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Statistical Mechanics of an Exactly Integrable System

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ABSTRACT

The equilibrium thermodynamics of the non-linear Schrödinger model at finite temperature is calculated by means of the quantum inverse method. Working directly in an infinite volume we derive the equation of state and the integral equation which determines the excitation spectrum. This integral equation is found to be closely related to the Gel'fand-Levitan expression for the charge density operator.

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I. INTRODUCTION

Exact solutions to certain completely integrable quantum field theories[1-4] have revealed that the vacuum and other physical states of these theories have a nontrivial structure which is most conveniently described in terms of many-body distributions. For example, in the massive Thirring model[1] the vacuum is a Dirac sea with a nonuniform distribution of filled negative energy modes while the description of excited states entails a "backflow" function which expresses the response of the Dirac sea to the excitation. The technique which has been used to determine these distribution functions, and thus calculate the energy spectrum of the theory, is patterned after the treatment of the δ-function Bose gas by Lieb.[5] First the system is placed in a box of length L and periodic boundary conditions (PBC's) are either imposed on the Bethe ansatz wave functions or obtained from the algebra of scattering data operators in the quantum inverse method.[6-10] For finite L, the PBC's are a complicated set of transcendental equations which restrict the allowed values of rapidity or momentum for the filled modes. Fortunately, in the limit $L \to \infty$, the PBC equations reduce to fairly simple linear integral equations which determine the vacuum distribution and backflow functions. In the relativistic models which have been studied, these integral equations can be solved explicitly by Fourier transformation, leading to exact spectral results.

Although the periodic boundary condition method is quite powerful and leads to exact results, there now appear to be compelling reasons to re-examine the details of the method with a view toward eliminating the use of a finite box entirely. In addition to the obvious aesthetic objection to using a box to compute quantities which ultimately have little to do with the presence or nature of the box, serious practical problems arise in the formulation of the quantum inverse method in a box of finite length. In the infinite volume case $L = \infty$, the algebra of the scattering data operators $a(\xi)$ and $b(\xi)$ is especially simple and leads to elegant properties for the reflection coefficient operator $R(\xi) \sim b(\xi)a^{-1}(\xi)$. [See Eqs. (1.6).] Although the algebra of a and b operators can be derived for finite L, it is more complicated than the $L = \infty$ case (e.g. an extra exchange term appears in the a - b commutator), and simple relations for the R-operators are not obtained. In fact the utility of the R operator seems to be entirely destroyed by the introduction of a finite box. Since the simple properties of the R operator are at the heart of the Gel'fand-Levitan transformation[10] (which is the inverse part of the quantum inverse method), it is apparent that the use of a finite box has serious drawbacks. It would be reassuring and perhaps enlightening if the spectral integral equations which are usually obtained from Bethe ansatz periodic boundary conditions could be derived directly in the infinite volume theory without resorting to a box. In this paper we will show that for the δ-function gas (quantum nonlinear Schrödinger model), such a derivation is not only possible but leads to new insight into the structure of the Gel'fand-Levitan transformation. We find that the Gel'fand-Levitan expression for the charge-density operator $j_o(x) = \phi^*(x)\phi(x)$ is closely related to the spectral integral equation for the finite temperature δ-function gas first derived by Yang and Yang.[11]

The connection between the spectral integral equation and certain almost-forward matrix elements of the charge density operator was pointed out some time ago in the course of a graphical calculation of the partition function (see Ref. 12, eq. (4.14) et seq.). At the time no means were available for studying these matrix elements directly, and the calculation was carried out by an indirect method using unitarity of the Møller wave operators. Although the calculation in Ref. 12

demonstrated that the spectral integral equation and partition function of the δ-function gas could be obtained without introducing a finite box or periodic boundary conditions, it required a delicate treatment of the $i\varepsilon \to 0$ limit in certain singular denominators. The method discussed in this paper utilizes a direct calculation of matrix elements of $j_o(x)$ starting from the Gel'fand-Levitan expression for that operator. It requires no delicacy in the treatment of $i\varepsilon$'s (which may in fact be ignored throughout) and exposes a remarkable correspondence between the expansion of $j_o(x)$ in powers of the R and R^* operators and the expansion of the spectral integral equation in powers of its kernel.

