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Simple QCD Parametrizations of Parton Distributions Beyond the Leading Order of Asymptotic Freedom

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ABSTRACT

It is shown that the Q^2 dependence of the non-singlet parton distributions in Quantum Chromodynamics with leading and next to leading order corrections included, can be adequately represented by simple analytic expressions, similar to those proposed by Gaemers and one of the authors. The pattern of higher order corrections to the Q^2 dependence of $F_2^{NS}(x,Q^2)$ and of $F_3(x,Q^2)$ is discussed.

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I. INTRODUCTION

During the last year the calculations of next-to the leading order QCD corrections to deep-inelastic structure functions have been completed.[1-4] These results are usually presented for the moments of the structure functions in question. In order to compare these predictions with the data one has to calculate the moments of the experimentally measured structure functions. Since for a finite ν and Q^2 the structure functions are not known in the whole range of Bjorken variable x, extrapolations have to be made in the calculations of moments. Therefore it seems useful to have theoretical predictions for the structure functions themselves so that they could be directly compared with the experimental data. In the leading order this problem has been solved. There exist already many inversion techniques[5,6] which have been applied to leading order expressions. In particular in Ref. 5 a procedure has been developed for obtaining analytic expressions for parton distributions with Q^2 dependence given by ASF. Having such analytic expressions for parton distributions at hand one can use them for quantitative predictions of scaling violations in all processes (inclusive, semi-inclusive) in which QCD effects can be reliably calculated. In this paper we generalize the method of Ref. 5 beyond the leading order. As a result we obtain analytic expressions for the Q^2 dependent parton distributions with next to leading order effects taken into account. Since the definition of parton distributions is not unique beyond the leading order[7,8] we shall discuss two such definitions: one in the main text and the second in an Appendix. In this paper we limit our discussion to non-singlet structure functions and non-singlet combinations of parton distributions as for instance valence distributions.

The paper is organized as follows. In Section II we recall all formal expressions for the moments of non-singlet structure functions calculated up to and including $\bar{g}^2(Q^2)$ corrections. Subsequently we discuss one definition of the parton distributions (still on the level of moments). In Section III, which contains the main results of the paper, we present generalization of the method of Ref. 5 beyond the leading order. Numerical estimates and various examples are given in Section IV. We end the paper with a brief summary of the results. Discussion of a second definition of parton distributions can be found in the Appendix.

II. BASIC FORMALISM

2.1. Formal Approach[9,4]

In QCD the Q^2 dependence of the moments of deep-inelastic non-singlet structure functions is given as follows[10]

$$M_k^{NS}(n,Q^2) = \int_0^1 dx\; x^{n-2} \mathscr{F}_k^{NS}(x,Q^2) \qquad k = 2,3 \tag{2.1}$$

$$= A_n^{NS}(\mu^2) C_{k,n}^{NS}\left(\frac{Q^2}{\mu^2}, g^2\right) \tag{2.2}$$

$$= A_n^{NS}(Q_0^2) \exp\left[-\int_{\bar{g}^2(Q_0^2)}^{\bar{g}(Q^2)} dg' \frac{\gamma_{NS}^n(g')}{\beta(g')}\right] C_{k,n}^{NS}(1,\bar{g}^2(Q^2)) \tag{2.3}$$

In Eq. (2.2) $A_n^{NS}(\mu^2)$ stands for the hadronic matrix element of a spin n non-singlet operator and $C_{k,n}^{NS}$ is the coefficient function of this operator in the Wilson expansion[11]. A_n's are incalculable by present methods and must be taken from the data at some arbitrary value of $Q^2=\mu^2=Q_0^2$. The $C_{k,n}^{NS}$'s, on the other hand, can be calculated in perturbation theory. Their Q^2 dependence is governed by certain equations called renormalization group equations. The solution of this equation is given in Eq. (2.3) where $\bar{g}^2(Q^2)$,

$\gamma_{NS}^{n}(g)$ and $\beta(g)$ are the effective coupling constant, anomalous dimensions of the non-singlet operator mentioned above and the standard β function, respectively. The latter function governs the Q^2 evolution of $\bar{g}^2(Q^2)$:

$$\frac{d\bar{g}^2}{dt} = \bar{g}\ \beta(\bar{g}) \quad ; \quad \bar{g}(t=0) = g \tag{2.4}$$

where $t = \ln(Q^2/\mu^2)$, and g is the quark-gluon coupling constant normalized at μ^2.

The functions $\gamma_{NS}^{n}(g)$, $\beta(g)$ and $C_{k,n}^{NS}(1,\bar{g}^2(Q^2))$ have the following perturbative expansions

$$C_{k,n}^{NS}(1,\bar{g}^2) = \delta_{NS}^{(k)}\left(1 + \frac{\bar{g}^2}{16\pi^2}\bar{B}_{k,n}^{NS} + \ldots\right) \quad , \tag{2.5}$$

$$\gamma_{NS}^{n}(g) = \gamma_{NS}^{(0),n}\frac{g^2}{16\pi^2} + \gamma_{NS}^{(1),n}\frac{g^4}{(16\pi^2)^2} + \ldots \tag{2.6}$$

$$\beta(g) = -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{16\pi^2} + \ldots \tag{2.7}$$

where $\delta_{NS}^{(k)}$ are weak or electromagnetic charge factors.

Inserting Eqs. (2.5)-(2.7) into Eq. (2.3) and expanding in $\bar{g}^2(Q^2)$ one obtains

$$M_n^{NS}(n,Q^2) = \delta_{NS}^{(k)} A_n^{NS}(Q_0^2)\left[1 + \frac{[\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)]}{16\pi^2} Z_n^{NS}\right] \cdot$$

$$\left[\frac{\bar{g}^2(Q^2)}{g^2(Q_0^2)}\right]^{d_{NS}^n} \cdot \left[1 + \frac{\bar{g}^2(Q^2)}{16\pi^2} \bar{B}_{k,n}^{NS}\right] \qquad (2.8)$$

where

$$Z_n^{NS} = \frac{\gamma_{NS}^{(1),n}}{2\beta_0} - \frac{\gamma_{NS}^{(0),n}}{2\beta_0^2}\beta_1 \quad ; \quad d_{NS}^n = \frac{\gamma_{NS}^{0,n}}{2\beta_0} \qquad (2.9)$$

and

$$\frac{\bar{g}^2(Q^2)}{16\pi^2} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} - \frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln^2(Q^2/\Lambda^2)} + O\left(\frac{1}{\ln^3(Q^2/\Lambda^2)}\right) \qquad (2.10)$$

Here Λ is a scale parameter and d_{NS}^n [9] and $\bar{B}_{k,n}^{NS}$ [1,2] are given as follows

$$d_{NS}^n = \frac{4}{(33-2f)}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n}\frac{1}{j}\right] \qquad (2.11)$$

$$\bar{B}^{NS}_{2,n} = \frac{4}{3}\left\{3 \sum_{j=1}^{n} \frac{1}{j} - 4 \sum_{j=1}^{n} \frac{1}{j^2} - \frac{2}{n(n+1)} \sum_{j=1}^{n} \frac{1}{j} \right. \tag{2.12}$$

$$\left. + 4 \sum_{s=1}^{n} \frac{1}{s} \sum_{j=1}^{s} \frac{1}{j} + \frac{3}{n} + \frac{4}{(n+1)} + \frac{2}{n^2} - 9 \right\}$$

and[1,2,8]

$$\bar{B}^{NS}_{3,n} = \bar{B}^{NS}_{2,n} - \frac{4}{3}\,\frac{4n+2}{n(n+1)} \quad . \tag{2.13}$$

where f is the number of flavors. The analytic expressions for $\gamma^{(1),n}_{NS}$ are very complicated and therefore we quote in Table I the numerical values for Z^{NS}_n. Also the numerical values for d^n_{NS}, $\bar{B}^{NS}_{2,n}$ and $\bar{B}^{NS}_{3,n}$ are collected there. Finally we give the expressions for β_0 [9] and β_1:[9,12]

$$\beta_0 = 11 - \frac{2}{3} f \quad ; \quad \beta_1 = 102 - \frac{38}{3} f \quad . \tag{2.14}$$

It should be recalled that,

i) $\gamma^{(1),n}_{NS}$ and $\bar{B}^{NS}_{k,n}$ depend on the renormalization scheme used to calculate these quantities.[1] This renormalization prescription dependence of $\gamma^{(1),n}_{NS}$ and $\bar{B}^{NS}_{k,n}$ cancel in Eq. (2.8) if these quantities are calculated in the same scheme. Numerical values in Table 1 correspond to the minimal subtraction scheme of 't Hooft.[13]

ii) The parameters $\bar{B}^{NS}_{k,n}$ depend also on the definition of $\bar{g}^2(Q^2)$.[2,14] The values in Table 1 correspond to so-called $\overline{MS}$ scheme[2] for $\bar{g}^2(Q^2)$. Of course the final answer for $M_k(n,Q^2)$ is independent of the definition of $\bar{g}^2(Q^2)$ since each redefinition of $\bar{B}^{NS}_{k,n}$ is compensated by the corresponding change of the values of Λ extracted from experiment.

