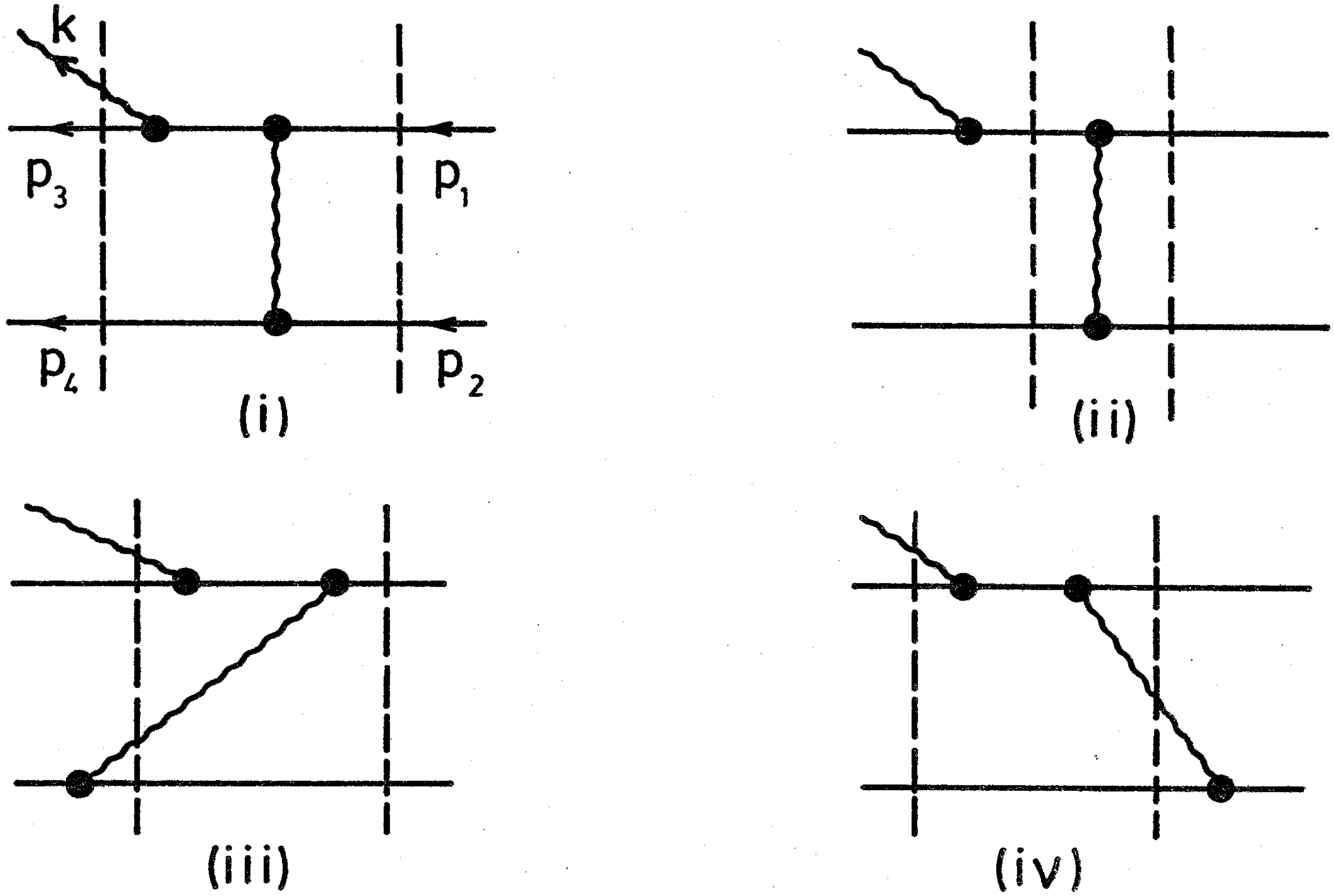

Fig. 2

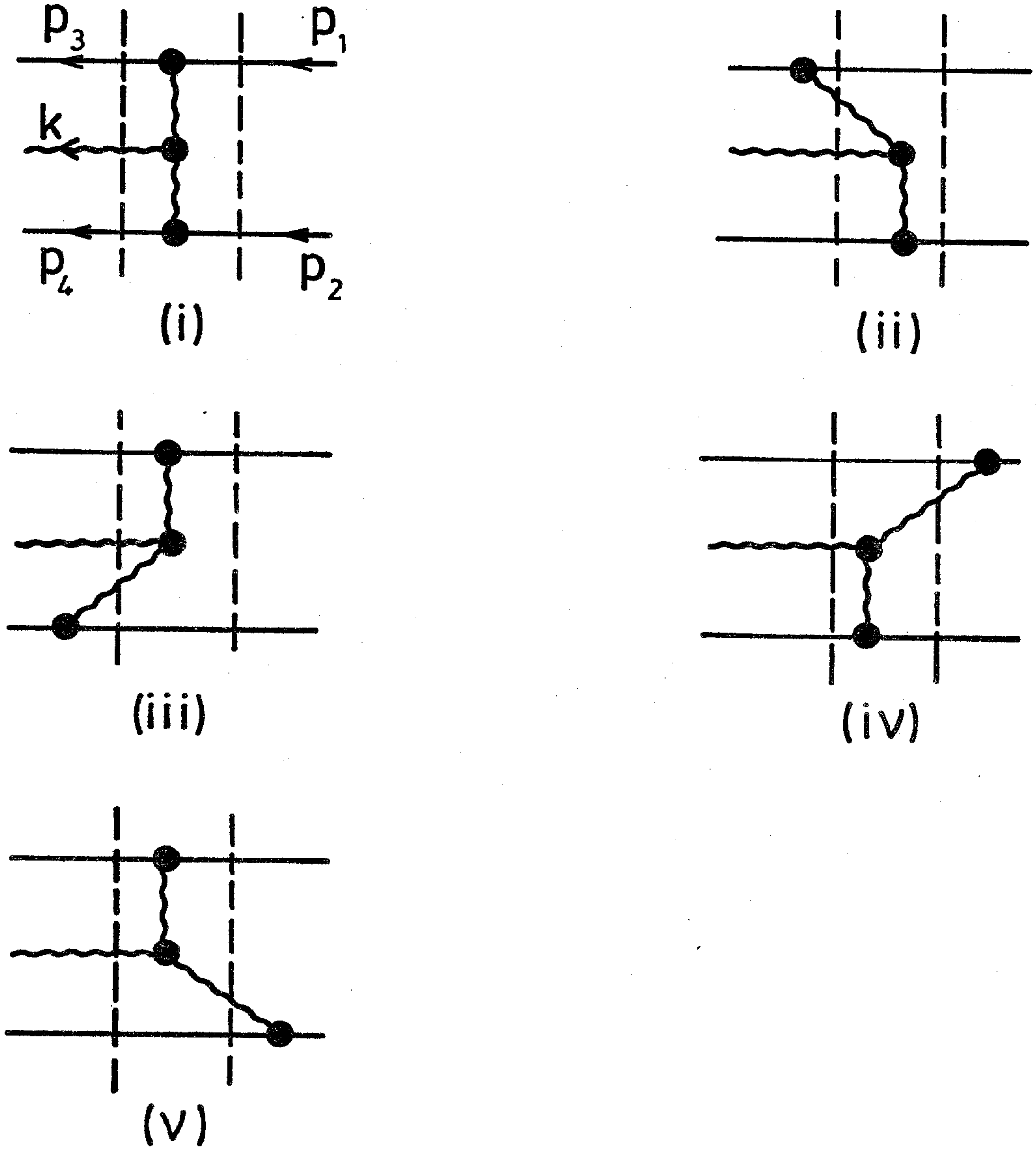

Fig. 3

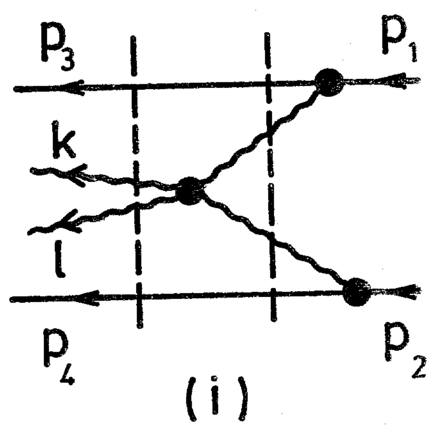

Fig. 4

C.A. Nelson, "Origin of Cancellation of Infrared Divergences in Coherent State Approach: Forward Process qq $\to$ qq + gluon

ERRATUM

p. 2 $\mathscr{H}_{as} = \exp[-\Omega(t)]\mathscr{H}_F$

p. 4 In this approach[1,3] the asymptotic S-matrix operator is defined as...

p. 4 $S_D = \exp(-iH_I(t_1 - t_2))$

$$\mathscr{H}_{as} = \lim_{t\to-\infty} \exp[-\Omega(t)]\mathscr{H}_F \qquad (4)$$

p. 8 $\Omega_f = -i\int^t V_f(\tau)d\tau = R_f(t)$

Change g to -g in Eq. (16).

p. 10 Add to first paragraph:

By Eq. (3), these Φ terms will generalize Eq. (19) and are expected to cancel generalizations of the Coulomb phase divergence. Such Φ contributions are implicit in Eq. (29) below.

p. 10 In Eq. (23)

$$\left[-\sum_{\ell=1}^{2} \tilde{S}^{(\ell)}(k)a_i(k) + \dots \right]$$

p. 11 Change i to -i in Eq. (25).

p. 12 Change Eq. (29) to read

$$S = \exp[\Omega(\infty)] S^{g}(\infty, -\infty) \exp[-\Omega(-\infty)]$$

$$= \exp[R]\, S_D \exp[-R], \text{ modulo } \Phi \text{ contributions}$$

p. 12 In text below Eq. (29):

$\exp[R]$ and $\exp[-R]$