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Gelfand-Levitan Method for Operator Fields

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ABSTRACT

The quantum generalization of the Gelfand-Levitan method is presented for the nonlinear Schrödinger model. The basic dispersion relation for operator Jost functions is derived, and the Heisenberg field operator is expressed in terms of scattering data operators. Construction of Green's functions in the zero-density vacuum is discussed. The four-point function is explicitly calculated from the expression for the field operator and compared with the result of a direct Feynman graph summation. In addition it is proven for any number of particles that the Hamiltonian eigenstates constructed from the quantized scattering data are identical with those previously obtained by means of Bethe's ansatz.

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I. INTRODUCTION

The development of the inverse scattering transform as an operator method in quantum field theory has provided a clearer understanding of the structure of exactly soluble models. The method has been used to formulate operator solutions of the quantum nonlinear Schrödinger[1-3] and sine-Gordon equations.[4] The Bethe ansatz solutions of certain two-dimensional fermion theories[5] may also be classified as indirect applications of the quantum inverse method. Comparing these methods with the classical inverse technique of Gardner, Greene, Kruskal, and Miura,[6] it is apparent that what has been developed in the quantum theory thus far represents only a part of the classical methodology. In the classical inverse method, one solves the initial value problem by first mapping the initial conditions into a set of scattering data (the direct problem) and at a subsequent time, reconstructing the field configuration from the scattering data (the inverse problem). Until now, the analysis of the quantum formulation has provided the generalization only of the direct problem. This suffices for the construction of the eigenstates and eigenvalues of the Hamiltonian. In order to discuss the Heisenberg field operator and the calculation of Green's functions, we must consider the quantum analog of the inverse problem, which is solved classically by the Gelfand-Levitan integral equation. This quantum generalization is the content of the present paper. In Section II we derive an operator form of the Gelfand-Levitan equation for the quantum nonlinear Schrödinger equation and then, in Section III, discuss some of its consequences. Section IV contains a short discussion.

In the remainder of this Section we collect some results from the direct problem analysis, following essentially the notation of Ref. 2 . The nonlinear Schrödinger model (delta-function gas) is defined by the normal ordered Hamiltonian

$$H = \int dx\{\partial_1\phi^*\partial_1\phi + c\ \phi^*\phi^*\phi\phi\} , \tag{1}$$

where the field ϕ obeys equal-time commutation relations

$$\left[\phi(x,t),\phi^*(y,t)\right] = \delta(x-y). \tag{2}$$

For simplicity, we will consider only the case of repulsive coupling ($c>0$) for which there are no bound states in the spectrum. The model is solved by considering the Zakharov-Shabat linear eigenvalue problem[7]

$$\left(i\ \frac{\partial}{\partial x} + \frac{1}{2}\ \xi\right)\ \Psi_1 = -\ \sqrt{c}\ \Psi_2\phi\ , \tag{3a}$$

$$\left(i\ \frac{\partial}{\partial x} - \frac{1}{2}\ \xi\right)\ \Psi_2 = \sqrt{c}\ \phi^*\ \Psi_1\ . \tag{3b}$$

The solutions Ψ of Eq. (3) are normal ordered operator functionals of the Heisenberg fields ϕ and ϕ^*. Particular solutions are chosen by specifying the asymptotic behavior in one direction at spatial infinity. Of special interest are the Jost solutions $\psi(x,\xi)$ and $\chi(x,\xi)$ defined by

$$\psi(x,\xi) \sim \begin{pmatrix}1\\0\end{pmatrix} e^{i\xi x/2} \quad \text{as } x \to -\infty , \tag{4a}$$

$$\chi(x,\xi) \sim \begin{pmatrix}0\\1\end{pmatrix} e^{-i\xi x/2} \quad \text{as } x \to +\infty . \tag{4b}$$