This completes the presentation of the formal expressions necessary for a phenomenological analysis of the moments of deep-inelastic non-singlet structure functions. The basic formula is given in Eq. (2.8) and the numerical values of the relevant parameters Z^{NS}_n, d^n_{NS} and $\bar{B}^{NS}_{k,n}$ are collected in Table 1. Comparison of the formula (2.8) with deep-inelastic data can be found in Refs. 2, 15.

We shall now express the formula (2.8) in terms of the moments of parton distributions. As discussed already in a few papers the definition of parton distributions beyond the leading order is not unique.[7,8] In order to simplify the presentation in this and subsequent sections we shall restrict our discussion to one specific definition[8] of parton distributions. Discussion of another definition[7] is presented in the Appendix.

2.2 Intuitive Approach

Our discussion applies to any non-singlet structure function but for definitness we shall here concentrate on the contributions involving valence quark distributions.[16] Their moments are defined as follows

$$<V(Q^2)>_n = \int_0^1 dx\ x^{n-1} V(x,Q^2) \tag{2.15}$$

In the leading order of asymptotic freedom

$$M_k^{NS}(n,Q^2)\Big|_{valence} = \delta_{NS}^{(k)} <V(Q^2)>_n \qquad k=2,3 \tag{2.16}$$

with

$$<V(Q^2)>_n = <V(Q^2)>_n \left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{NS}^n} \tag{2.17}$$

We shall now generalize Eqs. (2.16) and (2.17) beyond the leading order. To this end we use the definition of parton distributions of Altarelli et al.[8] to obtain

i) Generalization of Eq. (2.16):[17]

$$M_2^{NS}(n,Q^2) = \delta_{NS}^{(2)} <V(Q^2)>_n^{(a)} \tag{2.18}$$

$$M_3^{NS}(n,Q^2) = \delta_{NS}^{(3)} \langle V(Q^2)\rangle_n^{(a)} \left[1 + \frac{\bar{g}^2(Q^2)}{16\pi^2} (\bar{B}_{3,n}^{NS} - \bar{B}_{2,n}^{NS})\right] . \qquad (2.19)$$

ii) Generalization of Eq. (2.17):

$$\langle V(Q^2)\rangle_n^{(a)} = \langle V(Q_0^2)\rangle_n^{(a)} L_n^{(a)}(Q^2,Q_0^2) \qquad (2.20)$$

where

$$L_n^{(a)}(Q^2,Q_0^2) = \left[1 + \frac{\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)}{16\pi^2} Z_n^{NS}\right] \cdot \left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d_{NS}^n}$$

$$\cdot \left[1 + \frac{\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)}{16\pi^2} \bar{B}_{2,n}^{NS}\right] . \qquad (2.21)$$

The subscript "a" distinguishes the definition of valence quark distribution in question from the definition in the Appendix and from the leading order formula (2.17). Comparing Eqs. (2.18) and (2.21) with Eq. (2.8) we observe that $\langle V(Q^2)\rangle_n^{(a)}$ contains all next to leading order corrections to $M_2^{NS}(n,Q^2)$. Furthermore

$$\langle V(Q_0^2)\rangle_n^{(a)} = A_n^{NS}(Q_0^2)\left[1 + \frac{\bar{g}^2(Q_0^2)}{16\pi^2} \bar{B}_{2,n}^{NS}\right] . \qquad (2.22)$$

We note also that because $\bar{B}^{NS}_{3,n} \neq \bar{B}^{NS}_{2,n}$ the Q^2 evolution of $M^{NS}_3(n,Q^2)$ is different from $M^{NS}_2(n,Q^2)$. This is to be contrasted with the leading order formula (2.16). We shall discuss it in more detail in Section IV. Finally it should be remarked that for n=1

$$d^1_{NS} = Z^{NS}_1 = \bar{B}^{NS}_{2,1} = 0 \tag{2.23}$$

and consequently

$$\int_0^1 dx\ V^{(a)}(x,Q^2) = \text{const} = 3 \tag{2.24}$$

where the last equality is our input in accordance with parton model ideas.

III. SIMPLE PARAMETRIZATIONS

We shall now present a simple procedure for the inversion of the moment equations (2.18) - (2.21). This procedure is a straightforward generalization of the method of Ref. 5 which has been used successfully to invert the leading order formula (2.17). The present procedure consists of three steps[18]

i) write

$$xV^{(a)}(x,Q^2) = \frac{3\, x^{\eta_1^{(a)}(\bar{s},\Lambda)}\,(1-x)^{\eta_2^{(a)}(\bar{s},\Lambda)}}{B(\eta_1^{(a)}(\bar{s},\Lambda),1+\eta_2^{(a)}(\bar{s},\Lambda))} \tag{3.1}$$

where

$$\eta_i^{(a)}(\bar{s},\Lambda) = \eta_i^{(a)}(0) + \eta_i'^{(a)}(\Lambda)\bar{s} \qquad i=1,2 \tag{3.2}$$

and

$$\bar{s} = -\ln\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)} \tag{3.3}$$

with $\bar{g}^2(Q^2)$ given by Eq. (2.10). The "slopes" $\eta_i'(\Lambda)$ depend generally on Λ. The appearance of Euler's beta function B(,) is necessary if we want to satisfy the sum rule (2.24).

ii) Find $\eta_i^{(a)}(0)$'s from the data at $Q^2=Q_0^2$ and calculate the moments of $\langle V(Q_0^2)\rangle_n^{(a)}$. In this way the r.h.s. of Eq. (2.20) is known for any Q^2 up to the single parameter Λ which is to be determined by comparing the scaling violations predicted by the theory with those observed in experiment.

iii) Finally determine $\eta_i^{\prime(a)}$ by fitting the moments obtained from (3.1) - (3.3) to those predicted by the theory i.e., Eq. (2.20). As a result of this procedure we obtain analytic expression for F_2^{NS}:

$$F_2^{NS}(x,Q^2) = \delta_{NS}^{(2)} xV^{(a)}(x,Q^2) \tag{3.4}$$

with $xV^{(a)}(x,Q^2)$ given by Eq. (3.1). It is now a simple matter to invert Eq. (2.19) for $F_3^{NS}(x,Q^2)$. Applying convolution theorem to Eq. (2.19) we obtain

$$x\,F_3(x,Q^2) = \delta_{NS}^{(3)} \int_x^1 \frac{dy}{y}\left[yV^{(a)}(y,Q^2)\right] \sigma_3^{(a)}\left(\frac{x}{y},\bar{g}^2(Q^2)\right) \tag{3.5}$$

where

$$\sigma_3^{(a)}(x,\bar{g}^2(Q^2)) = \delta(1-x) - \frac{\bar{g}^2(Q^2)}{16\pi^2}\left[\frac{4}{3}\,x(1+x)\right] . \tag{3.6}$$

The moments of $\sigma_3^{(a)}$ are equal to the square bracket in Eq. (2.19). Before making explicit application of the procedure above let us make a few comments. The outlined procedure differs from that of Ref. 5 in replacing the leading order parameter.

