Fermi National Accelerator Laboratory

FERMILAB-Pub-80/17-THY
January 1980

Quantum Gel'fand-Levitan Method as a Generalized Jordan-Wigner Transformation

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ABSTRACT

The operator Gel'fand-Levitan method for the quantized nonlinear Schrödinger equation is shown to reduce to a Jordan-Wigner transformation in the limit of infinite repulsion. This result is used to obtain a representation for the finite density correlation function at zero temperature.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

The extension of the inverse scattering method to the domain of quantum field theory[1-5] has provided new insight into the structure of exactly integrable quantum systems. The most fully developed example of the quantum inverse method is the nonlinear Schrödinger model (δ-function gas), where the quantum versions of both the direct and inverse scattering transforms have been constructed. The direct transform employs an auxiliary linear eigenvalue problem to define quantized scattering data operators $a(k)$ and $b(k)$ as nonlinear functionals of the field operators $\phi(x)$ and $\phi^*(x)$. The inverse transform is accomplished by an operator generalization of the Gel'fand-Levitan method, which gives the field operator $\phi(x)$ as a functional of $R^*(k) \sim b(k)a^{-1}(k)$, the quantized reflection coefficient of the associated eigenvalue problem. In this paper we derive and discuss a remarkable simplification of the inverse transform for the nonlinear Schrödinger model in the strong coupling limit $c \to \infty$ (impenetrable bosons). In this limit the Gel'fand-Levitan expression for $\phi(x)$ exponentiates, becoming a Jordan-Wigner transformation which gives the boson field $\phi(x)$ as a functional of a free fermion field $\tilde{R}(x)$:

$$\phi(x) = \exp\left(i\pi\int_x^\infty \tilde{R}^+(y)\tilde{R}(y)dy \right)\tilde{R}(x) \quad . \tag{1}$$

Here $\tilde{R}(x)$ is the Fourier transform of the operator reflection coefficient $R(k)$. It is perhaps not surprising that the Jordan-Wigner transformation plays a role in the case $c = \infty$ in view of the equivalence of this case to an XY Heisenberg spin chain where this transformation is a standard technique. What we wish to emphasize here is that the Jordan-Wigner transformation (1) is obtained as a special case of the more general quantum inverse method. This result has some interesting implications. Relations of the form (1) between auxiliary fermion and boson fields play an important role in the treatment of Green's functions by deformation theory.[6] So far these methods have only been applied to "free fermion" theories like the $c = \infty$ nonlinear Schrödinger model, the XY spin chain, and the two-dimensional Ising model. The observation that (1) is a special case of a general operator transformation constructed by the Gel'fand-Levitan method may provide new insight into the problem of constructing Green's functions for general integrable systems. This result also suggests that there is a connection between integrability and self-duality. Although our present considerations are restricted to the $c = \infty$ nonlinear Schrödinger model, we expect a similar relation between the quantum inverse method and the Jordan-Wigner transformation to arise in other models, in particular the two-dimensional Ising model. For this model the Jordan-Wigner transformation describes the relationship between order, disorder, and fermion variables and is closely related to the Kramers-Wannier duality between the low and high temperature phases of the system. A more general connection between quantum inverse transformations and duality transformations would be a welcome development, since the latter are known to be useful in a broad range of physically interesting models[7] including four-dimensional gauge theories.[8,9]

The nonlinear Schrödinger model is described by the Hamiltonian

$$H = \int [\, \partial_x \phi^* \partial_x \phi + c\phi^* \phi^* \phi\phi]\, dx \qquad , \tag{2}$$

where ϕ is a nonrelativistic boson field,

$$[\, \phi(x), \phi^*(x') \,] = \delta(x - x') \qquad . \tag{3}$$

The quantum inverse method[1-3] provides an operator transformation from the original field $\phi(x)$ to an operator R(k) which creates eigenstates of the Hamiltonian. The R operator is given in terms of the asymptotic values of the Jost solutions to the Zakharov-Shabat linear eigenvalue problem,

$$\left(i \frac{\partial}{\partial x} + \frac{1}{2} \xi \right) \Psi_1 = -\sqrt{c}\, \Psi_2 \phi \tag{4a}$$

$$\left(i \frac{\partial}{\partial x} - \frac{1}{2} \xi \right) \Psi_2 = \sqrt{c}\, \phi^* \Psi_1 \quad . \tag{4b}$$