The nonlinear Schrödinger model is described by the Hamiltonian

$$H = \int \left[\partial_x \phi^* \partial_x \phi + c\phi^* \phi^* \phi\phi \right] dx \quad , \tag{1.1}$$

where $\phi(x)$ is a nonrelativistic boson field with canonical commutation relations

$$\left[\phi(x), \phi^*(x') \right] = \delta(x - x') \quad . \tag{1.2}$$

The quantum inverse method for this model is implemented through the linear Zakharov-Shabat eigenvalue problem[13]

$$\left(i \frac{\partial}{\partial x} + \frac{1}{2}\xi \right) \Psi_1 = - \sqrt{c}\, \Psi_2 \phi \tag{1.3a}$$

$$\left(i \frac{\partial}{\partial x} - \frac{1}{2}\xi \right) \Psi_2 = \sqrt{c}\, \phi^* \Psi_1 \quad . \tag{1.3b}$$

The scattering data operators $a(\xi)$ and $b(\xi)$ are defined in terms of the Jost solution $\psi(x, \xi)$ with the properties

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\xi x/2} \underset{x\to-\infty}{\longleftarrow} \begin{pmatrix} \psi_1(x,\xi) \\ \psi_2(x,\xi) \end{pmatrix} \underset{x\to+\infty}{\longrightarrow} \begin{pmatrix} a(\xi)e^{i\xi x/2} \\ b(\xi)e^{-i\xi x/2} \end{pmatrix} \quad . \tag{1.4}$$

The fundamental operators $R(\xi)$ are given by

$$R(\xi) = \frac{i}{\sqrt{c}} b(\xi)a^{-1}(\xi) \tag{1.5}$$

and may be shown to obey the following simple commutation relations:

$$\left[H, R^*(\xi)\right] = \xi^2 R^*(\xi) \quad , \tag{1.6a}$$

$$R(\xi)R(\xi') = S(\xi' - \xi)R(\xi')R(\xi) \quad , \tag{1.6b}$$

$$R(\xi)R^*(\xi') = S(\xi - \xi')R^*(\xi')R(\xi) + 2\pi\delta(\xi - \xi') \quad , \tag{1.6c}$$

where H is the Hamiltonian, and S is the two-body S-matrix

$$S(\xi - \xi') = \frac{\xi - \xi' - ic}{\xi - \xi' + ic} \quad . \tag{1.7}$$

From these relations we see that the states $|k_1 \ldots k_n>$ defined by

$$|k_1 \ldots k_n> = R^*(k_1)\ldots R^*(k_n)|0> \quad , \tag{1.8}$$

(where $|0>$ is the vacuum state with $\phi(x)|0> = 0$), are eigenstates of the Hamiltonian:

$$H|k_1 \ldots k_n> = \left(\sum_{i=1}^{n} k_i^2 \right) |k_1 \ldots k_n> \quad . \tag{1.9}$$

These states are identical with those obtained previously by means of Bethe's ansatz. The inner product between two such states may be easily obtained from the commutation relations (1.1).

The inverse transformation from the R operators back to the Heisenberg field $\phi(x)$ is accomplished by means of the quantized version of the Gel'fand-Levitan equation. By using the analytic properties in ξ of the Jost solution $\chi(x, \xi)$ with the behavior $\chi(x, \xi) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\xi x/2}$ as $x \to +\infty$, it was shown in Ref. 10 that the components χ_1 and χ_2 may be expressed as expansions in the operators $R(\xi)$ and $R^*(\xi)$. The asymptotic behavior of χ_1

$$\chi_1(x, \xi)\, e^{i\xi x/2} \underset{\xi \to \infty}{\longrightarrow} -\frac{\sqrt{c}}{\xi}\phi(x) + O\left(\frac{1}{\xi^2}\right)$$

yields a corresponding series expansion for the field operator $\phi(x)$. Some properties of this expression have been studied in Refs. 10 and 14. In this paper we use a similar series expansion for the charge density operator $j_o(x) = \phi^*(x)\phi(x)$ which comes from the asymptotic behavior of the other Jost solution component,

$$\chi_2(x, \xi)\, e^{i\xi x/2} \underset{\xi \to \infty}{\longrightarrow} 1 - \frac{ic}{\xi}\int_x^\infty j_o(x')dx' + O\left(\frac{1}{\xi^2}\right) \quad . \tag{1.10}$$