$$s = -\ln \frac{\bar{g}^2(Q^2)|_{L.O.}}{\bar{g}^2(Q_0^2)|_{L.O.}} = \ln \frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \tag{3.7}$$

by the parameter $\bar{s}$ which includes two-loop contributions to the β function, and in replacing the leading order exponents[20]

$$\eta_i(s) = \eta_i(0) + \eta_i' s \tag{3.8}$$

by $\eta_i^{(a)}(\bar{s},\Lambda)$ of Eq. (3.2). As discussed in the appendix the replacement of s by $\bar{s}$ almost entirely takes care of two-loop contributions to the Q^2 evolution of $\bar{g}^2(Q^2)$. The effect of Z_n^{NS} and $\bar{B}_{2,n}^{NS}$ being non-zero is then represented by the change of the slopes $\eta_i'^{(a)}$ relative to the leading order slopes η'_i. In spite of the success of the method of Ref. 5 as applied to the leading order formula (2.17) it is a priori not obvious that this method works beyond the leading order. This is due to the fact that the n and Q^2 dependence of next to leading order corrections are (in particular for large n or large x) quite different from that in the leading order. To our surprise however the procedure of Ref. 5 after the modification $s \to \bar{s}$ turns out to be a very useful tool for inversion of next to leading order corrections. We find that the expressions (3.1)-(3.3) reproduce up to the accuracy of 0.5% the Q^2 dependence of the first 20 moments in Eq. (2.20) for 5 $\mathrm{GeV}^2 < Q^2 < 200\ \mathrm{GeV}^2$ and up to the accuracy of 1%-2% for 5 $\mathrm{GeV}^2 < Q^2 < 5000\ \mathrm{GeV}^2$. We have verified (see Appendix) that the method reproduces correctly (1%-3% level) the next to leading order corrections in the range $0.02 < x < 0.80$. For $x > 0.80$ the formula (3.10) is less reliable and should not be used. Finally it should be remarked that the method can be trivially extended to the input distributions at $Q^2 = Q_0^2$ of the form

$$\sum_i A_i \, x^{B_i} (1-x)^{C_i} \tag{3.9}$$

by applying the procedure separately to each term in Eq. (3.9). We now turn to the explicit applications of our procedure.

IV. NUMERICAL ESTIMATES

To illustrate our procedure we have taken as the input the valence quark distribution found by the CDHS group[21] at $Q_0^2=5\ \mathrm{GeV}^2$. Choosing the same[22] input for the formula (3.1) and for the corresponding leading order expression with $\eta_i^{(a)}$ replaced by η_i of Eq (3.8) we obtain

$$\eta_1^{(a)}(0) = \eta_1(0) = 0.56 \quad ; \quad \eta_2^{(a)}(0) = \eta_2(0) = 2.71 \quad . \tag{4.1}$$

Using next the procedure of the preceding Section we have found the slopes $\eta_i'^{(a)}(\Lambda)$ for $\Lambda=0.3$ GeV and $\Lambda=0.5$ GeV. Correspondingly the method of Ref. 5 leads to leading order slopes η_i' of Eq. (3.8). The results are collected in Table 2. In obtaining these results we have used the moments from n=2 to n=20 in the range of Q^2 from 5 GeV^2 to 200 GeV^2. As the reader may check the formula (3.10) with the relevant parameters $\eta_i^{(a)}(0)$ of Eq. (4.1) and $\eta_i'^{(a)}(\Lambda)$ in the Table 2 reproduces very well the moment equations (2.20-2.21). The same comments apply to the corresponding leading order expression. In finding the slopes we have used here the moments up to n=20 [23] in order to reproduce well the large $x\,(0.6<x<0.8)$ behavior of the structure functions, where the higher order corrections are most important.

In Fig. 1 we plot $F_2^{NS}(x,Q^2)$ as function of Q^2 for various fixed values of x. We show the curves corresponding to various cases collected in the Table 2. In order to illustrate further the pattern of next to leading order corrections to the Q^2 dependence of $F_2^{NS}(x,Q^2)$ we plot in Fig. 2 the ratio

$$\frac{F_2^{NS}(x,Q^2)|_{H.O.}}{F_2^{NS}(x,Q^2)|_{L.O.}}$$

where H.O. and L.O. distinguish between the leading order (L.O.) Q^2 dependence and the one (H.O.) with next to leading order corrections taken into account. The following observations can be made on the basis of Figs. 1 and 2.:

a) Keeping the same value of $\Lambda(\Lambda=0.5)$ in the higher order expression as the one[24] in the leading order formula we observe that the next to leading order corrections increase the scaling violations at both small (faster increase) and large (faster decrease) values of x. The effect is considerably larger at large values of x. For $0.3<x<0.5$ the H.O and L.O. curves are for the Q^2 range considered very close to each other.

b) The decrease of Λ in the higher order expression to $\Lambda=0.3$ GeV has a very little effect for the Q^2 evolution at small values of x but changes considerable the scaling violations at moderate and large values of x as compared to the case $\Lambda=0.5$ GeV.

c) From this it follows that with a fixed (x independent) Λ it is impossible to bring the leading order and higher order predictions on top of each other. Therefore very accurate experiments which could measure 10-20% effects should be able to see the non-trivial x and Q^2 dependence of next to leading order corrections.

In Fig. 3 we plot the ratio

$$\frac{F_3^{NS}(x,Q^2)_{H.O.}}{F_2^{NS}(x,Q^2)_{H.O.}}$$

as function of x for various values of Q^2. In the leading order this ratio is equal unity. The effect of next to leading order corrections is mainly seen at small values of x, where the structure function F_3 is predicted to be slightly smaller than the non-singlet component of F_2^{NS}. For very large Q^2 the leading order result is reproduced. The Figure shows how this limit is approached with increasing Q^2. It should be remarked that in order to measure the ratio above one has to subtract first the singlet contribution from F_2, which is dominant at small values of x. This makes the experimental tests of predictions of Fig. 3 difficult.

V. SUMMARY

In this paper we have presented analytic expressions for non-singlet parton distributions with a Q^2-dependence given by asymptotic freedom with leading and next to leading order corrections taken into account. Since the parton distributions beyond the leading order can be defined in various ways we have discussed two examples; one in the main text and one in the Appendix. We hope that the simple inversion method presented here will be of help to both phenomenologist and experimentalists in their study of scaling violations and in particular in the study of higher order effects. The input distributions at $Q^2=Q_0^2$ discussed in our paper are not the only possible parametrizations of the data. Our method can also be used with more general input distributions of Eq. (3.9).

It is clear that having the Q^2 dependent parton distributions at hand it is a simple matter to estimate asymptotic freedom effects, and in particular higher order effects, in other processes, such as Drell-Yan process and other deep-inelastic processes. How the parton distributions discussed in our paper should be used in the QCD formulae for other processes has been discussed in some detail in Ref. 26, which the interested reader may consult.

We have seen that higher order effects introduce a non-trivial x and Q^2 dependence of the deep-inelastic structure functions which accurate experiments should be able to test. The comparison of our results with experimental data is beyond the scope of this paper.

It would be interesting to extend the method presented here to the singlet parton distributions. In the leading order simple analytic expressions for the Q^2 evolution of singlet parton distributions work only for $0.02<x<0.3$.[5] For $x>0.3$ one is led to very complicated analytic expressions, examples of which can be found in Ref. 27. The same situation is expected beyond the leading order. When completing this paper we received the papers by Gonzalez-Arroyo, Lopez and Yndurain[30] and Duke and Roberts,[31] in which the structure functions with next to leading order corrections have been obtained using different methods from the one presented here.

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APPENDIX

Second Definition of Parton Distributions

Here we shall discuss another definition[7] of parton distributions beyond the leading order. The generalizations i) and ii) of section 2.2 are now as follows

i) Generalization of Eq. (2.16):

$$M_k^{NS}(n,Q^2) = \delta_{NS}^{(k)} \langle V(Q^2)\rangle_n^{(b)} \left[1 + \frac{\bar{g}^2(Q^2)}{16\pi^2} \bar{B}_{k,n}^{NS}\right] \qquad k=2,3 \quad (A.1)$$

ii) Generalization of Eq. (2.17):

$$\langle V(Q^2)\rangle_n^{(b)} = \langle V(Q_0^2)\rangle_n^{(b)} L_n^{(b)}(Q^2,Q_0^2) \qquad (A.2)$$

where

$$L_n^{(b)}(Q^2,Q_0^2) = \left[1 + \frac{\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)}{16\pi^2} Z_n^{NS}\right]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d_{NS}^n} \qquad (A.3)$$

The subscript "b" distinguishes the definition of valence quark distribution in question from the definition "a" of Section 2.2 and from the leading order formula (2.17).