Both ψ and χ are analytic in the lower half ξ-plane, where analyticity of an operator is taken to be equivalent to analyticity of all its matrix elements between physical states. We will also have occasion to use the conjugate functions

$$\tilde{\psi} = \begin{pmatrix}\psi_2^*\\ \psi_1^*\end{pmatrix} \text{ and } \tilde{\chi} = \begin{pmatrix}\chi_2^*\\ \chi_1^*\end{pmatrix}, \tag{5}$$

which for real ξ are also solutions of Eq. (3) and are analytic in the upper half plane. The scattering data $a(\xi)$ and $b(\xi)$ are defined by the asymptotic behavior

$$\psi(x,\xi) \sim \begin{pmatrix}a(\xi)\, e^{i\xi x/2}\\ b(\xi)\, e^{-i\xi x/2}\end{pmatrix} \text{ as } x \to +\infty . \tag{6}$$

The fundamental commutators among the operators a and b may be obtained from the following properties (7) through (10) of the Jost functions ψ and χ.

From the defining relations (3) and (4) one finds that the non-vanishing commutators of ψ and χ with the fundamental fields ϕ at the same point x are given by

$$\left[\psi_1, \phi^*\right] = \frac{i\sqrt{c}}{2}\,\psi_2$$

$$\left[\psi_2, \phi\right] = \frac{i\sqrt{c}}{2}\,\psi_1$$

$$\left[\chi_1, \phi^*\right] = -\frac{i\sqrt{c}}{2}\,\chi_2$$

$$\left[\chi_2, \phi\right] = -\frac{i\sqrt{c}}{2}\,\chi_1\,. \tag{7}$$

In addition ψ and χ commute:

$$\left[\psi_i(x,\xi), \chi_j(x,\xi')\right] = 0\,. \tag{8}$$

The results (7) and (8) remain valid with ψ replaced by $\tilde{\psi}$ and/or χ replaced by $\tilde{\chi}$. Using (7) one may show that the Wronskian of ψ or $\tilde{\psi}$ with χ or $\tilde{\chi}$ is constant in x, and hence the scattering data may be expressed as

$$a = \psi_1\,\chi_2 - \psi_2\,\chi_1$$

$$b = \psi_2\,\tilde{\chi}_1 - \psi_1\,\tilde{\chi}_2\,. \tag{9}$$

Thus $a(\xi)$ is analytic in the lower half plane. From (9) we may also deduce the asymptotic behavior

$$\chi(x,\xi) \sim \begin{pmatrix} -\ b^*(\xi)e^{i\xi x/2} \\ a(\xi)e^{-i\xi x/2} \end{pmatrix} \text{ as } x \to -\infty \ . \tag{10}$$

Using these results it is possible to derive all the commutation relations among the Hamiltonian H and the scattering data $a(\xi)$ and $b(\xi)$. Expressing the results of Ref. 2 in terms of the reflection coefficient

$$R^*(\xi) = \frac{i}{\sqrt{c}}\, b(\xi)\ a^{-1}(\xi), \tag{11}$$

we have

$$[H, a(\xi)] = 0 \ , \tag{12}$$

$$[H, R^*(\xi)] = \xi^2\ R^*(\xi) \ , \tag{13}$$

$$[a(\xi), a(\xi')] = [a(\xi), a^*(\xi')] = 0 \ , \tag{14}$$

$$a(\xi)R^*(\xi') = \left(1 - \frac{ic}{\xi-\xi'-i\varepsilon}\right)R^*(\xi')a(\xi), \tag{15}$$

$$a^*(\xi)R^*(\xi') = \left(1 + \frac{ic}{\xi-\xi'+i\varepsilon}\right)R^*(\xi')a^*(\xi) \ , \tag{16}$$

$$R^*(\xi)R^*(\xi') = S(\xi',\xi)R^*(\xi')R^*(\xi) \ , \tag{17}$$

where $S(\xi,\xi')$ is given by

$$S(\xi,\xi') = \frac{\xi-\xi' - ic}{\xi-\xi' + ic} \ . \tag{18}$$

For $\xi>\xi'$ this is the two body S-matrix. The commutation relations (12) - (17) have also been derived, in extremely elegant fashion, by Sklyanin and Faddeev,[1] who obtain the additional result

$$R(\xi)R^*(\xi') = S(\xi,\xi')R^*(\xi')R(\xi) + 2\pi\ \delta(\xi-\xi') \ . \tag{19}$$