From this and eq. (40) of Ref. 10, we obtain the result

$$j_o(x) = \sum_{M=0}^{\infty} j_o^{(M)}(x) \quad , \tag{1.11a}$$

where

$$j_o^{(M)}(x) = (-c)^M \int \prod_{i=1}^{M+1} \left\{ \frac{dk_i dp_i}{(2\pi)^2} \right\} \frac{(\Sigma p - \Sigma k) e^{-i(\Sigma p - \Sigma k)x}}{\prod_{i=1}^{M} \left\{ (p_i - k_i - i\varepsilon)(p_i - k_{i+1} - i\varepsilon) \right\} (p_{M+1} - k_{M+1} - i\varepsilon)}$$

$$\times \; R^*(p_{M+1}) \ldots R^*(p_1) R(k_1) \ldots R(k_{M+1}) \quad . \tag{1.11b}$$

In Section II the expansion (1.11) will be used to calculate the partition function of a finite temperature gas. The combinatorics of the series is reduced to an integral equation which is just the equation of Yang and Yang. In the $T \to 0$ limit this reduces to the results of Lieb. Section III contains some concluding remarks.

II. PARTITION FUNCTION OF δ-FUNCTION GAS

The purpose of this section is to compute the partition function

$$Q(\beta, \mu) = \mathrm{Tr}\, e^{\beta(\mu N - H)} \quad , \tag{2.1}$$

where $N = \int dx \phi^*(x)\phi(x)$ is the number operator, H is the Hamiltonian, β is the inverse temperature, and μ is the chemical potential. Actually we will compute the extensive quantity ln Q, which was shown in Ref. 12 to have the representation

$$\ln Q = \lim_{q \to 0} \mathrm{Tr}\, Y(q) e^{\beta(\mu N - H)} \quad , \tag{2.2}$$

where the operator Y(q) is defined by

$$Y(q) = e^{-iqK} N^{-1} \int_{-\infty}^{\infty} dx j_o(x) e^{iNqx} \quad . \tag{2.3}$$

Here $K = \int dx\, x\phi^*(x)\phi(x)$ is the Galilean boost operator with the property $e^{iqK}R^*(k) = R^*(k+q)e^{iqK}$. Note that formally the limit $q \to 0$ of $Y(q)$ is the unit operator, and that in diagrammatic language the effect of taking the limit $q \to 0$ outside the trace is just to pick out the connected pieces which go to make up $\ln Q$. In the quantum inverse method this representation of $\ln Q$ is very convenient since equation (1.11) expresses $j_o(x)$ in terms of the fundamental operators R and R^* which have simple commutation relations with the Hamiltonian. The operator $Y(q)$ commutes with the total momentum operator $P = \frac{1}{2}i\int dx\, \phi^* \overleftrightarrow{\partial_x}\phi$, and so when we take the trace the x integration in (2.3) becomes trivial yielding a factor $2\pi\delta(0)$, which we interpret as the spatial extent L. Using the expansion (1.11) for $j_o(x)$ we then find that the pressure $\mathscr{P} = \beta^{-1}\partial \ln Q/\partial L$ may be written as

$$\mathscr{P} = \beta^{-1}\lim_{q\to 0} q \operatorname{Tr} e^{-iqK} J e^{\beta(\mu N - H)} \quad , \tag{2.4}$$

where the operator J is given by the expansion

$$J = \sum_{M=0}^{\infty} J^{(M)}$$

$$= \sum_{M=0}^{\infty}(-c)^M \int \prod_{i=1}^{M+1} \left\{\frac{dk_i dp_i}{(2\pi)^2}\right\} \frac{R^*(p_{M+1})\ldots R^*(p_1)R(k_1)\ldots R(k_{M+1})}{\prod_{i=1}^{M}\left\{(p_i - k_i)(p_i - k_{i+1})\right\}(p_{M+1} - k_{M+1})} \quad . \tag{2.5}$$