Comparing Eqs. (A.1)-(A.3) with Eq. (2.8) we observe that $\langle V(Q^2)\rangle_n^{(b)}$ contains next to leading order corrections related to $\gamma_{NS}^{(1),n}$ and β_1. The $\bar{g}^2$ corrections to $C_{k,n}^{NS}(1,\bar{g}^2)$ are explicitly factored out. Furthermore

$$\langle V(Q_0^2)\rangle_n^{(b)} = A_n^{NS}(Q_0^2) \quad . \tag{A.4}$$

Thus in this definition the moments of valence quark distribution at Q_0^2 are equal to the matrix elements of a non-singlet operator normalized at Q_0^2. Relation (A.4) is of course true for any Q^2. Note also that the sum rule (2.24) is satisfied by $V^{(b)}(x,Q^2)$. Before showing how Eqs. (A.1)-(A.3) can be inverted let us briefly compare the two definition "a" and "b".

The parton distributions in the definition a) are renormalization prescription independent. This is not the case in the example b) discussed here but the renormalization prescription dependence of $\langle V^{(b)}(Q^2)\rangle_n$ is cancelled by that of $\bar{B}_{k,n}^{NS}$. Since one can define parton distributions in many ways anyhow, one should not worry about this renormalization prescription dependence of parton distribution in example b).

We next notice that whereas the input distributions for k=2 at $Q^2=Q_0^2$ in the example a) will be the same as in the leading order[28] (i.e., the data for $F_2^{NS}(x,Q^2)$

does not change), the input distributions in the example b) will differ considerably at low Q^2 and large x from those used in the leading order phenomenology. The reason is that $C_n^{NS}(1,\bar{g}^2)$ differs considerably from 1 for low Q^2 and large n.

Needless to say expressions (A.1) and (2.18-2.19) are equivalent (through order $\bar{g}^2$) representations of next to leading order corrections to deep-ineleastic structure functions.

In order to invert Eqs. (A.1)-(A.3) we proceed as follows. We first apply the procedure of Section III to Eq. (A.2). Now from the beginning we expect this method of inversion to work well because Z_n^{NS} which enter Eq. (A.3) are small and the equation (A.2) for the Q^2-dependence of valence quark distribution is essentially the same as the leading order equation (2.17) except for the change in the formula for $\bar{g}^2(Q^2)$. This change is taken almost entirely into account by the modification $s \to \bar{s}$. Indeed applying the procedure of Section III to Eq. (A.2) and choosing for illustration the input distribution of Eq. (4.1) we obtain for the exponents $\eta_i^{(b)}$

$$\eta_1^{(b)}(\bar{s},\Lambda) = 0.56 - 0.185\,\bar{s}$$

$$\eta_2^{(b)}(\bar{s},\Lambda) = 2.71 + 0.745\,\bar{s} \qquad (A.5)$$

for both $\Lambda = 0.3$ and 0.5 GeV. As expected the slopes $\eta_i^{\prime(b)}$ are very close to the slopes $\eta_i^{\prime}$ relevant for the leading order (see Table II). In order to complete the inversion of Eq. (A.1) we apply the convolution theorem to Eq. (A.1) to obtain

$$F_2^{NS}(x,Q^2) = \delta_{NS}^{(2)} \int_x^1 \frac{dy}{y}\left[yV^{(b)}(y,Q^2)\right] \sigma_2^{(b)}\left(\frac{x}{y},\bar{g}^2\right) \tag{A.6}$$

and

$$xF_3^{NS}(x,Q^2) = \delta_{NS}^{(3)} \int_x^1 \frac{dy}{y}\left[yV^{(b)}(y,Q^2)\right] \sigma_3^{(b)}\left(\frac{x}{y},\bar{g}^2\right) \tag{A.7}$$

where[29]

$$\sigma_k^{(b)}(x,\bar{g}^2) = \delta(1-x) + \frac{\bar{g}^2(Q^2)}{16\pi^2}\, \bar{B}_k^{NS}(x) \quad , \tag{A.8}$$

$$\bar{B}_2^{NS}(x) = \frac{4}{3}\, x\left\{2(1-x)\ln\frac{(1-x)}{x} + 4x\left(\frac{\ln(1-x)}{1-x}\right)_+ \right.$$

$$- \frac{3x}{(1-x)_+} + 3 + 4x - 4x\,\frac{\ln x}{1-x} \tag{A.9}$$

$$\left. - \left(2\,\frac{\pi^2}{3} + 9\right)\delta(1-x)\right\}$$

and

$$\bar{B}_3^{NS}(x) = \bar{B}_2^{NS}(x) - \frac{4}{3}x(1+x) \quad . \tag{A.10}$$

The moments of $\bar{B}_2^{NS}(x)$ and of $\bar{B}_3^{NS}(x)$ are equal to $\bar{B}_{2,n}^{NS}$ and $\bar{B}_{3,n}^{NS}$ respectively. Notice that in this example the dominant part of higher order corrections is inverted exactly.

The symbol "+" in Eq. (A.9) is defined as follows

$$\int_0^1 \frac{h(x)}{(1-x)_+} dx \equiv \int_0^1 \frac{h(x)-h(1)}{(1-x)} \tag{A.11}$$

and

$$\int_0^1 dx\ h(x)\left(\frac{\ln(1-x)}{1-x}\right)_+ \equiv \int_0^1 dx(h(x)-h(1))\ \frac{\ln(1-x)}{(1-x)} \tag{A.12}$$

where h(x) is a function regular at the end points.

Since in this example the dominant part of higher order corrections is inverted exactly we can use it to test the accuracy of the inversion procedure of Section III. In fact if the procedure of Section III is good then the following equality should be satisfied

$$xV^{(a)}(x,Q^2) = \int_x^1 \frac{dy}{y}[yV^{(b)}(y,Q^2)]\tilde{\sigma}_2^{(b)}(\frac{x}{y},\bar{g}^2) \tag{A.13}$$

where

$$\tilde{\sigma}_2^{(b)}(x,\bar{g}^2) = \delta(1-x) + \frac{\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)}{16\pi^2}\bar{B}_2^{NS}(x) \tag{A.14}$$

with $\bar{B}_2^{NS}(x)$ given by Eq. (A.9). In testing (A.13) the same input at $Q^2=Q_0^2$ should be chosen for $xV^{(a)}$ and $xV^{(b)}$. We have verified that equation (A.13) is indeed very well satisfied (1-3% level) for $5<Q^2<200$ GeV^2 and $0.02<x<0.80$.

Finally we want to show with an example that, although $xV^{(a)}(x,Q^2)$ and $xV^{(b)}(x,Q^2)$ as extracted from the data differ from each other, the formulae (3.1) and (A.6) lead to a very good approximation, to the same Q^2 dependence of $F_2^{NS}(x,Q^2)$. To this end we first find $xV^{(b)}(x,Q^2)$ at $Q^2 = 5$ GeV^2 which through Eq. (A.6) leads to $F_2^{NS}(x,5)$ of Section IV. Taking $\Lambda = 0.5$ GeV and choosing for $xV^{(b)}$ the functional form of Eq. (3.1) we find (at $Q^2=5$ GeV^2)

$$\eta_1^{(b)}(0) = 0.66 \quad \text{and} \quad \eta_2^{(b)}(0) = 3.40 \tag{A.15}$$

Subsequently using this input and applying the procedure of Section III to Eq. (A.2) we obtain

$$\eta_1^{(b)}(\bar{s},0.5) = 0.66 - 0.229\bar{s}$$

$$\eta_2^{(b)}(\bar{s},0.5) = 3.40 + 0.762\bar{s} \quad . \tag{A.16}$$

$xV^{(a)}(x,Q^2) = F_2^{NS}(x,Q^2)\,|_a$ of Section IV, $xV^{(b)}(x,Q^2)$ given by (A.16) and $F_2^{NS}(x,Q^2)|_b$ given by (A.6) are shown in Figs. 4 and 5 for Λ=0.5 GeV and Q^2=5 GeV^2 and Q^2=100 GeV^2. We observe that although $xV^{(a)}(x,Q^2)$ and $xV^{(b)}(x,Q^2)$ differ, expecially for large x, from each other, $F_2^{NS}(x,Q^2|_a$ and $F_2^{NS}(x,Q^2)|_b$ are consistent with each other as it should be.