More precisely if $\psi(x, \xi)$ is the solution of (4) with the properties

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\xi x/2} \underset{x \to -\infty}{\longleftarrow} \begin{pmatrix} \psi_1(x, \xi) \\ \psi_2(x, \xi) \end{pmatrix} \underset{x \to +\infty}{\longrightarrow} \begin{pmatrix} a(\xi) e^{i\xi x/2} \\ b(\xi) e^{-i\xi x/2} \end{pmatrix} \tag{5}$$

then R^* is the quantum analog of the classical reflection coefficient,

$$R^*(\xi) = \frac{i}{\sqrt{c}} b(\xi) a^{-1}(\xi) \quad . \tag{6}$$

The operator algebra satisfied by the R-operators is determined by the structure of the eigenvalue problem:

$$[\, H, R^*(\xi)\,] = \xi^2 R^*(\xi) \quad , \tag{7}$$

$$R^*(\xi) R^*(\xi') = S(\xi', \xi) R^*(\xi') R^*(\xi) \quad , \tag{8}$$

$$R(\xi) R^*(\xi') = S(\xi, \xi') R^*(\xi') R(\xi) + 2\pi\delta(\xi - \xi') \quad , \tag{9}$$

where

$$S(\xi, \xi') = \frac{\xi - \xi' - ic}{\xi - \xi' + ic} \tag{10}$$

is the two-body S-matrix. From the commutation relation (7), we see that the state

$$| k_1, ..., k_n > = R^*(k_1)...R^*(k_n) | 0 > \tag{11}$$

is an eigenstate of the Hamiltonian (2),

$$H | k_1, ..., k_n > = \left(\sum_{i=1}^{n} k_i^2 \right) | k_1, ..., k_n > \quad . \tag{12}$$

In Ref. [4], and also in [10], it was shown that the states (11) are identical with those previously obtained by Bethe's ansatz.

The Gel'fand-Levitan method provides the inverse transformation from the R-operators back to the Heisenberg field. This transformation is accomplished by means of an operator integral equation for the Jost solution $\chi(x, \xi)$ with the asymptotic behavior $\chi(x, \xi) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\xi x/2}$ as $x \rightarrow +\infty$. Using the fact that $\chi_1(x, \xi) \sim -\sqrt{c}\,\phi(x)/\xi$ as $\xi \rightarrow \infty$ it was shown in [4] that the Heisenberg field $\phi(x)$ may be expressed as a series expansion in terms of R

$$\phi(x) = \int \frac{dk}{2\pi} R(k) e^{ikx} - c \int \frac{dp_1 dk_1 dk_2}{(2\pi)^3} \frac{R^*(p_1) R(k_1) R(k_2) e^{i(k_1 + k_2 - p_1)x}}{(p_1 - k_1 - i\epsilon)(p_1 - k_2 - i\epsilon)} + \cdots$$

$$\equiv \sum_{N=0}^{\infty} \phi^{(N)}(x) \quad , \tag{13}$$

where the general term has the form

$$\phi^{(N)}(x) = (-c)^N \int \prod_{i=1}^{N} \frac{dp_i}{2\pi} \prod_{j=1}^{N+1} \frac{dk_j}{2\pi} \frac{R^*(p_N)\ldots R^*(p_1)R(k_1)\ldots R(k_{N+1})e^{i(\Sigma k - \Sigma p)x}}{\prod_{m=1}^{N}(p_m - k_m - i\epsilon)(p_m - k_{m+1} - i\epsilon)} . \quad (14)$$

Here we will investigate the series (13) in the limit of infinite repulsion $c \to \infty$. In this limit we see from eqns. (8)-(10) that the operators R(k) become canonical fermion operators. At first sight it might seem that the presence of explicit positive powers of c in the series (13) would render it useless for studying the limit $c \to \infty$, but in fact we will see that the implicit c-dependence contained in the operators R(k) leads to a finite result. For the zero order term this is evident, since $\phi^{(0)}(x)$ is just $\tilde{R}(x)$, where

$$\tilde{R}(x) = \int \frac{dk}{2\pi} R(k) e^{ikx} . \quad (15)$$

Now consider the next term $\phi^{(1)}$. Symmetrizing over k_1 and k_2, using the commutation relations (8)-(9) and partial fractioning, we obtain

$$\phi^{(1)}(x) = -c \int \frac{dp_1 dk_1 dk_2}{(2\pi)^3} \frac{R^*(p_1)R(k_1)R(k_2)}{(p_1 - k_1 - i\epsilon)(p_1 - k_2 - i\epsilon)} \left\{ \frac{1 + S(k_1, k_2)}{2} \right\} e^{i(k_1 + k_2 - p_1)x}$$

$$= -2c \int \frac{dp_1 dk_1 dk_2}{(2\pi)^3} \frac{R^*(p_1)R(k_1)R(k_2)}{(k_1 - k_2 + ic)(p_1 - k_1 - i\epsilon)} e^{i(k_1 + k_2 - p_1)x} . \quad (16)$$