That the limit in (2.4) is nonzero is due to the denominators in (2.5), some of which become of order q when we take the trace. In order to compute this trace we need to evaluate the quantity

$$\Lambda^{(M)}(q) = \mathrm{Tr}\, e^{-iqK} R^*(p_{M+1})\ldots R^*(p_1)R(k_1)\ldots R(k_{M+1})e^{\beta(\mu N-H)} \quad , \quad (2.6)$$

which is shown in the Appendix to be given by

$$\Lambda^{(M)}(q) = (-1)^{M+1} \sum_{n_1\ldots n_{M+1}=1}^{\infty} \left[\prod_{i=1}^{M+1} (-1)^{n_i} e^{n_i\beta(\mu-k_i^2)} \right.$$

$$\left. \times \langle k_{M+1} + n_{M+1}q,\ldots k_1 + n_1 q \,|\, p_{M+1}\ldots p_1 \rangle \left\{ 1 + O(q) \right\} \right] \quad . \quad (2.7)$$

The additional terms of order q have no effect as $q \to 0$ and will be omitted in the following.

Let us use this result to evaluate the contribution to the pressure of the first few terms of (2.5). The zeroth term gives

$$\mathscr{P}^{(0)} = -\frac{1}{\beta}\int \frac{dk_1}{2\pi} \sum_{n_i=1}^{\infty} \frac{(-1)^{n_1}}{n_1} e^{n_1\beta(\mu-k_1^2)}$$

$$= -\frac{1}{\beta}\int \frac{dk_1}{2\pi} \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1} e^{n_1\beta(\mu-k_1^2)}}{n_1}$$

$$= \frac{1}{\beta}\int \frac{dk_1}{2\pi} \ln\left(1 + e^{\beta(\mu-k_1^2)} \right) \quad , \quad (2.8)$$

which is just the well-known expression for the pressure of a free fermi gas. This is at first surprising since the explicit powers of c in the expansion (2.5) might indicate that it is a small coupling expansion, so that the zeroth term should give the pressure for a free bose gas. In fact however we shall find that due to the implicit c-dependence of the operators R, the higher order terms are actually an expansion in the kernel $\Delta(p - q)$ given by

$$\Delta(p - q) = \frac{2c}{(p - q)^2 + c^2} \quad , \tag{2.9}$$

which vanishes as $c \to \infty$ but gives $2\pi\delta(p - q)$ as $c \to 0$.

To see this pattern begin, let us consider the next term $J^{(1)}$ in the series for J. Before taking the trace it is convenient to symmetrize the integrand of $J^{(1)}$ over k_1 and k_2 and over p_1 and p_2 and then use the commutation relations (1.6) to recover the original ordering of the R's and R^*'s in each term. In this way we obtain

$$J^{(1)} = \frac{c}{2}\int \frac{dk_1 dk_2 dp_1 dp_2}{(2\pi)^4} \frac{R^*(p_2)R^*(p_1)R(k_1)R(k_2)(p_1 + p_2 - k_1 - k_2)(k_1 - k_2)(p_2 - p_1)}{(p_1 - k_1)(p_1 - k_2)(p_2 - k_1)(p_2 - k_2)(k_1 - k_2 + ic)(p_2 - p_1 + ic)} . \tag{2.10}$$

After this symmetrization the contributions to the pressure coming from the two terms of the matrix element $\langle k_2 + n_2 q, k_1 + n_1 q | p_2 p_1 \rangle$ in the trace $\Lambda^{(1)}(q)$ are equal, so that we may replace this matrix element by $2(2\pi)^2\delta(p_1 - k_1 - n_1 q)\delta(p_2 - k_2 - n_2 q)$. We see that possible poles as $q \to 0$ are cancelled by two powers of q in the numerator so that the limit $q \to 0$ is finite and given by

$$\begin{aligned}
\mathscr{P}^{(1)} &= \frac{1}{2\beta}\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{2c}{(k_1 - k_2)^2 + c^2}\sum_{n_1,n_2=1}^{\infty}\frac{n_1 + n_2}{n_1 n_2}\times\prod_{i=1}^{2}(-1)^{n_i}e^{n_i\beta(\mu - k_i^2)} \\
&= \frac{1}{\beta}\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\Delta(k_1 - k_2)\sum_{n_1,n_2=1}^{\infty} n_2^{-1}\prod_{i=1}^{2}(-1)^{n_i}e^{n_i\beta(\mu - k_i^2)} \\
&= \frac{1}{\beta}\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\Delta(k_1 - k_2)\frac{\ln\left(1 + e^{\beta(\mu - k_2^2)}\right)}{\left(1 + e^{\beta(k_1^2 - \mu)}\right)} \quad .
\end{aligned} \tag{2.11}$$