FOOTNOTES AND REFERENCES

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16. $V(x,Q^2)=u_V(x,Q^2)+d_V(x,Q^2)$ where u_V and d_V are up and down valence quark distributions. Our method applies also to $u_V(x,Q^2)-d_V(x,Q^2)$ and other non-singlet structure which can be parametrized by Eq. (3.1).

17. In order to simplify the discussion we only include the contributions of $V(x,Q^2)$ to the structure functions in question and drop in the following the subscript "valence".

18. The method of Ref. 5 has been already applied in Ref. 19 without any modifications to invert the next to leading order corrections. The modification (3.3) of the method of Ref. 5 introduced here improves considerably the accuracy of inversion. Furthermore the next to leading order corrections used in Ref. 19 were unfortunately wrong.

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22. We have chosen the same inputs only for illustration. In practice the input distributions should be found from the best fit of the formula (3.1) to the data in some range of Q^2 with Λ and $\eta_i^{(a)}(0,\Lambda)$ being the free parameters. Because of errorbars and different Q^2 dependence predicted by the next to leading order corrections, the input distributions in the leading order formulae and higher order formulae could be slightly different.

23. We have also used as many moments in finding the leading order slopes η_i'. Therefore the small difference between the slopes found here and those in the Ref. 21 where only twelve moments have been used. The difference in question is also due to the smaller range in Q^2 used here in the inversion which improves the accuracy of the procedure.

24. The value Λ=0.5 GeV in the case of leading order is dictated by the CDHS data.[21]

25. Here we drop all charge factors $\delta_{NS}^{(k)}$.

26. See R.K. Ellis, H. Georgi, M. Machacek, H.D. Politzer and G.G. Ross, Nucl. Phys. B152, 453 (1979); S. Gupta and A.H. Mueller, Phys. Rev. D20, 118 (1979); A.J. Buras, FERMILAB-Conf-79/65-THY.

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FIGURE CAPTIONS

Fig. 1 The Q^2 behavior of $F_2^{NS}(x,Q^2) = xV^{(a)}(x,Q^2)$ for various values of x. The curves correspond to the leading order (L.O.) with Λ=0.5 GeV and to two cases (H.O. for Λ=0.3 and Λ=0.5) in which next to leading order corrections have been included.

Fig. 2 The x dependence of the ratio $F_2^{NS}(x,Q^2)|_{H.O.}/F_2^{NS}(x,Q^2)|_{L.O.}$ for various values of Q^2 and Λ. The notation is as in Fig. 1.

Fig. 3 The x dependence of the ratio $F_3^{NS}|_{H.O.}/F_2^{NS}|_{H.O.}$ for various values of Q^2 an Λ. In the leading order this ratio is equal unity.

Fig. 4 The valence quark distribution for the two definitions discussed in the main text (a)) and the appendix (b)) for Q^2=5 GeV^2. $F_2^{NS}(x,Q^2)|_b$ has been obtained on the basis of Eqs. (A.6), (A.8) and (A.9) with Λ=0.5 GeV.

Fig. 5 The cases discussed in Fig. 4 for Q^2=100 GeV^2 and Λ=0.5 GeV.

TABLE I. Numerical values of the parameters d^n_{NS}, Z^{NS}_n, $\bar{B}^{NS}_{2,n}$ and $\bar{B}^{NS}_{3,n}$ for various values of n and f=4.

n	d^n_{NS}	Z^{NS}_n	$\bar{B}^{NS}_{2,n}$	$\bar{B}^{NS}_{3,n}$
2	0.427	1.65	0.44	-1.78
3	0.667	1.94	3.22	-1.67
4	0.837	2.05	6.07	4.87
5	0.971	2.11	8.73	7.75
6	1.08	2.16	11.18	10.4
7	1.17	2.21	13.44	12.7
8	1.25	2.25	15.53	14.9
9	1.33	2.29	17.48	16.9
10	1.39	2.33	19.30	18.8
11	1.45	2.38	21.01	20.5
12	1.50	2.41	22.63	22.2
13	1.55	2.45	24.2	23.8
14	1.60	2.49	25.6	25.2
15	1.64	2.53	27.0	26.7
16	1.68	2.56	28.3	28.0
17	1.72	2.60	29.6	29.3
18	1.76	2.63	30.8	30.5
19	1.79	2.66	32.0	31.7
20	1.82	2.69	33.1	32.8

TABLE II. Parameters which enter Eq. (3.1) for the input of Eq. (4.1) at $Q_0^2=5$ GeV^2 and $\Lambda=0.3$ GeV and $\Lambda=0.5$ GeV.

Case	Λ	$\eta_1(0)$	$\eta_2(0)$	η_1'	η_2'
Leading	0.3	0.56	2.71	-0.170	0.745
Order	0.5	0.56	2.71	-0.170	0.745
Higher Order	0.3	0.56	2.71	-0.117	1.262
(definition a)	0.5	0.56	2.71	-0.085	1.490