That (19) is correct has also been shown in Ref. 8. From (12)-(16) it is seen that if we define a state

$$|k_1 \ldots k_n\rangle \equiv R^*(k_1)\ldots R^*(k_n)|0\rangle \ , \tag{20}$$

where $|0\rangle$ is the vacuum state with $\phi(x)|0\rangle=0$, then $|k_1\ldots k_n\rangle$ is a simultaneous eigenstate of H, $a(\xi)$ and $a^*(\xi)$:-

$$H|k_1\ldots k_n\rangle = \left(\sum_i k_i^2\right)|k_1\ldots k_n\rangle \ , \tag{21}$$

$$a(\xi)|k_1\ldots k_n\rangle = \prod_i\left(1 - \frac{ic}{\xi-k_i-i\varepsilon}\right)|k_1\ldots k_n\rangle \ , \tag{22}$$

$$a^*(\xi)|k_1\ldots k_n\rangle = \prod_i\left(1 + \frac{ic}{\xi-k_i+i\varepsilon}\right)|k_1\ldots k_n\rangle \ . \tag{23}$$

The particular significance of the operator $R^*(\xi)$ is that if $k_1<k_2<\ldots<k_n$ then $|k_1\ldots k_n\rangle$ is a normalized in-state, while if $k_1>k_2\ldots>k_n$ it is a normalized out-state. In Section III it is proven that these states are identical with those previously obtained by means of Bethe's ansatz,[9,10] a result till now checked only up to n=3.

II. QUANTUM GELFAND-LEVITAN EQUATION

In this Section we derive the quantum version of the Gelfand-Levitan integral equation which relates the field $\phi(x)$ to the reflection coefficient $R(\xi)$. To begin let us review the derivation of Zakharov and Shabat[7] for the classical case. These authors consider the equation $\psi=a\tilde{\chi}+b\chi$ (for ξ real) and use the definition (11) to write it in the form

$$\psi a^{-1} = \tilde{\chi} - i\sqrt{c}\, R^* \chi \ . \tag{24}$$

This suggests a piecewise analytic function Φ defined by

$$\Phi(x,\xi) = \begin{cases} \tilde{\chi}\, e^{-i\xi x/2} & \mathrm{Im}\xi>0 \\ \psi a^{-1}\, e^{-i\xi x/2} & \mathrm{Im}\xi<0 \end{cases} \ . \tag{25}$$

From (24) the discontinuity of Φ across the real axis is $i\sqrt{c}\, R^*\chi$, while as $|\xi|\to\infty$, Φ has the behavior

$$\Phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\left(\frac{1}{\xi}\right) . \tag{26}$$

Thus we may write a dispersion relation for Φ which when evaluated just above the real axis reads

$$\tilde{\chi} e^{-i\xi x/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} d\xi' \, \frac{R^*(\xi')\chi(x,\xi')e^{-i\xi' x/2}}{\xi'-\xi-i\varepsilon} \ . \tag{27}$$

This equation enables the Jost function $\chi(x,\xi)$ to be determined in terms of the reflection coefficient $R(\xi)$; the field ϕ may then be obtained by means of the asymptotic behavior

$$\chi_1(x,\xi)e^{i\xi x/2} \sim -\frac{\sqrt{c}\phi(x)}{\xi}, \text{ as } |\xi|\to\infty , \tag{28}$$

which follows directly from the defining equations (3) and (4).