Note that by combining $\mathscr{P}^{(0)}$ and $\mathscr{P}^{(1)}$ and expanding in the fugacity $z = e^{\beta\mu}$ (and multiplying by $2\pi\delta(0)\beta$) we recover the well-known result for the second virial coefficient

$$2\pi\delta(0)\int\frac{dk_1}{2\pi}\,\frac{dk_2}{2\pi}\left\{\Delta(k_1-k_2)-\pi\,\delta(k_1-k_2)\right\}e^{-\beta(k_1{}^2+k_2{}^2)} \quad . \tag{2.12}$$

Let us now consider the general Mth term in the series. Although the details are somewhat complicated the essential pattern of the computation is the same; after symmetrizing over $k_1 \ldots k_{M+1}$ and over $p_1 \ldots p_{M+1}$ the limit $q \to 0$ is seen to be finite and the c-dependence appears only in the form of M kernels $\Delta(k_i - k_j)$. Explicitly we find[15]

$$\mathscr{P}(M) = \frac{1}{\beta}\int\frac{dk_1\ldots dk_{M+1}(-1)^{M+1}}{(2\pi)^{M+1}(M+1)!}\sum_{\mathscr{C}_M}\sum_{n_i=1}^{\infty}\prod_{i=1}^{M+1}(-1)^{n_i}\,\frac{e^{n_i\beta(\mu-k_i{}^2)}}{n_i}\,n_i^{\,m_i-1}$$

$$\times\left(\sum_{1}^{M+1}n_i\right)\prod_{\{k_i,k_j\}\in\mathscr{C}_M}\Delta(k_i-k_j) \quad , \tag{2.13}$$

where $\mathscr{C}_M$ is the set of all collections of M pairs $\{k_i, k_j\}$ with $i \neq j$ such that each k_i appears in at least one of the pairs $\{k_i, k_j\}$. The integers $m_i \geq 1$ indicate the number of pairs $\{k_i, k_j\}$ which contain k_i. For example, the M = 2 term may be written explicitly as

$$\mathscr{P}(2) = -\frac{1}{\beta}\int\frac{dk_1 dk_2 dk_3}{(2\pi)^3 3!}\sum_{n_1,n_2,n_2=1}^{\infty}\prod_{i=1}^{3}(-1)^{n_i}e^{n_i\beta(\mu-k_i{}^2)}$$

$$\times\frac{n_1+n_2+n_3}{n_1 n_2 n_3}\left(n_1\Delta_{12}\Delta_{13}+n_2\Delta_{23}\Delta_{21}+n_3\Delta_{31}\Delta_{32}\right) \tag{2.14}$$

where $\Delta_{ij} \equiv \Delta(k_i - k_j)$. Summing (2.13) over M from zero to infinity and using the symmetry in the k_i and n_i to replace Σn_i by $(M+1)n_1$ we obtain the desired expression for the pressure:

$$\mathscr{P} = \frac{1}{\beta} \sum_{M=0}^{\infty} \frac{(-1)^{M+1}}{M!} \sum_{\mathscr{C}_M} \int \frac{dk_1 \ldots dk_{M+1}}{(2\pi)^{M+1}} \sum_{n_i=1}^{\infty} \prod_{i=1}^{M+1} (-1)^{n_i} e^{n_i \beta(\mu - k_i^2)} n_i^{m_i - 2}$$

$$\times \quad n_1 \prod_{\{i,j\} \in \mathscr{C}_M} \Delta(k_i - k_j) \quad . \tag{2.15}$$

We now show that this result may be expressed in terms of a certain non-linear integral equation. Let us define $\mathscr{P}(k_1, n_1)$ to be the above expression but with the integral $dk_1/2\pi$ and the sum over n_1 suppressed, and let $\mathscr{P}(k_1)$ be $\sum_{n_1=1}^{\infty} \mathscr{P}(k_1, n_1)$. Also we introduce a quantity $\sigma(k_1)$ which is essentially those terms of $\mathscr{P}(k_1, n_1)$ with $m_1 = 1$ and external factors omitted, i.e.