Now we see that the limit $c \to \infty$ is finite with

$$\phi^{(1)}(x) \xrightarrow[c \to \infty]{} 2i \int \frac{dp_1 dk_1 dk_2}{(2\pi)^3} \frac{R^*(p_1)R(k_1)R(k_2)}{(p_1 - k_1 - i\epsilon)} e^{i(k_1 + k_2 - p_1)x} = \Delta(x)\tilde{R}(x) , \quad (17)$$

where

$$\Delta(x) \equiv 2i\int \frac{dp_1 dk_1}{(2\pi)^2} \frac{R^*(p_1)R(k_1)}{p_1 - k_1 - i\varepsilon}$$

$$= -2\int_x^\infty \tilde{R}^*(y)\tilde{R}(y)dy \quad . \qquad (18)$$

The operators R in (17) and (18) are understood to be in the limit $c \to \infty$, i.e. they are canonical anti-commuting operators in both k- and x-space. Thus, the first two terms of the series give $(1 + \Delta(x))\tilde{R}(x)$. The central result of this note is that the series when summed to all orders exponentiates, so that

$$\phi(x) = N_R\{[\exp \Delta(x)]\tilde{R}(x)\} \quad , \qquad (19)$$

where the symbol N_R means normal ordering with respect to the fermion operators $\tilde{R}$. We now sketch a proof of this exponentiation by considering the general term $\phi^{(N)}$ of the series in (14). Symmetrizing over the N variables p_i and over the N + 1 variables k_j, we obtain an expansion of the form

$$\phi^{(N)}(x) = \int \prod_{i=1}^{N} \frac{dp_i}{2\pi} \prod_{j=1}^{N+1} \frac{dk_j}{2\pi} f_N(p_i, k_j; c)\, R^*(p_N)\ldots R^*(p_1)R(k_1)\ldots R(k_{N+1})e^{i(\Sigma k - \Sigma p)x} \quad . \qquad (20)$$

It is easy to see that f_N must vanish if any two p's or k's coincide and that f_N may therefore be expressed in the form

$$f_N(p_i, k_j; c) = \frac{\prod_{1\le i<j\le N+1} (k_i - k_j) \prod_{1\le j<i\le N} (p_i - p_j)}{\prod_{i=1}^{N} \prod_{j=1}^{N+1} (p_i - k_j)} g_N(p_i, k_j; c) \quad , \qquad (21)$$

where g_N is a dimensionless function which is given by a ratio of homogeneous multinomials in its arguments. (In our units c has the same dimension as p_i and k_j). By considering the action of $\phi(x)$ on the Bethe's ansatz states (10) it is not hard to convince oneself that each of the $f_N(p_i, k_j; c)$ is finite and nonzero in the limit $c \to \infty$. Thus in this limit g_N must become a numerical constant. Equation (20) may then be simplified to

$$f_N(p_i, k_j; \infty) = B_N \operatorname*{Antisym}_{p,k} \prod_{i=1}^{N} \left(\frac{1}{p_i - k_i - i\varepsilon} \right) \quad , \tag{22}$$

where B_N is a constant, simply related to g_N, and the right-hand side is to be antisymmetrized over $p_1, \ldots, p_N$ and over $k_1, \ldots, k_{N+1}$. Given (22), the constant B_N may be determined by considering the case $p_i = k_i + \delta$, $i = 1,2,\ldots,N$, and picking out the leading $1/\delta^N$ contributions on each side. Even for c finite, the task of calculating $f_n(p, k; c)$ is greatly simplified by this device, since only $(N + 1)!$ of the possible $N!(N + 1)!$ permutations contribute to order $1/\delta^N$. By this procedure we obtain

$$f_N(p, k; c) \underset{\delta \to 0}{\sim} \frac{(-1)^N}{N!(N+1)!} \frac{c^N}{\delta^N} \prod_{i=1}^{N} \left\{ \frac{1 + S(k_i, k_{N+1})}{k_i - k_{N+1}} \right\} \quad . \tag{23}$$