The main result of this paper is to demonstrate that (27) is maintained in the quantum theory and that the correct ordering is as shown.[8] This is not a trivial extension of the classical result since, as we shall see explicitly below, the motivating equation (24) is no longer valid in the quantum theory. Let us instead, for real ξ, define an operator function g by

$$g(x,\xi) = \tilde{\chi}(x,\xi) - i\sqrt{c}\; R^*(\xi)\chi(x,\xi) \ . \tag{29}$$

Since the functions $\tilde{\chi}$ and χ on the right satisfy the Zakharov-Shabat equation (3) it follows that g satisfies the coupled equations

$$\left(i\frac{\partial}{\partial x} + \frac{1}{2}\xi\right)g_1 = -\sqrt{c}\; g_2\phi , \tag{30a}$$

$$\left(i\frac{\partial}{\partial x} - \frac{1}{2}\xi\right)g_2 = \sqrt{c}\;\phi^*\; g_1 - ic\left[R^*(\xi),\phi^*(x)\right]\chi_1 \ . \tag{30b}$$

The second term in (30b) arises from the quantum orderings. Using Eqs. (7), (8), (9) and (11) the commutator $[R^*(\xi),\phi^*(x)]$ may be evaluated, yielding

$$\left[R^*(\xi),\phi^*\right] = (\tilde{\chi}_2 - i\sqrt{c}\; R^*\chi_2)\psi_2 a^{-1}, \tag{31}$$

so that equations (30) become

$$\left(i \frac{\partial}{\partial x} + \frac{1}{2}\xi\right) g_1 = -\sqrt{c}\; g_2 \phi \;, \tag{32a}$$

$$\left(i \frac{\partial}{\partial x} - \frac{1}{2}\xi\right) g_2 = \sqrt{c}\; \phi^* g_1 - ic\, g_2\, \psi_2 a^{-1} \chi_1 \;. \tag{32b}$$

From its definition (29) the function g has the asymptotic behavior

$$g \sim e^{i\xi x/2} \begin{pmatrix} \tilde{a} \\ 0 \end{pmatrix} \text{ as } x \to -\infty \;, \tag{33}$$

where, for real ξ, the operator $\tilde{a}$ is defined by

$$\tilde{a}(\xi) = a^*(\xi) - c\, R^*(\xi) a^*(\xi) R(\xi) \;. \tag{34}$$

Classically we could use the unitarity relation $|a|^2 - |b|^2 = 1$ to conclude that $\tilde{a}(\xi)$ equals $a^{-1}(\xi)$, as indeed it would be if g were ψa^{-1}. To gain some feeling for the quantum case let us use the results (16)-(19) to evaluate $\tilde{a}(\xi)$ on a one-particle state

$$\tilde{a}(\xi)|k\rangle = \left(1 + \frac{ic}{\xi - k + i\varepsilon} - 2\pi\, c\, \delta(\xi - k)\right)|k\rangle$$

$$= \left(1 + \frac{ic}{\xi - k - i\varepsilon}\right)|k\rangle \;. \tag{35}$$

We see that only to first order in c is this the same as $a^{-1}(\xi)|k\rangle$, since the latter is $\left(1 - \frac{ic}{\xi - k - i\varepsilon}\right)^{-1}|k\rangle$. More generally we may show

$$\tilde{a}(\xi)|k_1 \ldots k_n\rangle = \prod_{i=1}^{n} \left(1 + \frac{ic}{\xi - k_i - i\varepsilon}\right)|k_1 \ldots k_n\rangle \ . \tag{36}$$

Since the states $|k_1 \ldots k_n\rangle$ are complete, we may regard (36) as an alternative definition of $\tilde{a}(\xi)$ and conclude that it is analytic in the lower half ξ-plane with the property

$$\tilde{a}(\xi) = 1 + 0\left(\frac{1}{\xi}\right) \text{ as } |\xi| \to \infty \ . \tag{37}$$