$$\sigma(k_1) = \frac{1}{\beta} \sum_{M=1}^{\infty} \frac{(-1)^{M+1}}{M!} \sum_{\mathscr{C}'_M} \int \frac{dk_2 \ldots dk_{M+1}}{(2\pi)^M} \sum_{n_2 \ldots n_{M+1}=1}^{\infty} \prod_{i=2}^{M+1} (-1)^{n_i} e^{n_i \beta(\mu - k_i^2)} n_i^{m_i - 2}$$

$$\times \prod_{\{i,j\} \in \mathscr{C}'_M} \Delta(k_i - k_j) \quad , \tag{2.16}$$

where $\mathscr{C}'_M$ is that subset of $\mathscr{C}_M$ with $m_1 = 1$. Then it is a simple combinatorical exercise to verify that these quantities obey the coupled equations

$$\sigma(k_1) = -\frac{1}{\beta}\int\frac{dk_2}{2\pi}\,\Delta(k_1-k_2)\mathscr{P}(k_2) \quad , \tag{2.17}$$

and

$$\mathscr{P}(k_1, n_1) = -\frac{1}{\beta}\,\frac{(-1)^{n_1}e^{n_1\beta(\mu-k_1^2)}}{n_1}\sum_{m_1=0}^{\infty}\frac{\{-n_1\beta\sigma(k_1)\}^{m_1}}{m_1!}$$

$$= -\frac{1}{\beta}\,\frac{(-1)^{n_1}}{n_1}\,e^{n_1\beta\left\{\mu-k_1^2-\sigma(k_1)\right\}} \quad , \tag{2.18}$$

so that

$$\mathscr{P}(k_1) = \ln\left[1+e^{\beta\left\{\mu-k_1^2-\sigma(k_1)\right\}}\right] \quad . \tag{2.19}$$

Combining these results we find that the pressure $\mathscr{P}$ is given by

$$\mathscr{P} = \frac{1}{\beta}\int\frac{dk}{2\pi}\ln\left[1+e^{\beta\{\mu-k^2-\sigma(k)\}}\right] \quad , \tag{2.20}$$

where $\sigma(k)$ obeys the non-linear integral equation

$$\sigma(k) = -\frac{1}{\beta}\int\frac{dq}{2\pi}\,\Delta(k-q)\ln\left[1+e^{\beta\{\mu-q^2-\sigma(q)\}}\right] \quad , \tag{2.21}$$

in agreement with the result first obtained by Yang and Yang using a variational method. The quantity $\epsilon(k)$ used by these authors is related to our $\sigma(k)$ by

$$\epsilon(k) = k^2 - \mu + \sigma(k) \quad . \tag{2.22}$$

We emphasize that our original expansion (2.4), (2.5) for the pressure, which follows directly from the Gel'fand-Levitan expression for the charge density, corresponds term by term to an expansion of (2.20) in the kernel $\Delta(k_i - k_j)$.

III. DISCUSSION

To summarize, we have found that the structure of the charge density operator $j_o(x)$, when expressed in terms of R-operators by the quantum inverse method, is directly related to the spectral equation which describes the thermodynamics of the system at finite temperature and density first derived by Yang and Yang. Using the Gel'fand-Levitan expression (1.11) for $j_o(x)$, the partition function was calculated from (2.2) and (2.3). The expansion (1.11) or (2.5) leads to the expansion (2.15) for the pressure in powers of the kernel Δ. This result was reduced to a single integral (2.20), where $\sigma(k)$ is the solution of the nonlinear integral equation (2.21). The function $\sigma(k)$ is simply related, by eq. (2.22), to the quantity $\varepsilon(k)$ introduced by Yang and Yang.

Some perspective may be added to these results by recalling that the function $\varepsilon(k)$ describes not only the pressure but the complete excitation spectrum of the theory. As shown in Ref. 5, the excitations are of two types, particles with energy $\varepsilon(k)$ and holes with energy $-\varepsilon(k)$. For multiple excitations the energies are additive. In the zero temperature limit these considerations are found to be equivalent to the method of Lieb for computing excitation energies above the ground state. (Note that eq. (2.21) becomes linear in the limit $\beta \to \infty$. See Ref. 11.) In fact the Yang and Yang method provides a convenient simplification of Lieb's result, with the bare energy of an excited mode and the backflow energy associated with the excitation combined into a single quantity $\varepsilon(k)$. A similar simplification may be noted in the calculation of the massive Thirring model spectrum.[1,2] In this case, the Lieb method is used, and the integral equation for the backflow distribution can be solved explicitly. The backflow energy associated with each excited mode combines nicely with the corresponding bare energy to produce a simple result. Again the energy spectrum may be expressed as additive combinations of a single-particle function $\varepsilon(\alpha)$, which gives the energy of a mode with rapidity α. From this similarity one suspects that a method like the one described in Sec. II might provide an alternative derivation of the spectral integral equation for the massive Thirring model which does not rely on periodic boundary conditions in a finite box. Such calculations must await an extension of the quantum Gel'fand-Levitan method to this theory.