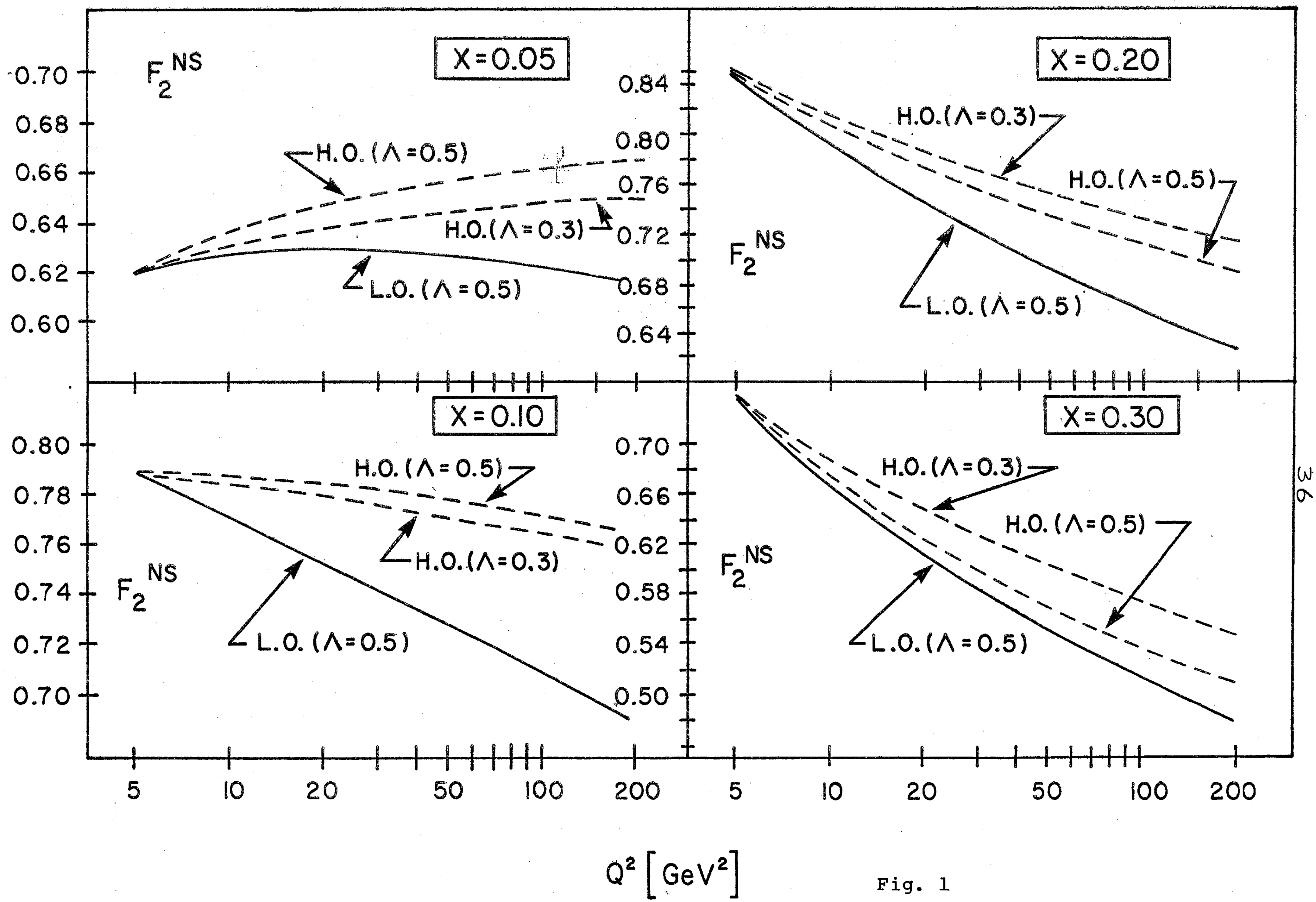

X=0.05
X=0.20
X=0.10
X=0.30
F_2^{NS}
H.O. (Λ=0.5)
H.O. (Λ=0.3)
L.O. (Λ=0.5)
0.70
0.68
0.66
0.64
0.62
0.60
0.84
0.80
0.76
0.72
0.68
0.64
0.80
0.78
0.76
0.74
0.72
0.70
0.70
0.66
0.62
0.58
0.54
0.50
5
10
20
50
100
200
Q^2 [GeV^2]

Fig. 1

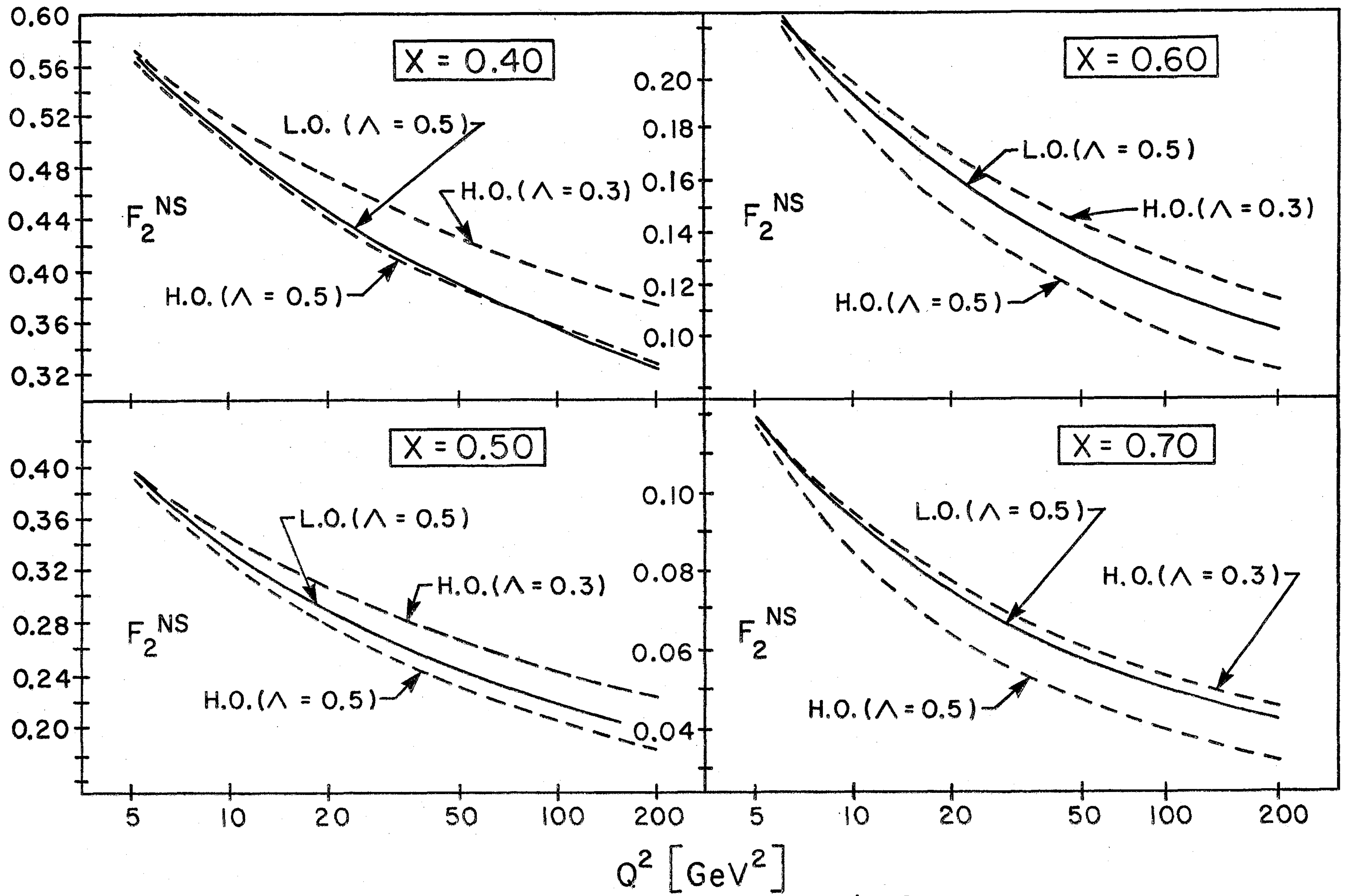
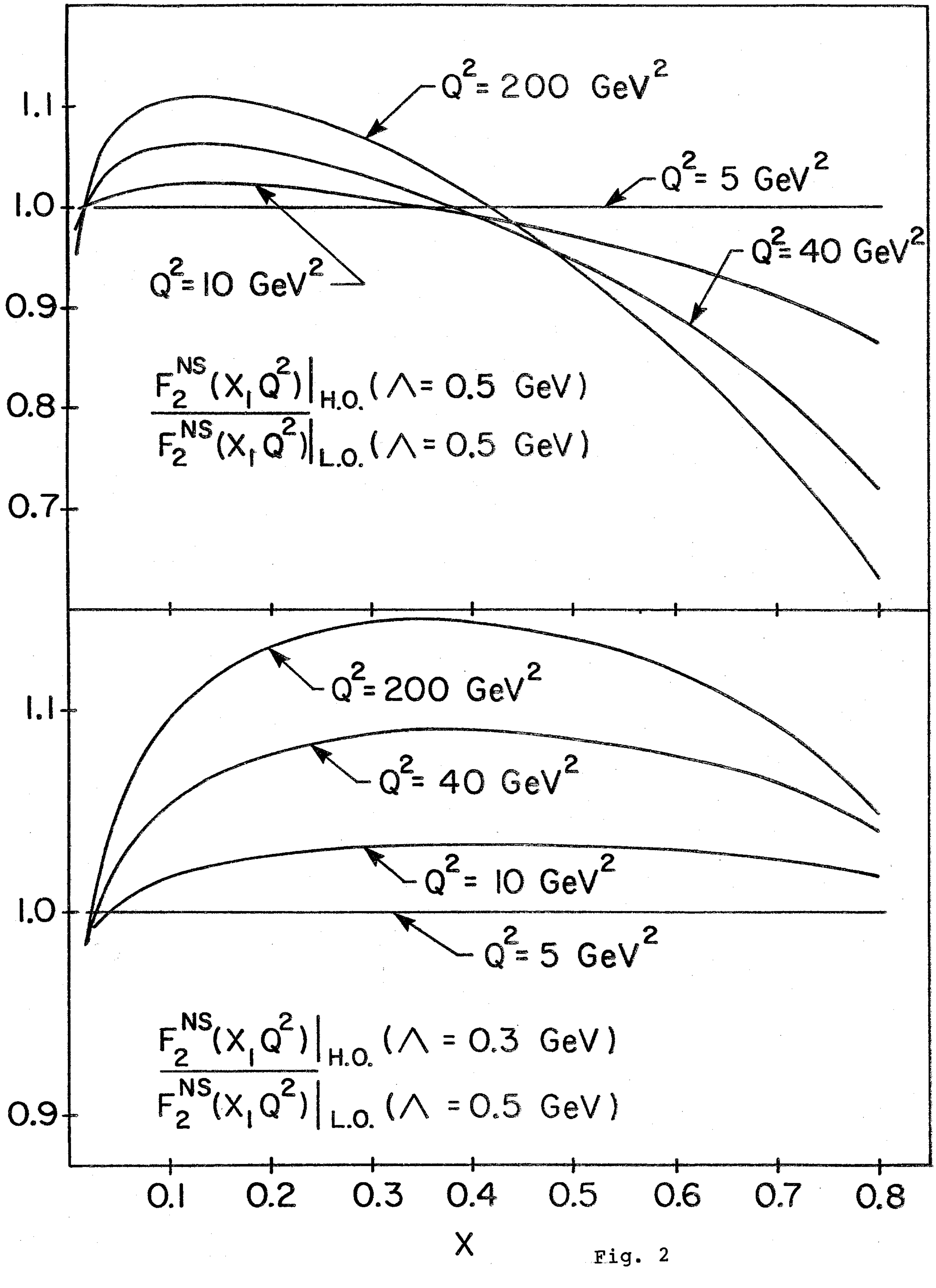