We now make the crucial observation that both the differential equation (32) for g and its asymptotic behavior in (33) may be continued into the lower half ξ-plane without singularities. Thus the operator function $g(x,\xi)$ may itself be continued into the lower half ξ-plane, and is analytic there. We are now in a position to mimic the classical derivation, defining a function Φ by

$$\Phi(x,\xi) = \begin{cases} \tilde{\chi}\ e^{-i\xi x/2} & \text{Im } \xi > 0 \\ g\ e^{-i\xi x/2} & \text{Im } \xi < 0 \ . \end{cases} \tag{38}$$

By construction Φ is piecewise analytic with discontinuity across the real axis given by $i\sqrt{c}\ R^*(\xi)\chi(x,\xi)e^{-i\xi x/2}$. In addition, as $|\xi| \to \infty$, Φ has the property

$$\Phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underline{0}\left(\frac{1}{\xi}\right) , \tag{39}$$

which holds in the lower half plane by virtue of (37). Thus we may write a dispersion relation for Φ which yields the desired result

$$\tilde{\chi} e^{-i\xi x/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} d\xi' \ \frac{R^*(\xi')\chi(x,\xi')e^{-i\xi' x/2}}{\xi' - \xi - i\varepsilon} \ . \tag{40}$$

Iterating this equation and its hermitian conjugate, and using the asymptotic behavior (28), which holds also in the quantum theory, we obtain

$$\phi(x) = \int \frac{d\xi_1}{2\pi} R(\xi_1) e^{i\xi_1 x}$$

$$+ c \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} \frac{d\xi_3}{2\pi} \frac{R^*(\xi_2) R(\xi_1) R(\xi_3) e^{i(\xi_1 - \xi_2 + \xi_3)x}}{(\xi_2 - \xi_1 - i\varepsilon)(\xi_3 - \xi_2 + i\varepsilon)}$$

$$+ \ldots \quad . \tag{41}$$

All the above results refer to some fixed time, say t=0. However from (13) we see that the time dependence of $R(\xi)$ is simple so that (40) and (41) may be generalized to arbitrary times by the replacement

$$R(\xi) \rightarrow R(\xi,t) = e^{-i\xi^2 t} R(\xi) \quad . \tag{42}$$

Equations (40) and (41) are the fundamental results of this paper. However since the above proof is somewhat formal it may be useful to provide a simple test of these results. Note that equation (40) holds for all values of the spatial variable x. In particular it is valid in the asymptotic regime $x \rightarrow -\infty$ where it produces only one non-trivial result:

$$a(\xi) = 1 + ic \int \frac{d\xi'}{2\pi} \frac{R^*(\xi') a(\xi') R(\xi')}{\xi' - \xi + i\varepsilon} \quad . \tag{43}$$

Since $a(\xi)$ is diagonal on the n-particle states (20) we can verify this equation directly. By acting on the state $|k_1 \ldots k_n\rangle$ it is easily seen using (17),(19) and (22) that (43) is equivalent to the identity

$$\prod_{i=1}^{n}\left(1-\frac{ic}{\xi-k_i}\right)=1-ic\sum_{j=1}^{n}\left\{\frac{1}{\xi-k_j}\prod_{\substack{i=1\\ i\neq j}}^{n}\left(1-\frac{ic}{k_j-k_i}\right)\right\}. \tag{44}$$

(Here, all denominators are understood to have a negative imaginary part.) For n=2 the validity of (44) is seen by noting that

$$\frac{1}{\xi-k_1}\left(1-\frac{ic}{k_1-k_2}\right)+\frac{1}{\xi-k_2}\left(1-\frac{ic}{k_2-k_1}\right)=\frac{1}{\xi-k_1}+\frac{1}{\xi-k_2}-\frac{ic}{(\xi-k_1)(\xi-k_2)}\ , \tag{45}$$

and thus the right hand side of (44) reduces to $(1-ic/(\xi-k_1))\times(1-ic/(\xi-k_2))$. Using algebraic manipulations similar to the above, the identity (44) may be proven for arbitrary n by induction, thus verifying (43).