In recent investigations of exactly integrable relativistic theories,[1-4] calculations have been carried out at zero temperature with emphasis on the construction of eigenstates. The method developed in this paper provides an interesting counterpoint to the usual approach. Here neither periodic boundary conditions nor explicit properties of the eigenstates were used to obtain the integral equation which determines the spectrum. Only the Gel'fand-Levitan formula (1.11) and the algebra of R-operators (1.6) are used. The role of eigenstates is greatly diminished. This may be a useful shift of emphasis for integrable relativistic boson theories (e.g. nonlinear sigma models) where the explicit construction of eigenstates has not been accomplished. It is also worth noting that the use of finite temperature is essential to the results described in this paper. This is apparent from the repeated use of the cyclic property of the Hilbert space trace for the derivation of eq. (2.7) given in Appendix A. Corresponding expressions at zero temperature would involve vacuum expectation values which have no such cyclic property. Moreover, the series expansion for $\varepsilon(k)$ which emerges from the Gel'fand-Levitan approach is not term-by-term finite at zero temperature. Only after summing the series in the form of an integral equation can the $\beta \to \infty$ limit be taken. It may be that, in further investigations of exactly integrable theories, finite temperature calculations can provide a useful tool even for the study of zero-temperature theories.

APPENDIX

In this appendix we show that $\Lambda^{(M)}(q)$, the trace of a product of R's and R^*'s, defined in (2.6) is Eq. (2.7). One way to do this is to use Bethe ansatz states (1.8) to perform the trace at fixed particle number n and then sum over n. Here we will adopt a more formal, but equivalent, method which employs the cyclic property of the trace. First look at the case M = 0:

$$\Lambda^{(0)}(q; p; k) = \mathrm{Tr}\, e^{-iqK} R^*(p) R(k) e^{\beta(\mu N - H)} \quad . \tag{A.1}$$

Using the cyclic property of the trace and the algebra of the R operators (1.6) we secure the relation

$$\Lambda^{(0)}(q; p, k) = z e^{-\beta k^2} \{2\pi\, \delta(k + q - p) \mathrm{Tr}\, e^{-iqK} e^{\beta(\mu N - H)} + $$

$$+ S(k + q - p) \mathrm{Tr}\, e^{-iqK} R^*(p) R(k + q) e^{\beta(\mu N - H)} \} \quad , \tag{A.2}$$

where $z = e^{\beta\mu}$ is the fugacity. Owing to momentum conservation only the zero particle state contributes to the trace in the first term so that

$$\Lambda^{(0)}(q; p, k) = e^{\beta(\mu - k^2)} \{ \langle k + q | p \rangle + S(k + q - p)\, \Lambda^{(0)}(q; p; k + q) \} \quad , \tag{A.3}$$

where we have used $<k + q|p> = 2\pi\delta(k + q - p)$. Iterating this result we obtain eq. (2.7) for the case $M = 0$:

$$\Lambda^{(0)}(q; p; k) = \sum_{n=1}^{\infty} f_n^{(0)}(q, k) <k + nq|p> \quad , \tag{A.4}$$

where

$$f_n^{(0)}(q, k) = z^n e^{-\beta k^2} \prod_{\ell=1}^{n-1} S(\ell q - nq) e^{-\beta(k+\ell q)^2}$$

$$= -(-z)^n e^{-n\beta k^2} + O(q) \quad . \tag{A.5}$$

We will prove a similar formula for arbitrary M by induction. Assume $\Lambda^{(M-1)}(q; p_1 ... p_M, k_1 ... k_M)$ has the form

$$\Lambda^{(M-1)}(q; \{p_i\}, \{k_i\}) = \sum_{\{n_i\}=1}^{\infty} f^{(M-1)}_{\{n_i\}}(q, \{k_i\}) \times < k_M + n_M q ... k_1 + n_1 q | p_M ... p_1 > \;, \tag{A.6}$$

where

$$f^{(M-1)}_{\{n_i\}}(q, \{k_i\}) = (-1)^M \prod_{i=1}^{M} (-z)^{n_i} e^{-n_i \beta k_i^2} + O(q) \quad . \tag{A.7}$$