We see that the limit $c \to \infty$ is indeed finite with

$$f_N(p, k; \infty) \underset{\delta \to 0}{\sim} \frac{1}{N!(N+1)!} \frac{(2i)^N}{\delta^N} \quad . \tag{24}$$

Comparing this with the leading behavior of the right-hand side of (22) determines B_N to be $(2i)^N/N!$. Substituting (22) into (20), replacing the $i\varepsilon$'s, and recalling that the R's anticommute, we find

$$\phi^{(N)}(x) = \frac{(2i)^N}{N!}\int \prod_{i=1}^{N} \frac{dp_i}{2\pi} \prod_{j=1}^{N+1} \frac{dk_i}{2\pi} \frac{R^*(p_N)\dots R^*(p_1)R(k_1)\dots R(k_{N+1})}{(p_1 - k_1 - i\varepsilon)\dots(p_N - k_N - i\varepsilon)} e^{i(\Sigma k - \Sigma p)x}$$

$$= N_R\{\ [\Delta(x)]^N\tilde{R}(x)/N!\} \tag{25}$$

thus establishing the exponential form (19) for the full series. Since R(x) is a canonical field, we may convert the normal ordered exponential to an unordered exponential to obtain the Jordan-Wigner transformation eq. (1).

As an application of this result, eq. (1) or eq. (25), we will now relate the two point correlation function for the finite density impenetrable Bose gas to the solution of a certain integral equation. This result was first obtained by Schultz[11] who exploited the correspondence between this model and the XY Heisenberg spin chain. A determinantal representation of all the 2n-point correlation functions was later provided by Lenard.[12] The quantity of interest is the two-point function at fixed time,

$$\rho(x - y) \equiv \langle\Omega|\phi^*(x)\phi(y)|\Omega\rangle \tag{26}$$

where $|\Omega\rangle$ denotes the physical ground state at zero temperature and finite density. Recently this problem has been studied in more detail by Vaidya and Tracy,[13] who obtained both large and short distance expansions of the function $\rho(x)$, and by Jimbo, Miwa, Mōri and Sato[6] who showed how $\rho(x)$ may be simply expressed in terms of the solution to a certain non-linear ordinary differential equation. At $c = \infty$, the ground state is a uniform distribution of filled states up to some limiting momentum k_F, with the property

$$R(k)|\Omega> = 0 \quad \text{for } |k| > k_F \quad ,$$

$$R^*(k)|\Omega> = 0 \quad \text{for } |k| < k_F \quad . \tag{27}$$

To evaluate the two-point correlation function $\rho(x - y)$, we use (1) to write

$$\begin{aligned} \phi^*(x)\phi(y) &= \tilde{R}^*(x)\exp\left\{\lambda\pi\int_x^y \tilde{R}^*(z)\tilde{R}(z)dz\right\}\tilde{R}(y) \\ &= N_R\left[\tilde{R}^*(x)\exp\left\{-2\int_x^y \tilde{R}^*(z)\tilde{R}(z)dz\right\}\tilde{R}(y)\right] \quad . \end{aligned} \tag{28}$$

Using the properties of the vacuum, eq. (27), we see that

$$\begin{aligned} <\tilde{R}^*(x)\tilde{R}(y)>_\Omega &= \frac{\sin[k_F(x - y)]}{\pi(x - y)} \\ &\equiv \tfrac{1}{2}K(x, y) \quad . \end{aligned} \tag{29}$$

Expanding the exponential in (28) and using (29), we find

$$\begin{aligned} \rho(x - y) &= \frac{1}{2}\left\{K(x, y) - \int dz \begin{vmatrix} K(x, y) & K(x, z) \\ K(z, y) & K(z, z) \end{vmatrix}\right. \\ &\quad \left. + \frac{1}{2!}\int dzdz' \begin{vmatrix} K(x, y) & K(x, z) & K(x, z') \\ K(z, y) & K(z, z) & K(z, z') \\ K(z', y) & K(z', z) & K(z', z) \end{vmatrix} + \ldots\right\} \\ &= \frac{1}{2}D(x, y; \lambda = 1) \end{aligned} \tag{30}$$

where $|A|$ denotes the determinant of the matrix A. Here $D(x, y; \lambda)$ is the first Fredholm minor associated with the linear integral equation

$$K(x, y) = f(x, y) - \int_y^x K(x, z) f(z, y) dz \quad . \tag{31}$$

The result (30) is particularly well-suited for computing successive terms in a short distance expansion of the correlation function and is found to reproduce in this region the results of Vaidya and Tracy and Jimbo, Miwa, Mōri and Sato.

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