III. APPLICATIONS

In this section we present two applications of Eqs. (41) and (42) for the Heisenberg field $\phi(x,t)$. First we observe that together with the commutator (19) these equations may be used to compute any 2n-point Green's function

$$\langle 0|T\{\phi(x_1't_1')\ldots\phi(x_n't_n')\phi^*(x_1t_1)\ldots\phi^*(x_nt_n)\}|0\rangle \quad ,$$

where T denotes the usual time ordering operation. Since for each term in the expansion (41), the annihilation operators $R(\xi)$ appear to the right of the creation operators $R^*(\xi)$, it is evident that for the 2n-point function above we need consider at most the first n terms in the expansion. Thus the evaluation of any Green's function is reduced to a finite calculation. The two-point function $\langle 0|T\{\phi(x,t)\phi^*(0,0)\}|0\rangle$ is easily seen to be just the free propagator, so that the first non-trivial application of (41) is for the 4-point function. Denoting by $G^4(\omega_i' k_i' \omega_i k_i)$ the Fourier transform

$$\int \prod_{i=1}^{2} \left\{ dx_i dt_i dx_i' dt_i' \ \exp\ i(\omega_i t_i - \omega_i' t_i' - k_i x_i + k_i' x_i') \right\}$$

$$\times \ <0|T\left\{\phi(x_1' t_1')\phi(x_2' t_2')\phi^*(x_1 t_1)\phi(x_2 t_2)\right\}|0> \ , \tag{46}$$

we find, after some computation, that the connected part G_c^4 is given by

$$G_c^4 = \left[\frac{4\pi^2 \delta(\omega_1' + \omega_1' - \omega_1 - \omega_2)\,\delta(k_1' + k_2' - k_1 - k_2)}{\prod_{i=1}^{2} (\omega_i - k_i^2 + i\varepsilon)(\omega_i' - k_i'^2 + i\varepsilon)} \right] \left[\frac{-4ic}{1 + ic/\sigma} \right] , \tag{47}$$

with

$$\sigma \equiv \left[2(\omega_1 + \omega_2) - (k_1 + k_2)^2 \right]^{1/2} . \tag{48}$$

The result (47) may be compared with a direct Feynman graph calculation. Apart from external factors [the first bracket in (47)] the tree graph is just -4ic, while the one-loop term is given by

$$(-4ic)^2 \int \frac{d\omega dk}{(2\pi)^2} \ \frac{i^2}{(\omega - k^2 + i\varepsilon)(\omega_1 + \omega_2 - \omega - (k_1 + k_2 - k)^2 + i\varepsilon)} , \tag{49}$$

which reduces to $-4ic(-ic/\sigma)$. Since this contribution depends only on the total energy and momentum, and not on any relative momentum, we see that the full connected 4-point function, which is just a sum of bubbles, is a geometric series which sums to give the result (47). As an aside, it is interesting to note how the 4-point function (47) leads on the mass shell, $\omega_i = k_i^2, \omega_i' = k_i'^2$, to the two-body S-matrix (18). Writing the scattering operator as S=1+T, and noting that on shell the quantity σ in (48) is just $k_{12} \equiv |k_1 - k_2|$, we conclude that the matrix element of T between asymptotic states is

$$\langle k_1' k_2' | T | k_1 k_2 \rangle = (2\pi)^2 \delta(k_1' + k_2' - k_1 - k_2)\, \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2') \left[\frac{-4ic}{1 + ic/k_{12}}\right]$$

$$= (2\pi)^2 \delta(k_1' + k_2' - k_1 - k_2) \left[\delta(k_1 - k_1') + \delta(k_1 - k_2')\right] \left[\frac{-2ic/k_{12}}{1 + ic/k_{12}}\right]$$

$$= -\frac{2ic}{k_{12} + ic} \langle k_1' k_2' | k_1 k_2 \rangle \ . \tag{50}$$

Thus

$$\langle k_1' k_2' | S | k_1 k_2 \rangle = \frac{k_{12} - ic}{k_{12} + ic} \langle k_1' k_2' | k_1 k_2 \rangle \ , \tag{51}$$

in agreement with (18).