X = 0.40
X = 0.60
X = 0.50
X = 0.70
F_2^{NS}
L.O. (Λ = 0.5)
H.O. (Λ = 0.3)
H.O. (Λ = 0.5)
0.60
0.56
0.52
0.48
0.44
0.40
0.36
0.32
0.20
0.18
0.16
0.14
0.12
0.10
0.40
0.36
0.32
0.28
0.24
0.20
0.10
0.08
0.06
0.04
5
10
20
50
100
200
Q^2 [GeV^2]

Fig. 1 Cont.



$Q^2 = 200\ GeV^2$
$Q^2 = 5\ GeV^2$
$Q^2 = 40\ GeV^2$
$Q^2 = 10\ GeV^2$
1.1
1.0
0.9
0.8
0.7
$\frac{F_2^{NS}(X, Q^2)|_{H.O.} (\Lambda = 0.5\ GeV)}{F_2^{NS}(X, Q^2)|_{L.O.} (\Lambda = 0.5\ GeV)}$
$Q^2 = 200\ GeV^2$
$Q^2 = 40\ GeV^2$
$Q^2 = 10\ GeV^2$
$Q^2 = 5\ GeV^2$
1.1
1.0
0.9
$\frac{F_2^{NS}(X, Q^2)|_{H.O.} (\Lambda = 0.3\ GeV)}{F_2^{NS}(X, Q^2)|_{L.O.} (\Lambda = 0.5\ GeV)}$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
X

Fig. 2

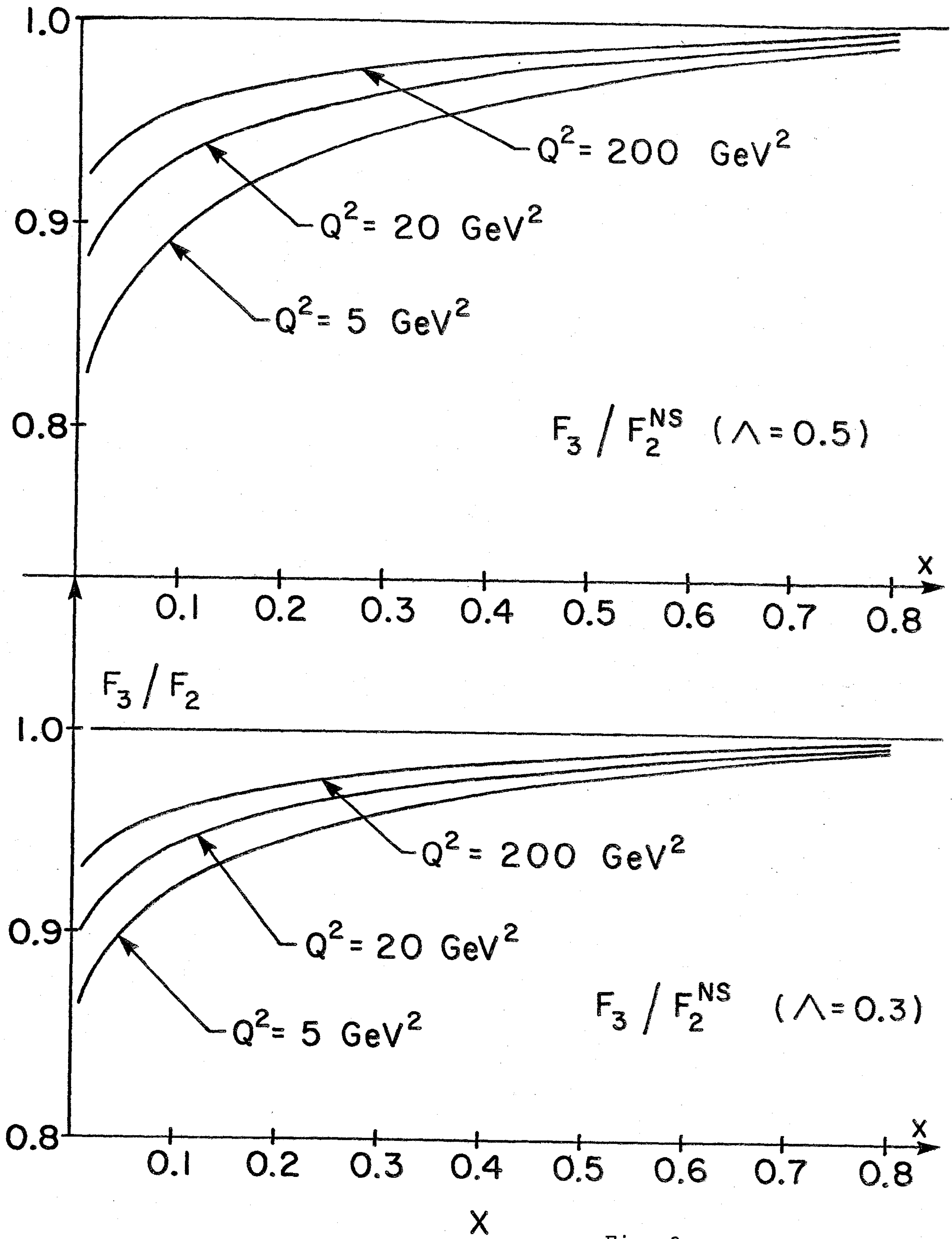

1.0
0.9
0.8
$Q^2 = 200\ GeV^2$
$Q^2 = 20\ GeV^2$
$Q^2 = 5\ GeV^2$
F_3 / F_2^{NS} ($\Lambda = 0.5$)
x
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
F_3 / F_2
1.0
0.9
0.8
$Q^2 = 200\ GeV^2$
$Q^2 = 20\ GeV^2$
$Q^2 = 5\ GeV^2$
F_3 / F_2^{NS} ($\Lambda = 0.3$)
x
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
X