Consider now

$$\Lambda^{(M)}(q, \{p_i\}, \{k_i\}) \equiv \mathrm{Tr}\, e^{-iq\mathbf{K}} R^*(p_{M+1}) ... R^*(p_1) R(k_1) ... R(k_{M+1}) e^{\beta(\mu N - H)} \quad . \tag{A.8}$$

Generalizing the previous technique, we cycle $R(k_{M+1})$ to obtain

$$\Lambda^{(M)}(q, \{p_i\}, \{k_i\}) = \sum + ze^{-\beta k_{M+1}^2} \prod_{j=1}^{M+1} S(k_{M+1} + q - p_j)$$

$$\times \prod_{\ell=1}^{M} S^{-1}(k_{M+1} + q - k_\ell)\Lambda^{(M)}(q; \{p_i\}; k_1 \ldots k_M, k_{M+1} + q) \ , \quad \text{(A.9)}$$

where

$$\sum = ze^{-\beta k_{M+1}^2} \sum_{j=1}^{M+1} 2\pi\,\delta(k_{M+1} + q - p_j) \prod_{\ell=j+1}^{M+1} S(k_{M+1} + q - p_\ell)$$

$$\times \quad \Lambda^{(M-1)}(q; p_1 \ldots p_{j-1}, p_{j+1} \ldots p_{M+1}; k_1 \ldots k_M)$$

$$= ze^{-\beta k_{M+1}^2} \sum_{\substack{\{n_i\}=1 \\ i=1\ldots M}}^{\infty} f^{(M-1)}_{\{n_i\}}(q, \{k_i\})$$

$$\times \quad \langle k_{M+1} + q, k_M + n_M q \ldots k_1 + n_1 q \,|\, p_{M+1} \ldots p_1 \rangle \quad . \quad \text{(A.10)}$$

The second equality follows from the inductive hypothesis coupled with the identity

$$\langle q_{M+1} \ldots q_1 \,|\, p_{M+1} \ldots p_1 \rangle = \sum_{j=1}^{M+1} 2\pi\,\delta(q_{M+1} - p_j) \prod_{\ell=j+1}^{M+1} S(q_{M+1} - p_\ell)$$

$$\times \quad \langle q_M \ldots q_1 \,|\, p_{M+1} \ldots p_{j+1}, p_{j-1} \ldots p_1 \rangle \quad , \quad \text{(A.11)}$$

which itself follows from the algebra of the R operators (1.6). Equation (A.9) can now be iterated yielding an infinite series for $\Lambda^{(M)}$. If we note that when multiplied by the inner product $\langle k_{M+1} + n_{M+1} q \ldots k_1 + n_1 q \,|\, p_{M+1} \ldots p_1 \rangle$ we have the relation

$$\prod_{j=1}^{M} S(k_{M+1} + nq - p_j) \prod_{\ell=1}^{M} S^{-1}(k_{M+1} + nq - k_\ell) = -1 + O(q) \quad , \qquad (A.12)$$

then we find

$$\Lambda^{(M)}(q; \{p_i\} \{k_i\}) = \sum_{\{n_i\}=1}^{\infty} f^{(M)}_{\{n_i\}}(q, \{k_i\}) < k_{M+1} + n_{M+1}q \ldots k_1 + n_1 q | p_{M+1} \ldots p_1 > \quad , (A.13)$$

with

$$f^{(M)}_{\{n_i\}} = -(-z)^{n_{M+1}} e^{-n_{M+1}\beta k^2_{M+1}} f^{(M-1)}_{\{n_i\}} + O(q)$$

$$= (-1)^{M+1} \prod_{i=1}^{M+1} (-z)^{n_i} e^{-n_i \beta k_i} + O(q) \quad . \qquad (A.14)$$

This completes the induction from M-1 to M.

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[15] We have not managed to construct a general proof of this result but have verified it for the first four terms of the series. The symmetrization for the $M = 2$ and $M = 3$ terms was carried out using the algebraic manipulation program MACSYMA.