As our second application of the Heisenberg field expression (41), we will prove for any number of particles that the states (20) are identical with those obtained previously by Bethe's ansatz.[9,10] We first introduce the Fourier transform

$$R(x) = \int \frac{d\xi}{2\pi} e^{i\xi x} R(\xi) \ , \tag{52}$$

and define states $|x_1 \ldots x_n\rangle$ by

$$|x_1 \ldots x_n\rangle = R^*(x_1) \ldots R^*(x_n) |0\rangle \ . \tag{53}$$

Then if $x>x_i$ for $i=1,2\ldots n$ we find using (41) that

$$\phi^*(x)|x_1\ldots x_n> = R^*(x)|x_1\ldots x_n> , \tag{54}$$

i.e. only the first term in (41) survives; the contribution of the remaining terms may be shown to vanish by simple contour integration. Applying (54) repeatedly we conclude that if $x_1>x_2>\ldots>x_n$ then

$$\phi^*(x_1)\phi^*(x_2)\ldots\phi^*(x_n)|0> = R^*(x_1)R^*(x_2)\ldots R^*(x_n)|0> . \tag{55}$$

That this result follows from identifying the Bethe ansatz states with those in (20) has already been noted in Ref. 8 . Here we will show the converse—that (55) actually implies this identification. To see this let us consider the coordinate space wave function

$$f(k_i,x_i) = <0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|k_1\ldots k_n> . \tag{56}$$

Since this is symmetric in $x_1\ldots x_n$ it is sufficient to consider it in the region $x_1<x_2<\ldots<x_n$. In this region we have from (55)

$$\begin{aligned} f(k_i x_i) &= <0|R(x_1)R(x_2)\ldots R(x_n)|k_1\ldots k_n> \\ &= \int \prod_i \left(\frac{dp_i}{2\pi} e^{ip_i x_i}\right) <0|R(p_1)\ldots R(p_n)R^*(k_1)\ldots R^*(k_n)|0> . \end{aligned} \tag{57}$$

Using the commutator (19) to evaluate the matrix element we recover the usual Bethe ansatz expression for the wave function.

IV. DISCUSSION

We have shown that in the nonlinear Schrödinger equation, the operator transformation which expresses $a(\xi)$ and $b(\xi)$ in terms of the field operators $\phi(x)$ and $\phi^*(x)$ may be inverted by a generalization of the Gelfand-Levitan method. The derivation of the basic integral equation in Section II employed an analyticity argument similar to the classical treatment of Zakharov and Shabat, although the problem of operator ordering led to some essential differences in the quantum analysis. Using the Gelfand-Levitan equation (41), the field operators are expressed in terms of the reflection operator $R(\xi)$ which has a simple time dependence. This is a central result of the present analysis. To gain experience with this formalism we carried out a number of calculations, some of which are reported in Section III. By considering vacuum expectation values and few-body matrix elements of fields, exact calculations may be carried out by keeping only the first few terms of the series expansion (41). The detailed combinatorics of these calculations is very reminiscent of the graphical formalism which was developed for this model by studying Feynman graphs.[10] All of these results should be appended with the remark that in this model, the vacuum state is structureless from the point of view of Bethe's ansatz, i.e. it is the reference state upon which the Bethe eigenstates are built. A more meaningful application of the methods we have developed here is provided by considering correlation functions in a finite density gas. This problem is more closely analogous to the calculation of Green's functions in a relativistic theory, where the physical vacuum is a complicated many-body state. By such considerations, one might also hope to establish a connection with the work of Vaidya and Tracy[11] and Jimbo, Miwa, Mōri and Sato,[12] who have obtained exact results for the correlation functions in the case of infinitely repulsive coupling ($c=\infty$). These questions are presently being investigated and will be discussed in a subsequent paper.

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