Fig. 3

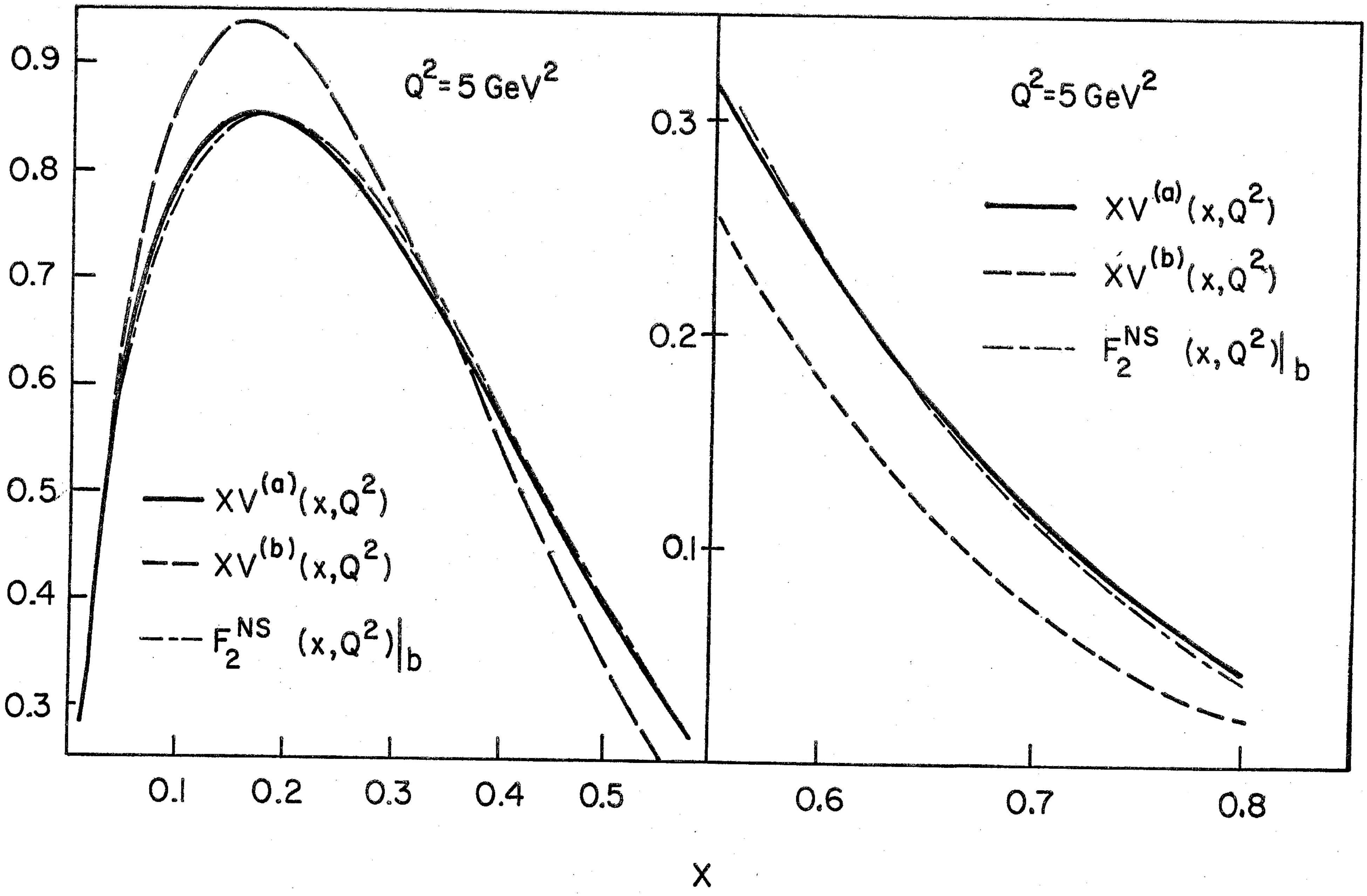
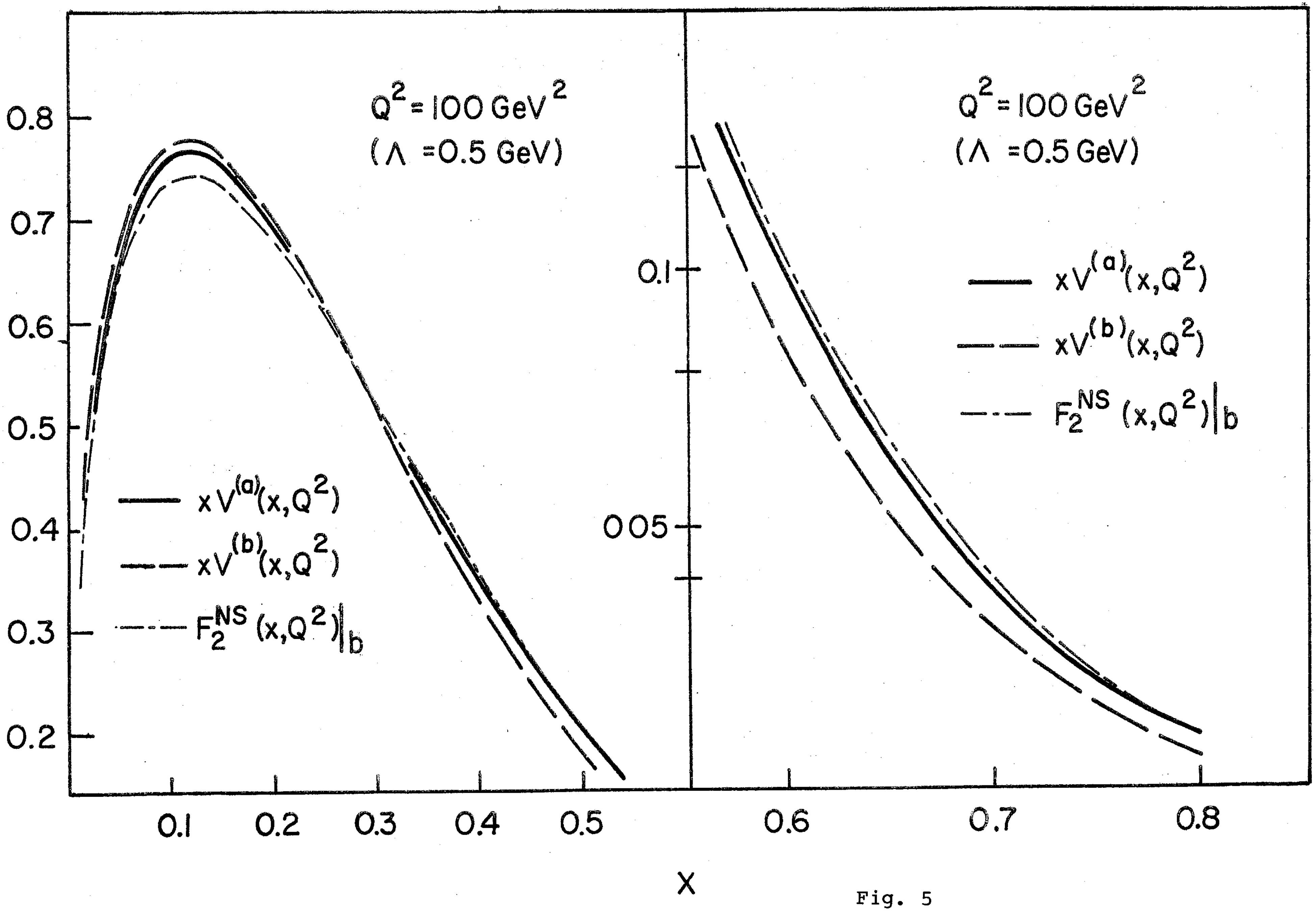

0.9
0.8
0.7
0.6
0.5
0.4
0.3
$Q^2 = 5\ GeV^2$
$XV^{(a)}(x,Q^2)$
$XV^{(b)}(x,Q^2)$
$F_2^{NS}\ (x,Q^2)|_b$
0.1
0.2
0.3
0.4
0.5
0.3
0.2
0.1
$Q^2 = 5\ GeV^2$
$XV^{(a)}(x,Q^2)$
$XV^{(b)}(x,Q^2)$
$F_2^{NS}\ (x,Q^2)|_b$
0.6
0.7
0.8
X

Fig. 4



$Q^2 = 100\ \mathrm{GeV}^2$
$(\Lambda = 0.5\ \mathrm{GeV})$
$xV^{(a)}(x,Q^2)$
$xV^{(b)}(x,Q^2)$
$F_2^{NS}(x,Q^2)|_b$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.2
0.3
0.4
0.5
$Q^2 = 100\ \mathrm{GeV}^2$
$(\Lambda = 0.5\ \mathrm{GeV})$
$xV^{(a)}(x,Q^2)$
$xV^{(b)}(x,Q^2)$
$F_2^{NS}(x,Q^2)|_b$
0.1
005
0.6
0.7
0.8
X

Fig. 5